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# A Locking-Free Weak Galerkin Finite Element Method for Linear Elasticity Problems Based on a Reconstruction Operator

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**Abstract.** The weak Galerkin (WG) finite element method has shown great potential in solving various type of partial differential equations. In this paper, we propose an arbitrary order locking-free WG method for solving linear elasticity problems, with the help of an appropriate $H(div)$-conforming displacement reconstruction operator. Optimal order locking-free error estimates in both the $H^1$-norm and the $L^2$-norm are proved, i.e., the error is independent of the *Lamé* constant $\lambda$. Moreover, the term $\lambda\|\nabla\cdot\mathbf{u}\|_k$ does not need to be bounded in order to achieve these estimates. We validate the accuracy and the robustness of the proposed locking-free WG algorithm by numerical experiments.

**AMS subject classifications**: 65N30, 65N15, 74S05, 74B05, 35J50

**Key words**: Weak Galerkin finite element methods, linear elasticity problems, $H(div)$-conforming displacement reconstruction, locking-free.

## 1 Introduction

In this paper, we consider the linear elasticity problems as follows: Find displacement vector $\mathbf{u}$ satisfying

$$-\nabla\cdot\sigma(\mathbf{u})=\mathbf{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\mathbf{u}=\widehat{\mathbf{g}}, \quad \text{on } \Gamma_D, \tag{1.2}$$

$$\sigma(\mathbf{u})\mathbf{n}=\widehat{\mathbf{t}}, \quad \text{on } \Gamma_N, \tag{1.3}$$

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where $\Omega$ is an open bounded, connected domain in $\mathbb{R}^d$ $(d=2,3)$, and the boundary $\Gamma=\partial\Omega$ is Lipschitz continuous. $\mathbf{f}$ is the body force, $\widehat{\mathbf{g}}$ is the boundary displacement function, $\widehat{\mathbf{t}}$ is the traction force, $\mathbf{n}$ is the unit outward normal direction on boundary $\Gamma$. Here $\Gamma_D$ and $\Gamma_N$ are two subsets of $\Gamma$, satisfying $|\Gamma_D|>0$, $\Gamma_D\bigcap\Gamma_N=\emptyset$, and $\Gamma_D\bigcup\Gamma_N=\Gamma$. In addition, $\sigma(\mathbf{u})$ is the Cauchy stress tensor given by

$$\sigma(\mathbf{u})=2\mu\varepsilon(\mathbf{u})+\lambda(\nabla\cdot\mathbf{u})\mathbf{I},$$

where $\varepsilon(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)$ is the strain tensor, $\mu$ and $\lambda$ are *Lamé* constants, satisfying $0<\mu_1\leq\mu\leq\mu_2\ll\infty$ and $0<\lambda<\infty$.

The weak formulation of (1.1)-(1.3) can be written as: Find $\mathbf{u}\in[H^1(\Omega)]^d$ satisfying $\mathbf{u}=\widehat{\mathbf{g}}$ on $\Gamma_D$ and

$$2\mu(\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))+\lambda(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})=(\mathbf{f},\mathbf{v})+\langle\widehat{\mathbf{t}},\mathbf{v}\rangle_{\Gamma_N},\quad\forall\mathbf{v}\in[H^1_D(\Omega)]^d,\tag{1.4}$$

where $H^1(\Omega)$ and $H^1_D(\Omega)$ are the standard Sobolev spaces defined as follows:

$$\begin{aligned}&H^1(\Omega)=\{v\in L^2(\Omega):\ \nabla v\in[L^2(\Omega)]^d\},\\&H^1_D(\Omega)=\{v\in H^1(\Omega):\ v|_{\Gamma_D}=0\}.\end{aligned}$$

In elasticity theory, it is known that the "locking" phenomenon [1,3,8] arises when the *Lamé* constant $\lambda$ approaches infinity. Conventional finite element scheme often fails to converge to the exact solution or does not reach optimal convergence in such cases. This phenomenon is primarily attributed to the dependence of the finite element error estimates on the *Lamé* constant $\lambda$. Consequently, the coefficients of the error estimates tend towards infinity when $\lambda\to\infty$, significantly impacting the computational accuracy and efficiency of the finite element scheme. In order to overcome the locking phenomenon, some effective techniques have been proposed in various discretizations, such as, the mixed finite element method [19], the nonconforming finite element method [11], the discontinuous Galerkin method [13,32], the virtual element method [2,12], and the weak Galerkin (WG) finite element method [25,33], etc. Among them, methods based on the primal formulation generally require regularity assumptions.

The main purpose of this paper is to propose an arbitrary order locking-free WG method for linear elasticity problems (1.1)-(1.3). The WG method is an extension of the classical Galerkin finite element method. It employs weak functions and introduces weak differential operators to replace the traditional differential operators. A stabilizer is added to ensure the weak continuity of the numerical solution. Comparing to the standard finite element method, it is usually much more convenient to design and implement high-order WG schemes. The WG method was first proposed by J. Wang and X. Ye for solving second order elliptic problems [26], and later applied to Navier-Stokes equations [34], Brinkman equations [20], Maxwell's equations [22], biharmonic equations [21,35], eigenvalue problems [6], Stokes-Darcy problems [7,24], lower regularity problems [31], convective Brinkman-Forchheimer equations [30], wave equation [9], etc.