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A Complete Error Analysis of PINNs for Elliptic Equations Using Projected Stochastic Gradient Descent

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Abstract. Physics-informed neural networks (PINNs) have recently gained attention as a powerful and efficient tool for solving partial differential equations (PDEs). Despite their empirical success, the theoretical understanding of PINNs, especially in the context of over-parameterization, remains incomplete. This paper presents a complete error analysis of over-parameterized PINNs for elliptic equations using projected stochastic gradient descent (PSGD) optimization. Our analysis rigorously examines the interplay of approximation error, statistical error, and optimization error, offering a unified framework for understanding the convergence behavior of PINNs. By leveraging the properties of PSGD, we establish convergence rates and derive conditions on neural network architecture, training sample requirements, and optimization parameters to ensure specified accuracy.

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Key words: Physics-informed neural networks, projected stochastic gradient descent, complete error analysis, over-parameterization.

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1 Introduction

Traditional numerical methods, such as the finite element method [9, 12], have proven to be very effective for solving low-dimensional PDEs. However, they encounter significant difficulties when applied to high-dimensional problems. The impressive success of deep learning in handling high-dimensional data has paved the way for employing deep neural networks in solving high-dimensional PDEs [1, 8, 16, 17, 19, 31, 35, 45, 47, 52]. Due to the excellent approximation power of deep neural networks, several numerical schemes have been proposed for solving PDEs, including the deep Ritz method [16], PDE-net [34], PINNs [45] and weak adversarial networks [52]. Among the various techniques developed, PINNs have emerged as a particularly powerful approach [45]. PINNs not only leverage the robust approximation abilities of deep learning but also seamlessly incorporate the underlying physical laws of the PDEs, making them highly effective for solving high-dimensional problems [25, 43, 44]. The success of PINNs has spurred deeper theoretical analysis, highlighting the need for comprehensive error analysis in deep learning, including approximation, generalization, and optimization errors.

While several studies have investigated the theoretical mechanisms of PINNs [13, 21–23, 26, 32, 36, 38, 39, 41, 46, 49, 50], these analyses exhibit two key limitations. First, they are typically conducted in scenarios where the number of neural network parameters is smaller than the number of training samples. Second, these analyses often do not address optimization errors, which are crucial for a comprehensive understanding of model performance. Specifically, over-parameterized deep neural networks, where the number of parameters significantly exceeds the sample size, are frequently employed in real-world applications due to their computational efficiency during training. Although extensive research has examined the role of over-parameterization in linear and kernel models, particularly in relation to the double descent phenomenon [2–7, 20, 33, 42], the underlying reasons for the effectiveness of over-parameterized deep neural networks remain unclear. Providing theoretical guarantees in such regimes continues to be a fundamental yet challenging problem. Recent studies have reported convergence results for over-parameterized norm-controlled neural networks in both regression and PDE settings [11, 29, 30, 51]. However, these analyses typically assume that optimization algorithms such as gradient descent (GD) or stochastic gradient descent (SGD) yield the empirical risk minimization (ERM) estimator, the theoretically optimal solution, thus neglecting the influence of optimization errors introduced during the training process.

In this work, we establish a complete error analysis of PINNs for elliptic equations in the over-parameterized setting using projected stochastic gradient descent (PSGD) optimization. Our analysis accounts for all three key error components: approximation error, statistical (generalization) error, and optimization error. By integrating these error components within a unified theoretical framework, we derive explicit convergence rates and precise conditions for achieving specified accuracy levels when solving elliptic boundary value problems. This represents a significant advancement in the theoretical understanding of PINNs.