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An Enhanced Finite Element Method for a Class of Variational Problems Exhibiting the Lavrentiev Gap Phenomenon

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Abstract. This paper develops an enhanced finite element method for approximating a class of variational problems which exhibits the *Lavrentiev gap phenomenon* in the sense that the minimum values of the energy functional have a nontrivial gap when the functional is minimized on the spaces $W^{1,1}$ and $W^{1,\infty}$. To remedy the standard finite element method, which fails to converge for such variational problems, a simple and effective cut-off procedure is utilized to design the (enhanced finite element) discrete energy functional. In essence the proposed discrete energy functional curbs the gap phenomenon by capping the derivatives of its input on a scale of $\mathcal{O}(h^{-\alpha})$ (where h denotes the mesh size) for some positive constant α. A sufficient condition is proposed for determining the problem-dependent parameter α. Extensive 1-D and 2-D numerical experiment results are provided to show the convergence behavior and the performance of the proposed enhanced finite element method.

AMS subject classifications: 65N35

Key words: Energy functional, variational problems, minimizers, singularities, Lavrentiev gap phenomenon, finite element methods, cut-off procedure.

1 Introduction

This paper concerns with finite element approximations of variational problems whose solutions (or minimizers) exhibit the so-called *Lavrentiev gap phenomenon* – a defect from the singularities of the solutions. Such problems are often encountered in materials sciences, nonlinear elasticity, and image processing (cf. [4,7,13] and the references therein).

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These variational problems can be abstractly stated as follows:

$$u = \underset{v\in\mathcal{A}}{\operatorname{argmin}} \mathcal{J}(v), \tag{1.1}$$

where the energy functional $\mathcal{J}:\mathcal{A}\to\mathbb{R}\cup\{\pm\infty\}$ is defined by

$$\mathcal{J}(v)=\int_\Omega f(\nabla v,v,x)\,\mathrm{d}x. \tag{1.2}$$

Here $\Omega\subset\mathbb{R}^n$ is an open and bounded domain and $f:\mathbb{R}^n\times\mathbb{R}\times\Omega\to\mathbb{R}$, called the density function of $\mathcal{J}$, is assumed to be a continuous function. The space $\mathcal{A}:=\{v\in W^{1,1}(\Omega): v= g \text{ on } \partial\Omega\}$ is known as the admissible set with $g\in L^1(\partial\Omega)$ being some given function.

Let $\mathcal{A}_\infty:=\mathcal{A}\cap W^{1,\infty}(\Omega)$. Since Ω is bounded, then $\mathcal{A}_\infty\subset\mathcal{A}$ and consequently there holds

$$\inf_{v\in\mathcal{A}_1}\mathcal{J}(v)\le \inf_{v\in\mathcal{A}_\infty}\mathcal{J}(v). \tag{1.3}$$

Problem (1.1) is said to exhibit *the Lavrentiev gap phenomenon* whenever

$$\inf_{v\in\mathcal{A}_1}\mathcal{J}(v)< \inf_{v\in\mathcal{A}_\infty}\mathcal{J}(v), \tag{1.4}$$

in other words, when the strict inequality holds in (1.3).

The gap between the minimum values on both sides of (1.4) suggests that the the minimizer of the left-hand side must have some singularity which causes the gap. Such a singularity often corresponds to a defect in a material or an edge in an image. It has been known in the literature [4,7,13] that the gap phenomenon could happen not only for nonconvex energy functionals but also for strictly convex and coercive energy functionals. As a result, it is a very complicated phenomenon to characterize and to analyze as well as to approximate (see below for details). This is because the gap phenomenon can be triggered by quite different mechanisms and the definition of *the Lavrentiev gap phenomenon* is a very broad concept which covers many different types of singularities. To the best of our knowledge, so far there are no known general sufficient conditions which guarantee the existence of the gap phenomenon.

The simplest and best known example of the gap phenomenon is Maniá's 1-D problem [11], where one minimizes the functional

$$\mathcal{J}(v)=\int_0^1 v'(x)^6\big(v(x)^3-x\big)^2\mathrm{d}x \tag{1.5}$$

over all functions $v\in W^{1,1}(0,1)$ satisfying $v(0)=0$ and $v(1)=1$. By inspection it is easy to see that $u(x)=x^{\frac{1}{3}}$ minimizes (1.5) with a minimum value of zero. However, it can be shown that the minimum over the space $W^{1,\infty}(0,1)$ (i.e., the space of all Lipschitz functions) is strictly larger than zero. As a result, Maniá's problem does exhibit *the Lavrentiev*