Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2019-0047

Vol. **27**, No. 3, pp. 639-671
March 2020

Entropies and Symmetrization of Hyperbolic Stochastic Galerkin Formulations

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Received 25 March 2019; Accepted (in revised version) 21 August 2019

Abstract. Stochastic quantities of interest are expanded in generalized polynomial chaos expansions using stochastic Galerkin methods. An application to hyperbolic differential equations does in general not transfer hyperbolicity to the coefficients of the truncated series expansion. For the Haar basis and for piecewise linear multiwavelets we present convex entropies for the systems of coefficients of the one-dimensional shallow water equations by using the Roe variable transform. This allows to obtain hyperbolicity, wellposedness and energy estimates.

AMS subject classifications: 35L65, 54C70, 58J45, 70S10, 37L45, 35R60

Key words: Hyperbolic partial differential equations, uncertainty quantification, stochastic Galerkin, shallow water equations, wellposedness, entropy, Roe variable transform.

1 Introduction

Wellposedness is an important property that systems of partial differential equations (PDEs) should fulfil. Wellposedness means the solution exists, it is unique and the solution depends continuously on initial conditions [35]. Classical solutions to most hyperbolic conservation laws have this property, which explains, why these equations are widely used to model fluid dynamics [49] and other applications like traffic flow [50]. Most physically motivated systems are endowed with an entropy that describes the decay of energy, which in turn guarantees well-posed classical solutions [9,29,45]. A famous example is the physical entropy for Euler and shallow water equations, see e.g. [18].

Classical solutions, however, exist in finite time only up to the possible occurrence of shocks [65]. Therefore, weak solutions are considered which are not necessarily unique. Existence and uniqueness of bounded weak entropy solutions have been shown in [45]

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using entropy-entropy flux pairs. All of these entropy-entropy flux pairs must satisfy an entropy inequality. In the *scalar* case a strictly convex flux function and *one* entropy-entropy flux pair is sufficient to characterize the entropy solution uniquely [47, 59]. This result could not been extended to arbitrary systems, when entropies rarely exist or remain unknown [47]. A single entropy-entropy flux pair, however, manages to weed out all but one weak solution, as long as a classical solution exists [18]. Thus, an entropy transfers the wellposedness of classical solutions to a weak formulation.

When initial data are not known exactly, but are given by their probability law or by statistical moments, the deterministic entropy concepts should be extended to the stochastic case. A mathematical framework for random entropy solutions of scalar random hyperbolic equations is developed in [55, 75]. It is shown that existing statistical moments in the initial conditions are transferred to the solution. In this non-intrusive point of view, first *pointwise* entropy solutions are determined, then the statistics of interest are computed. If there is only interest in the statistics, non-intrusive methods have been proven successful in previous works [1, 3, 13, 17, 56, 66, 68, 73, 74, 78] and are often preferred in practice, since deterministic solvers can be used. In particular for shallow water equations, results are available in several spatial and random dimensions [57].

Desirable deterministic schemes are in smooth regions high-order accurate, but can also resolve singularities in an essentially nonoscillatory (ENO) fashion. WENO schemes consist of a weighted combination of local reconstructions on different stencils. Some schemes allow unstructured grid in higher dimensions [37, 69]. In particular for balance laws, centered CWENO schemes [15, 16, 43] can reconstruct also the source term.

In contrast to non-intrusive methods, we investigate the intrusive stochastic Galerkin method. Stochastic processes are represented as orthogonal functions, for instance orthogonal polynomials and multiwavelets. These representations are known as generalized polynomial chaos (gPC) expansions [7, 25, 30, 79, 82]. Expansions of the stochastic input are substituted into the governing equations and they are projected to obtain deterministic evolution equations for the gPC coefficients. The applications of this procedure have been proven successful for diffusion [22, 83] and kinetic equations [8, 38, 42, 70, 85]. In general, results for hyperbolic systems are not available [20, 21, 53], since desired properties like hyperbolicity and the existence of entropies are not transferred to the intrusive formulation. A problem is posed by the fact that the deterministic Jacobian of the projected system differs from the random Jacobian of the original system and therefore not even real eigenvalues, which are necessary for hyperbolicity, are guaranteed in general.

