A Monotone Finite Volume Scheme with Fixed Stencils for 3D Heat Conduction Equation

Hui Xie[1], Chuanlei Zhai[1], Xuejun Xu[2,3], Jun Peng[1] and Heng Yong[1,*]

[1] *Institute of Applied Physics and Computational Mathematics, Beijing, 100094, China.*
[2] *LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, P.O. Box 2719, Beijing, 100190, China.*
[3] *School of Mathematical Sciences, Tongji University, Shanghai, China.*

Abstract. In this paper, a new nonlinear finite volume scheme preserving positivity for three-dimensional (3D) heat conduction equation is proposed. Being different from the traditional monotone schemes, the flux on each 3D non-planar cell-face is entirely approximated by the so-called effective directional flux firstly, then the effective directional flux is decomposed by the fixed stencils. Fixed stencil means the decomposition is just conducted on this face such that searching the convex decomposition stencil over all cell-faces is avoided. This feature makes our scheme more efficient than the traditional monotone ones based on the adaptive stencils for convex decompositions, especially in 3D. In addition, similar to other schemes based on the fixed stencils, there is also no assumption of the non-negativity of the interpolated cell-vertex unknowns. Some benchmark examples are presented to demonstrate the second-order accuracy. Two anisotropic diffusion problems show that not only can our schemes maintain the positivity-preserving property, but also they are more efficient than the scheme based on the adaptive stencil.

AMS subject classifications: 65M08, 35R05, 76S05

Key words: Anisotropic diffusion tensor, distorted mesh, positivity-preserving, fixed stencils.

1 Introduction

In the simulation of Lagrangian radiation hydrodynamics arising from some applications such as inertial confinement fusion (ICF) [19, 22], both the accuracy and positivity-preserving property are the key factors to any discrete scheme for simulating the energy

---

diffusion process. For example, the negative electron temperature T_e can cause its conduction coefficient $\kappa_e \propto T_e^{5/2}$ meaningless. In addition, robustness and efficiency of the scheme are also key factors for stable simulations within the finite time, especially in 3D. However, with the multi-material and fluid motion in the Lagrangian scheme [48], the resulting strong discontinuous and anisotropic diffusion coefficients and distorted meshes bring great challenges to achieve these goals.

There are many literatures [2, 6, 9, 20, 29–33] devoted to propose some restrictive conditions on mesh geometry and diffusion operator for traditional numerical methods (e.g. finite element and finite volume) to achieve the second-order accuracy and positivity-preserving property on the distorted meshes either with or without material discontinuities, also some preprocessing or postprocessing methods are proposed to repair the possible non-positive numerical solution [1, 21, 28, 46]. Note that the positivity-preserving property does not imply the scheme is free of spurious oscillations. To obtain a consistent approximation with M-matrix, some new discretization schemes [11, 12] based on full pressure support are proposed to exploit much larger range of quadrature points to minimize the numerical spurious oscillations and possess a conditional discrete maximum principle. Two reviews of these improved numerical methods are present in [10] and [30] for the finite volume and finite element method, respectively.

Besides the traditional finite volume and finite element methods, the popular lattice Boltzmann method (LBM) for fluid dynamic simulations, which solves the Boltzmann equation instead of solving the continuum diffusion-type equation, has been extended to advection-diffusion problem [3, 41]. Furthermore, several multiple-relaxation-time lattice Boltzmann methods [17, 50] have been proposed to solve advection-diffusion equations with *anisotropic* diffusion tensor. The advantages of LBM consist of easy parallelization and implementation for complicated flow model and irregular domains.

However, it was shown in [31] that no linear nine-point scheme can unconditionally satisfy the positivity-preserving property with the second-order accuracy on any distorted quadrilateral mesh or for any diffusion tensor. Also some restrictive conditions can hardly be fulfilled in practical applications. As for lattice Boltzmann method, it was also shown in [18] that they violate the non-negativity constraint and the maximum principle for *anisotropic* diffusion, and may not be removed by refining the time-step and the lattice mesh-size. Nonlinear discretization for diffusion operator might be a price to pay for the construction of positivity-preserving finite volume scheme with the second-order accuracy. Note that discrete maximum principle is the stringent requirement for the nonlinear discrete schemes, we only concern the positivity-preserving property in this paper.

In the last decade, some nonlinear finite volume methods have been proposed [23, 35, 51]. For easy coupling with hydrodynamics computation, the nonlinear scheme was developed for the general star-shaped 2D polygonal meshes in [36, 49] by using adaptive convex decomposition stencil, there the auxiliary unknowns (mesh vertices, edge centers) are introduced. They proved the nonlinear scheme is monotone under the assumption of these interpolated auxiliary unknowns should be nonnegative. Similar idea was extended to the nonequilibrium radiation diffusion problem [40]. This idea with some