Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2024-0283

Vol. **40**, No. 1, pp. 199-238
July 2026

An Exponential-Free Runge–Kutta Framework for Developing Third-Order Unconditionally Energy Stable Schemes for the Cahn–Hilliard Equation

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Received 19 November 2024; Accepted (in revised version) 22 August 2025

Abstract. In this work, we develop a class of up to third-order energy-stable schemes for the Cahn–Hilliard equation. Building on Lawson's integrating factor Runge–Kutta method, which is widely used for stiff semilinear equations, we discuss its limitations, such as the inability to preserve the equilibrium state and the oversmoothing of interfacial layers in the solution's profile because of the exponential damping effects. To overcome this drawback, we approximate the exponential term using a class of sophisticated Taylor polynomials, leading to a novel Runge–Kutta framework called exponential-free Runge–Kutta. By incorporating stabilization techniques, we analyze the energy stability of the proposed schemes and demonstrate that they preserve the original energy dissipation without time-step restrictions. Furthermore, we conduct an analysis of the linear stability and establish an error estimate in the ℓ^2 norm. A series of numerical experiments validate the high-order accuracy, mass conservation, and energy dissipation of our schemes.

AMS subject classifications: 65M12, 65M15, 65M70

Key words: Cahn–Hilliard equation, exponential-free Runge–Kutta scheme, unconditional energy stability, convergence analysis.

1 Introduction

The Cahn–Hilliard (CH) equation, which belongs to the class of phase field models, was originally introduced by Cahn and Hilliard in [3] to describe phenomena of phase

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separation and coarsening in non-uniform systems, such as glasses, alloys, and fixed-temperature polymer blends. While the CH equation offers an improvement over the Allen–Cahn (AC) equation by ensuring mass conservation, it also introduces higher-order spatial derivatives that increase stiffness and pose challenges for numerical computation. As a result, despite its theoretical advantages, creating a high-order, efficient, and stable numerical scheme that can deal with the stiffness from these spatial derivatives for the CH equation while maintaining the original characteristics is crucial in practical applications.

In this work, we consider the CH equation with the following form:

$$\begin{cases} u_t = \Delta(-\epsilon^2\Delta u + f(u)), & \mathbf{x}\in\Omega,\ t\in(0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x}\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega=\prod_{i=1}^{d}(a_i,b_i)$ is a rectangle when $d=2$ or a cuboid when $d=3$, and the unknown solution u is subject to the periodic boundary condition that represents the difference between two phase proportions. The diffusion coefficient ϵ characterizes the interfacial thickness of the transition region. The nonlinear function $f(\cdot)$ is defined as the derivative of the double-well potential function:

$$F(u) := \frac{1}{4}(u^2-1)^2, \quad f(u) := F'(u) = u^3 - u.$$

As is well known, the CH equation (1.1) is the H^{-1} gradient flow [15] with respect to the Ginzburg-Landau free energy functional $E(u)$, i.e.,

$$u_t = \Delta\Big(\frac{\delta E(u)}{\delta u}\Big),$$

where $\frac{\delta E(u)}{\delta u}$ denotes the standard variational derivative, and energy is represented as:

$$E(u) := \int_\Omega \left(\frac{\epsilon^2}{2}|\nabla u|^2 + F(u)\right)\mathrm{d}\mathbf{x}, \tag{1.2}$$

with the symbol $|u|$ denoting the Euclidean norm. Therefore, the CH equation inherently satisfies the energy dissipation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u) = \Big(u_t, \frac{\delta E(u)}{\delta u}\Big) = -\|(-\Delta)^{-\frac{1}{2}}\frac{\partial u}{\partial t}\|_{L^2}^2 < 0, \quad \forall t>0,$$

where $(\cdot,\cdot)$ and $\|\cdot\|_{L^2}$ are the standard L^2 inner product and norm, respectively, and are defined as

$$(u,v) = \int_\Omega uv\mathrm{d}\mathbf{x}, \quad \|u\|_{L^2} = \Big(\int_\Omega |u|^2\mathrm{d}\mathbf{x}\Big)^{\frac{1}{2}}, \quad \forall u,v\in L^2(\Omega).$$

In addition, the CH equation conserves mass during the time evolution:

$$\int_\Omega u(\mathbf{x},t)\mathrm{d}\mathbf{x} = \int_\Omega u(\mathbf{x},0)\mathrm{d}\mathbf{x}, \quad t>0, \tag{1.3}$$