In particular for a stochastic Galerkin formulation of shallow water equations, the loss of hyperbolicity and hence the loss of all entropy-entropy flux pairs is proven in [21, Prop. 2]. Also stochastic Galerkin formulations for isothermal Euler equations are in general not hyperbolic [24, 40].

So far, a serious problem for both non- and intrusive methods remains the convergence in the stochastic space. Methods are desirable that allow estimates and convergence results for a smooth dependency on the stochastic input. Convergence results in previous works are based on smoothness assumptions, although solutions to hyperbolic differential equations admit discontinuities in both physical and random space. Convergence results for smooth solutions of the *linear* wave equation with random velocity have been established for stochastic Galerkin [31] as well as stochastic collocation methods [58, 84]. For the nonlinear Burgers' equation an entropy concept is used to obtain convergence results [21] and it is used for an a posteriori error estimate [27], which enables adaptivity also in the stochastic space. This ansatz, however, has not been extended to stochastic Galerkin formulations of nonlinear hyperbolic systems and it is valid for smooth solutions only. Convergence of weak solutions remains still an open question [21].

Still, stochastic Galerkin methods applied to hyperbolic systems is an active field of research. At least initial values can be represented in an optimal way. Furthermore, the knowledge of the whole random space is available in contrast to non-intrusive methods, where only the solution corresponding to a particular sample or quadrature point is available. This allows adaptivity in the stochastic space [5, 27] and numerical schemes were constructed that are well-balanced at any location in the random space [41]. A non-intrusive method can enforce this property only at the collocation nodes.

It is possible for some systems to first transform the partial differential equations into non-conserved variables and then apply the Galerkin method. See [80] for quasilinear systems, [20, 21] for entropy variables, [24, 61] for Euler equations using the Roe variable transform from [67]. Also, formulations of hyperbolic systems with eigenvectors that are independent of the uncertainty remain hyperbolic [77]. However, for classical fluid-dynamic equations, like the shallow water equations, eigenvectors are stochastic.

Even if the Jacobian has real eigenvalues, which implies unique classical solutions, the system may not be well-posed in the sense that the solution does not depend on initial conditions in a stable way. For this property at least a complete set of eigenvectors must exist. Eigenvalues of most physically motivated hyperbolic systems are separated. Then, there is a complete set of eigenvectors, which implies wellposedness [71]. This argument, however, cannot be used for stochastic Galerkin formulations, when eigenvalues are no more separated.

So far, general results on wellposedness can be established for the gPC systems of scalar conservation laws only [10, 31, 41, 60, 64], since the resulting Jacobian is symmetric and hence diagonalizable with real eigenvalues and a complete set of eigenvectors. In fact, an entropy-entropy flux pair exists for these symmetric systems [28]. This wellposedness result can be extended to hyperbolic systems that do not have necessarily a symmetric Jacobian, but admit an entropy. Then, the system is symmetrizable and hence classical solutions are well-posed [9, 28, 29]. Therefore, an expansion in entropy variables [20, 21] transfers the entropy to the stochastic Galerkin formulation. An entropy-entropy flux pair for the Roe variables, however, has not been established, yet. There are attempts to make the transforms for entropy variables more competitive [46], but Roe variables lead to numerically efficient and stable transforms [61].

This research gap makes the entropy variables and Roe variables be complementary. Furthermore, entropy variables can be used for *arbitrary* gPC expansions. The use of Roe variables is so far restricted to a certain class only, which we will call $\mathcal{A}$gPC bases. This class includes the Wiener-Haar basis and linear multiwavelets. These wavelet expansions are motivated by a robust expansion for solutions that depend on the stochastic input in a non-smooth way and are used for stochastic multiresolution as well as adaptivity in the stochastic space [2,5,6,44,54,63,76]. This restriction could be weakened in the case of isothermal Euler equations [24] only.

