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A Weak Galerkin Finite Element Method Coupled with Mortar Spectral Element Method for Schrödinger Eigenvalue Problem with an Inverse Square Potential

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Abstract. In this paper, we introduce a weak Galerkin (WG) finite element method coupled with mortar spectral element method (MSEM) to solve the Schrödinger eigenvalue problem with an inverse square potential. For the domain around the inverse square potential, we use the mortar spectral element method to simulate the singularities in eigenfunctions caused by the inverse square potential, while we employ the WG method in the remaining domain. This coupled method can effectively handle the singularity arising from the inverse square potential. Notably, hanging nodes are allowed on the coupled interface. Compared to the conforming finite element method coupled with MSEM, our approach is not constrained by the mesh size of the mortar spectral element. This flexibility permits the use of fine meshes in the WG domain, thereby enhancing accuracy. We provide hp error analysis for both eigenfunctions and eigenvalues. Numerical experiments demonstrate the hp convergence of the theoretical results.

AMS subject classifications: 65N15, 65N25, 65N30, 65N35
Key words: Weak Galerkin finite element method, mortar spectral element method, Schrödinger eigenvalue problem, error estimates.

1 Introduction

The Schrödinger operator is extremely important in science and exists in several different forms. The Schrödinger operator with an inverse square singular potential has recently garnered significant attention due to its fundamental role in both mathematics and

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physics. Mathematically, the inverse square potential exhibits the same homogeneity or "differential order" as the Laplacian, however, it typically induces strong singularities in the Schrödinger eigenfunctions, and, as such, cannot be considered as a lower-order perturbation term [9, 13, 14, 20]. On the other hand, in non-relativistic quantum mechanics, the inverse square potential represents an intermediate threshold between regular and singular potentials [11, 15]. Additionally, the inverse square potential figures prominently in nuclear physics, molecular physics, and quantum cosmology [11, 15].

Based on the variational form, Galerkin-type numerical schemes can be designed. However, even with adaptive schemes, low-order methods exhibit only limited convergence rates [21,22,31]. Similarly, due to the strong singularities of eigenfunctions induced by the singular potential, classical high-order methods, including spectral/spectral element methods, generally fail to achieve exponential convergence rates [8, 17, 23]. In the literature, the idea of incorporating singular terms into the basis functions has been used, which comes at the expenses of compromising the sparsity of the resulting algebraic matrix system. Although this approach improves the convergence rate to some extent, its effectiveness depends on the number of singular terms introduced in [16].

In [23], the authors proposed a new method: the mortar spectral element method (MSEM), which can effectively handle all singularities and construct orthogonal basis functions, thereby achieving very sparse (sometimes diagonal) matrices. Specifically, at each singular point including the origin, an additional disk or sector element is used. This element employs a class of non-polynomial spectral basis functions to model the singularity. The remaining domain are divided into quadrilaterals or triangles, with some boundaries being curved. Conforming basis functions are then applied to the elements in the rest of subdomains. These two types of elements are connected using mortar techniques. Exponential convergence rates $\mathrm{e}^{-\sigma\sqrt{DoF}}$ are observed for different eigenvalues in various numerical experiments, with σ being nearly uniform. Numerical tests were also conducted for the Schrödinger eigenvalue problem on the whole plane and other types of inverse square potential problems [24, 25]. Additionally, Jia et al. [19] conducted a rigorous theoretical analysis of the numerical results in [23] and obtained optimal error estimates.

The mortar element method is a flexible technique for connecting different variational discretizations on each subdomain. In addition to the h finite element method and the spectral element method studied in [6], it can be applied within the frameworks of the hp finite element method [7], and the finite volume method [1]. Consequently, the mortar method finds significant applications in various areas, including those involving the curl and divergence operators [2, 5], as well as fourth-order problems [4]. The weak Galerkin (WG) finite element method is an effective numerical method for solving partial differential equations. In [32], Wang and Ye first proposed the WG method for solving second-order elliptic problems. The WG method has also been applied to a range of different equations, such as: second-order elliptic problems [33, 35], biharmonic equations [37, 39], Stokes equations [26], Brinkman equations [27], Maxwell equations [29, 36], eigenvalue problems [10, 38], elasticity equations [18, 34], and others.