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Integrated Linear Reconstruction for Finite Volume Scheme on Arbitrary Unstructured Grids

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Abstract. In [L. Chen and R. Li, Journal of Scientific Computing, Vol. 68, pp. 1172–1197, (2016)], an integrated linear reconstruction was proposed for finite volume methods on unstructured grids. However, the geometric hypothesis of the mesh to enforce a local maximum principle is too restrictive to be satisfied by, for example, locally refined meshes or distorted meshes generated by arbitrary Lagrangian-Eulerian methods in practical applications. In this paper, we propose an improved integrated linear reconstruction approach to get rid of the geometric hypothesis. The resulting optimization problem is a convex quadratic programming problem, and hence can be solved efficiently by classical active-set methods. The features of the improved integrated linear reconstruction include that i). the local maximum principle is fulfilled on arbitrary unstructured grids, ii). the reconstruction is parameter-free, and iii). the finite volume scheme is positivity-preserving when the reconstruction is generalized to the Euler equations. A variety of numerical experiments are presented to demonstrate the performance of this method.

AMS subject classifications: 65M08, 65M50, 76M12, 90C20

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1 Introduction

The high-order finite volume schemes can be summarized by a reconstruct-evolve-average (REA) process, i.e. a piecewise polynomial is reconstructed in each cell with given cell averages, then the governing equation is evolved according to those polynomials, and

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finally the cell averages are recalculated. Among these three stages, reconstruction plays an important role in giving high-order solutions without numerical oscillations. Nowadays, second-order methods have been the workhorse for computational fluid dynamics [1]. To achieve second-order accuracy, a prediction of the gradient is obtained first. Due to the possible non-physical flow caused by the underestimation or overestimation of the gradient, it is necessary to limit the gradient in a proper way, and thus the prediction-limiting algorithm arises. In one-dimensional case, total variation diminishing (TVD) limiters are commonly used in designing high-resolution schemes for conservation laws. Unfortunately, it is very difficult to implement the TVD limiter for multi-dimensional problems, especially on unstructured grids. To get around this negative result, a new class of positive schemes has been proposed [2] which ensures a local maximum principle. Since the introduction of this idea, a large number of limiters have then been developed. These limiters include, among others, the Barth's limiter [3], the Liu's limiter [4], the maximum limited gradient (MLG) limiter [5] and the projected limited central difference (PLCD) limiter [6]. See [6] for a comprehensive comparison of these limiters. More recently, Park *et al.* successfully extended the multi-dimensional limiting process (MLP) introduced in [7] from structured grids to unstructured grids [8]. Li *et al.* proposed the weighted biased averaging procedure (WBAP) limiter [9] based upon the biased averaging procedure (BAP) limiter which was introduced in [10]. Towards the positivity-preserving property, which is crucial for the stability on solving the Euler equations, there have also been several pioneering works. For instance, Perthame and Shu [11] developed a general finite volume framework on preserving the positivity of density and pressure when solving the Euler equations. Motivated by this work, the framework was extended to the discontinuous Galerkin method on rectangular meshes [12] and on triangular meshes [13], and to the Runge-Kutta discontinuous Galerkin method [14]. More recently, a parametrized limiting technique was proposed in [15] to preserve the positivity property on solving the Euler equations on unstructured grids.

Besides the above prediction-limiting algorithm in the reconstruction, there are also methods which deliver the limited gradient in a single process. For example, Chen and Li [16] introduced the concept of integrated linear reconstruction (ILR), in which the limited gradient is computed by solving a linear programming on each cell using an efficient iterative method. A similar approach was given by May and Berger [17]. However, the fulfillment of local maximum principle of these methods requires certain geometric hypothesis on the grids [16]. Buffard and Clain [18] considered a monoslope MUSCL method, where a least-squares problem subjected to maximum principle constraints was imposed. They solved the optimization problem explicitly. However, the involved cases for triangular grids, discussed in the article, were rather complicated, let alone irregular grids with hanging nodes appeared in the computational practice such as mesh adaptation. This motivates us to discard the unsatisfactory geometric hypothesis on unstructured grids by imposing constraints on the quadrature points, and to solve the optimization problem iteratively. In our linear reconstruction, the gradient is indeed obtained by solving a quadratic programming problem using the active-set method. It can be verified