In general, entropy solutions exist on finite time domains only. For deterministic Euler and shallow water equations, which have distinct eigenvalues and genuinely nonlinear or linearly degenerate characteristic fields, an entropy solution exists for all fixed time domains as long as the total variation of initial values is sufficiently small [4, Th. 7.1]. Although the assumption of genuine nonlinearity can be weakened [4,51,52], the eigenvalues of stochastic Galerkin formulations may coincide and the total variation of initial values may be not sufficiently small. Therefore, we expect that weak solutions exist in finite time only and we study the following setting:

A weak formulation on a bounded *time domain* $[0,T)$ *is considered, when an entropy solution exists in the weak sense. If additionally a classical solution exists it should be well-posed also in the weak sense. Furthermore, the solution should coincide in the deterministic case with a physically relevant entropy solution.*

The main contribution of this paper is to close a gap between the Roe variable and entropy variable transform by introducing the required entropy-entropy flux pair for a Roe variable based stochastic Galerkin formulation of shallow water equations.

These Roe variables are used as auxiliary variables that depend in a conservative formulation on the gPC modes of the physical states and can be obtained as stochastic Galerkin root. First, we define the transform to these variables as an energy minimization problem to obtain a stable and bijective transform. Then, we consider a hyperbolic stochastic Galerkin formulation and endow this system with an entropy. Our main results are restricted to $\mathcal{A}$gPC bases. Hence, we state examples and an approach to determine if a basis belongs to this set.

To make the presented formulation of shallow water equations more competitive to non-intrusive methods, we quantify truncation errors at least for smooth solutions by using the similar entropy framework from [18,21,26,27].

We illustrate numerically theoretical results for the Wiener-Haar expansion. The results show the hyperbolic character of the system, the smoothness properties of truncated gPC expansions and wellposedness, which follows from the decay of entropies. We applied a third-order central, weighted, essentially nonoscillatory (CWENO) reconstruction from [16], since it enables also an higher-order reconstruction of the source term.

2 Cauchy problem and weak solutions

We briefly recall basic results from [4,18,48,49]. A function

$$y : [0,T)\times\mathbb{R}\to\mathbb{R}^n, \quad (t,x)\mapsto y(t,x)$$

is a **weak solution** to the **Cauchy problem**

$$y_t+f(y)_x=0 \quad \text{with} \quad y(0,x)=\mathcal{I}(x) \quad \text{for} \quad x\in\mathbb{R} \tag{2.1}$$

if the map $t\mapsto y(t,\cdot)$ is continuous with values in L^1_{loc}, the initial condition is satisfied and the solution satisfies

$$\int_0^T\int_{\mathbb{R}}\Big[y(t,x)\varphi_t(t,x)+f\big(y(t,x)\big)\varphi_x(t,x)\Big]\,\mathrm{d}x\mathrm{d}t=0$$

for every C^1-function φ with compact support contained in the open strip $(0,T)\times\mathbb{R}$, where the integral is interpreted componentwise. Given a **strictly convex entropy** η with **entropy flux** μ, a weak solution is called **η-admissible** if the **entropy inequality**

$$\eta(y)_t+\mu(y)_x\leq 0 \tag{2.2}$$

is satisfied in the distributional sense. For all non-negative testfunctions we have

$$\int_0^T\int_{\mathbb{R}}\Big[\eta\big(y(t,x)\big)\varphi_t(t,x)+\mu\big(y(t,x)\big)\varphi_x(t,x)\Big]\,\mathrm{d}x\mathrm{d}t\geq 0.$$

