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An Adaptive ANOVA-Based Data-Driven Stochastic Method for Elliptic PDEs with Random Coefficient

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Abstract. In this paper, we present an adaptive, analysis of variance (ANOVA)-based data-driven stochastic method (ANOVA-DSM) to study the stochastic partial differential equations (SPDEs) in the multi-query setting. Our new method integrates the advantages of both the adaptive ANOVA decomposition technique and the data-driven stochastic method. To handle high-dimensional stochastic problems, we investigate the use of adaptive ANOVA decomposition in the stochastic space as an effective dimension-reduction technique. To improve the slow convergence of the generalized polynomial chaos (gPC) method or stochastic collocation (SC) method, we adopt the data-driven stochastic method (DSM) for speed up. An essential ingredient of the DSM is to construct a set of stochastic basis under which the stochastic solutions enjoy a compact representation for a broad range of forcing functions and/or boundary conditions.

Our ANOVA-DSM consists of offline and online stages. In the offline stage, the original high-dimensional stochastic problem is decomposed into a series of low-dimensional stochastic subproblems, according to the ANOVA decomposition technique. Then, for each subproblem, a data-driven stochastic basis is computed using the Karhunen-Loève expansion (KLE) and a two-level preconditioning optimization approach. Multiple trial functions are used to enrich the stochastic basis and improve the accuracy. In the online stage, we solve each stochastic subproblem for any given forcing function by projecting the stochastic solution into the data-driven stochastic basis constructed offline. In our ANOVA-DSM framework, solving the original high-dimensional stochastic problem is reduced to solving a series of ANOVA-decomposed stochastic subproblems using the DSM. An adaptive ANOVA strategy is also provided to further reduce the number of the stochastic subproblems and speed up our method. To demonstrate the accuracy and efficiency of our method, numerical examples are presented for one- and two-dimensional elliptic PDEs with random coefficients.

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1 Introduction

Over the past few decades, there has been growing interest and significant progress in modeling complex physical and engineering systems with uncertainties. Many physical and engineering applications involving uncertainty quantification can be described by stochastic partial differential equations (SPDEs). One of the essential challenges in these applications is how to solve SPDEs efficiently when the dimension of stochastic input variables is high. These problems are computationally prohibitive for some of the existing numerical methods, such as stochastic finite element method [14], Wiener chaos expansion method [19, 28], generalized polynomial chaos (gPC) methods [29, 35, 37, 38], and stochastic collocation method [1, 39]. One of the reasons is that these methods use a problem-independent basis, which produces a very large coupled system when the dimension of the input stochastic variables is high.

For stochastic problems with high stochastic input dimensions, we employ the functional Analysis of Variance, or ANOVA method [4, 16] as a dimension-reduction technique. This is motivated by the observation that for many real physical systems, only a relatively small number of stochastic dimensions is important and will significantly impact the stochastic systems' outputs. The ANOVA decomposition was introduced by Fisher [9]. Later in 1948, Hoeffding successfully applied ANOVA decomposition to study U-statistics [17]. ANOVA also was used for uncertainty quantification in [36] and was employed in gPC for solving high-dimensional stochastic PDE systems in [5, 10, 12, 26, 27, 40, 42]. In [10] ANOVA was integrated with a multi-element stochastic collocation method. In [26], an adaptive version of ANOVA was developed to automatically detect the important dimensions. In [40], adaptive ANOVA methods based on three different adaptive criteria were proposed and compared.

ANOVA decomposition of the original high-dimensional stochastic problem results in a set of low-dimensional subproblems in stochastic space, which are efficiently solved by the sparse-grid stochastic collocation method. The stochastic collocation method was first introduced by Tatang and McRae in [34]. The properties of stochastic collocation method have been extensively studied in the past 10 years. In [3,30,31], the errors of integrating or interpolating functions with Sobolev regularity were analyzed for Smolyak constructions based on one-dimensional (1D) nested Clenshaw-Curtis rules. In [31], the degree of exactness of the Smolyak quadrature using Clenshaw-Curtis and Gaussian one-dimensional rules was investigated. In 2003, Gerstner and Griebel [13] introduced the dimension-adaptive tensor-product quadrature method. Recently Xiu and Hesthaven [39] have used Lagrange polynomial interpolation to construct high-order