For this homogeneous system the entropy and the entropy flux are smooth functions defined on an open ball $\mathbb{H}\subset\mathbb{R}^n$ [18] and satisfy the **compatibility condition**

$$\mathrm{D}_y\mu(y)=\mathrm{D}_y\eta(y)\mathrm{D}_yf(y). \tag{2.3}$$

3 Stochastic Galerkin and Roe variable transform

Similar to [19, 32, 53, 62, 72, 81] we extend the Cauchy problem (2.1) to have initial conditions depending on a possibly multidimensional random parameter ξ, which we call similar to [53] **germ**. We consider the weak formulation

$$\int_0^T\int_{\mathbb{R}}\mathbb{E}\Big[\Big(y(t,x;\xi)\varphi_t(t,x)+f\big(y(t,x;\xi)\big)\varphi_x(t,x)\Big)\phi_k(\xi)\Big]\,\mathrm{d}x\mathrm{d}t=0$$

for all $k=0,\cdots,K$, where the orthogonal functions ϕ_k form a basis of

$$\begin{aligned}&\mathbb{L}^2(\Omega,\mathbb{P}):=\Big\{Z\;\Big|\;Z:\Omega\to\mathbb{R}\;\text{ measurable, }\;\|Z\|<\infty\Big\}\\ &\text{with}\quad\langle Z_1,Z_2\rangle:=\int Z_1Z_2\,\mathrm{d}\mathbb{P}.\end{aligned}$$

We introduce a **generalized polynomial chaos** (gPC) as a set of orthogonal subspaces $\hat{\mathcal{S}}_k \subset \mathbb{L}^2(\Omega,\mathbb{P})$ with

$$\mathcal{S}_K := \bigoplus_{k=0}^{K} \hat{\mathcal{S}}_k \to \mathbb{L}^2(\Omega,\mathbb{P}) \quad \text{for } K\to\infty.$$

We refer to an orthogonal basis of $\mathcal{S}_K$ as a gPC basis $\{\phi_k(\xi)\}_{k=0}^K$ with germ $\xi\sim\mathbb{P}$. We assume $y(t,x;\cdot)\in\mathbb{L}^2(\Omega,\mathbb{P})$ and approximate for any fixed (t,x) the solution by

$$\mathcal{G}_K[y](t,x;\xi) := \sum_{k=0}^{K} \hat{y}_k(t,x)\phi_k(\xi), \quad \hat{y}_k(t,x) := \frac{\langle y(t,x;\cdot),\phi_k(\cdot)\rangle}{\|\phi_k\|^2}, \tag{gPC}$$

where $\mathcal{G}_K$ denotes the projection operator of the stochastic process $y(t,x;\xi)$ onto the gPC basis of degree $K\in\mathbb{N}_0$. Using a multi-index notation $\boldsymbol{k}:=(k_1,\cdots,k_M)\in\mathbb{K}$ with an index set $\mathbb{K}\subseteq\mathbb{N}_0^M$, we may extend definition (gPC) to the multidimensional case as

$$\mathcal{G}_K[y](t,x;\xi) := \sum_{\boldsymbol{k}\in\mathbb{K}} \hat{y}_{\boldsymbol{k}}(t,x)\phi_{\boldsymbol{k}}(\xi) \quad \text{with} \quad \phi_{\boldsymbol{k}}(\xi) := \phi_{k_1}(\xi_1)\cdot\dots\cdot\phi_{k_M}(\xi_M).$$

In the following, we use the notation (gPC) with $K+1=|\mathbb{K}|$. The expansion converges in the sense $\|\mathcal{G}_K[y](t,x;\cdot)-y(t,x;\cdot)\|\to 0$ for $K\to\infty$ [7,14,23]. We will assume normed basis functions with $\|\phi_k\|=1$. Then, the **Galerkin product** is defined as

$$\begin{aligned}
&\hat{\mathcal{G}}_K[y,z](t,x;\xi) := \sum_{k=0}^{K} (\hat{y}*\hat{z})_k(t,x)\phi_k(\xi) \\
&\text{with} \quad (\hat{y}*\hat{z})_k(t,x) := \sum_{i,j=0}^{K} \hat{y}_i(t,x)\hat{z}_j(t,x)\langle \phi_i\phi_j,\phi_k\rangle.
\end{aligned}$$

The **third and fourth moment** are approximated by

$$\begin{aligned}
&\hat{\mathcal{G}}_k^{(\ell)}[y](t,x;\xi) := \sum_{k=0}^{K} \hat{y}_k^{\ell*}(t,x)\phi_k(\xi) \\
&\text{with} \quad \hat{y}^{3*} := \big((\hat{y}*\hat{y})*\hat{y}\big) \quad \text{and} \quad \hat{y}^{4*} := \big((\hat{y}*\hat{y})*(\hat{y}*\hat{y})\big).
\end{aligned} \tag{3.1}$$

Similar to [61,62], we express these terms with the symmetric matrix

$$\mathcal{P}(\hat{y}) := \sum_{\ell=0}^{K} \hat{y}_\ell \mathcal{M}_\ell \quad \text{with} \quad \mathcal{M}_\ell := \Big(\langle \phi_\ell,\phi_i\phi_j\rangle\Big)_{i,j=0,\cdots,K} \tag{3.2}$$

such that $\hat{y}*\hat{z}=\mathcal{P}(\hat{y})\hat{z}$. The Galerkin product is symmetric, but not associative [19,72], i.e. $(\hat{y}*\hat{z})*\hat{q}\neq\hat{y}*(\hat{z}*\hat{q})$. An intuitive explanation would be the truncation errors that arise from disregarding the components of the product (yz) which are orthogonal to $\mathcal{S}_K$.

Therefore, the definitions (3.1) are rather arbitrary and we refer the interested reader to [19, 53], where other approximations of the moments are discussed. In particular, the choice (3.1) allows for an extension of desired properties, e.g. hyperbolicity and entropy, to the stochastic case. To this end we adopt the idea of **Roe variables** [48, 61, 67] by stating a precise characterization in Lemma 3.1, which allows an interpretation as an energy minimization.

Definition 3.1 (Roe Variables)**.** With velocity $u(y) := q/h$ as auxiliary variable the Roe variables are defined as $\omega := (\alpha,\beta) := \big(\sqrt{h}, \sqrt{h}u(y)\big)$ and the gPC modes as $\hat{\omega} := (\hat{\alpha},\hat{\beta})$ for $\hat{\alpha} \in \mathbb{H}^+$ defined on the set

$$\mathbb{H}^+ := \Big\{ \hat{\alpha} \in \mathbb{R}^{K+1} \;\Big|\; \mathcal{P}(\hat{\alpha}) \text{ is strictly positive definite} \Big\}.$$

The mapping between Roe and conserved variables is

$$\begin{aligned}
\mathcal{Y} : \mathbb{R}^+ \times \mathbb{R} &\to \mathbb{R}^+ \times \mathbb{R}, & \hat{\omega} &\mapsto \begin{pmatrix} \alpha^2 \\ \alpha\beta \end{pmatrix} = y \quad \text{for } K = 0, \\
\hat{\mathcal{Y}} : \mathbb{H}^+ \times \mathbb{R}^{K+1} &\to \big(\mathbb{R}^+ \times \mathbb{R}^K\big) \times \mathbb{R}^{K+1}, & \hat{\omega} &\mapsto \begin{pmatrix} \hat{\alpha} * \hat{\alpha} \\ \hat{\alpha} * \hat{\beta} \end{pmatrix} = \hat{y} \quad \text{for } K \in \mathbb{N}.
\end{aligned}$$

Note that the expected water height $\hat{h}_0 = (\hat{\alpha} * \hat{\alpha})_0 = \|\hat{\alpha}\|_2^2 > 0$ is positive. The inverse mapping $\hat{\mathcal{Y}}^{-1}$ involves the **Galerkin square root** of gPC modes $\hat{h} \in \mathbb{R}^{K+1}$, which is introduced e.g. in [53] as the solution of the nonlinear system $\hat{\alpha} * \hat{\alpha} = \hat{h}$. It is already remarked in [19] that the representation of positive physical quantities is difficult. To illustrate the point, we consider an expansion with Hermite polynomials for $K = 1$. The solutions read as

$$\begin{aligned}
\hat{\alpha}^+ &:= \frac{1}{2}\begin{pmatrix} \sqrt{\hat{h}_0 + \hat{h}_1} + \sqrt{\hat{h}_0 - \hat{h}_1} \\ \sqrt{\hat{h}_0 + \hat{h}_1} - \sqrt{\hat{h}_0 - \hat{h}_1} \end{pmatrix}, &
\hat{\alpha}^- &:= -\frac{1}{2}\begin{pmatrix} \sqrt{\hat{h}_0 + \hat{h}_1} + \sqrt{\hat{h}_0 - \hat{h}_1} \\ \sqrt{\hat{h}_0 + \hat{h}_1} - \sqrt{\hat{h}_0 - \hat{h}_1} \end{pmatrix}, \\
\tilde{\alpha}^{(1)} &:= \frac{1}{2}\begin{pmatrix} \sqrt{\hat{h}_0 + \hat{h}_1} - \sqrt{\hat{h}_0 - \hat{h}_1} \\ \sqrt{\hat{h}_0 + \hat{h}_1} + \sqrt{\hat{h}_0 - \hat{h}_1} \end{pmatrix}, &
\tilde{\alpha}^{(2)} &:= -\frac{1}{2}\begin{pmatrix} \sqrt{\hat{h}_0 + \hat{h}_1} - \sqrt{\hat{h}_0 - \hat{h}_1} \\ \sqrt{\hat{h}_0 + \hat{h}_1} + \sqrt{\hat{h}_0 - \hat{h}_1} \end{pmatrix}.
\end{aligned}$$

We observe that the solution of the nonlinear system $\hat{\alpha} * \hat{\alpha} = \hat{h}$ may neither be unique nor real, which is similar to the deterministic case. In previous works [11,12,24] it is assumed that the inverse $\mathcal{P}(\hat{\alpha})$ exists. Then, the implicit function theorem guarantees invertibility at least *locally*. We introduce in Lemma 3.1 a more precise *global* characterization, which is based on the following observations:

(i) Let a deterministic state with $\hat{h}_0 > 0$, $\hat{h}_1 = 0$ be given. The solutions $\tilde{\alpha}^{(1)}$ and $\tilde{\alpha}^{(2)}$ yield stochastic expansions, which are not meaningful. The solution $\hat{\alpha}^+$ gives the positive and $\hat{\alpha}^-$ the negative root.

(ii) The matrix $\mathcal{P}(\hat{\alpha}^+)$ is positive definite and $\mathcal{P}(\hat{\alpha}^-)$ is negative definite. Both solutions are related by $\mathcal{P}(\hat{\alpha}^-) = -\mathcal{P}(\hat{\alpha}^+)$. On the other hand, the solutions $\tilde{\alpha}^{(1)}$, $\tilde{\alpha}^{(2)}$ yield indefinite matrices.

(iii) If the variance $\hat{h}_1^2$ is sufficiently large, there is no real valued solution.

Lemma 3.1 generalizes these observations for arbitrary expansions by identifying the positive square root as the unique minimum of a strictly convex function.

Lemma 3.1 (Stochastic Galerkin Square Root)**.** *Let a state $\hat{h} \in \mathbb{R}^+ \times \mathbb{R}^K$ be given such that there is $\hat{\alpha} \in \mathbb{H}^+$ satisfying $\hat{\alpha} * \hat{\alpha} = \hat{h}$. Then, the minimum*

$$\hat{\alpha}^+ := \underset{\hat{\alpha} \in \mathbb{H}^+}{\operatorname{argmin}} \left\{ \eta_{\hat{h}}(\hat{\alpha}) \right\} \quad \text{of the convex function} \quad \eta_{\hat{h}}(\hat{\alpha}) := \frac{\hat{\alpha}^{\mathrm{T}} \mathcal{P}(\hat{\alpha}) \hat{\alpha}}{3} - \hat{h}^{\mathrm{T}} \hat{\alpha}$$

on the convex set $\mathbb{H}^+$ is unique and it is a solution of stochastic Galerkin square roots, i.e. $\hat{\alpha}^+ \in \left\{ \hat{\alpha} \in \mathbb{R}^{K+1} \,\middle|\, \hat{\alpha} * \hat{\alpha} = \hat{h} \right\}$.

Proof. The set $\mathbb{H}^+$ is convex, since for arbitrary $\hat{\alpha}, \hat{\beta} \in \mathbb{H}^+$ the matrix

$$\mathcal{P}\Big(\lambda\hat{\alpha} + (1-\lambda)\hat{\beta}\Big) = \sum_{k=0}^{K} \Big(\lambda\hat{\alpha} + (1-\lambda)\hat{\beta}\Big)_k \mathcal{M}_k = \lambda \mathcal{P}(\hat{\alpha}) + (1-\lambda)\mathcal{P}(\hat{\beta})$$

is strictly positive definite for all $\lambda \in [0,1]$ as a sum of strictly positive definite matrices. The gradient of the auxiliary function $\tilde{\eta}(\hat{\alpha}) := \frac{1}{3}\hat{\alpha}^{\mathrm{T}} \mathcal{P}(\hat{\alpha})\hat{\alpha}$ satisfies

$$\begin{aligned}
3\nabla_{\hat{\alpha}} \tilde{\eta}(\hat{\alpha}) &= \nabla_{\tilde{\alpha}} \left[\sum_{k=0}^{K} \tilde{\alpha}_k \hat{\alpha}^{\mathrm{T}} \mathcal{M}_k \hat{\alpha} \right] \Bigg|_{\tilde{\alpha} = \hat{\alpha}} &&+ \sum_{k=0}^{K} \hat{\alpha}_k \nabla_{\hat{\alpha}} \left[\hat{\alpha}^{\mathrm{T}} \mathcal{M}_k \hat{\alpha} \right] \\
&= \Big(\hat{\alpha}^{\mathrm{T}} \mathcal{M}_k \hat{\alpha} \Big)_{k=0,\cdots,K} &&+ 2\sum_{k=0}^{K} \hat{\alpha}_k \mathcal{M}_k \hat{\alpha} \\
&= \Big(\hat{\alpha}_k^{2*} \Big)_{k=0,\cdots,K} &&+ 2\mathcal{P}(\hat{\alpha})\hat{\alpha} \\
&= 3\hat{\alpha} * \hat{\alpha}.
\end{aligned}$$

Since the Hessian $\nabla^2_{\hat{\alpha}} \eta_{\hat{h}}(\hat{\alpha}) = 2\mathcal{P}(\hat{\alpha})$ is strictly positive definite, the function $\eta_{\hat{h}}(\hat{\alpha})$ is strictly convex on $\mathbb{H}^+$ and its unique minimum is attained for

$$0 = \nabla_{\hat{\alpha}} \eta_{\hat{h}}(\hat{\alpha}) = \hat{\alpha}^+ * \hat{\alpha}^+ - \hat{h} \quad \Leftrightarrow \quad \hat{\alpha}^+ * \hat{\alpha}^+ = \hat{h}.$$

This completes the proof. □

Fig. 1 shows the sets $\mathbb{H}^{\pm}$ in terms of $\hat{\alpha}$, where $\mathbb{H}^-$ denotes states of a negative definite matrix $\mathcal{P}(\hat{\alpha})$. For given $\hat{h}$ the function $\eta_{\hat{h}}(\hat{\alpha})$, introduced in Lemma 3.1, and contours are

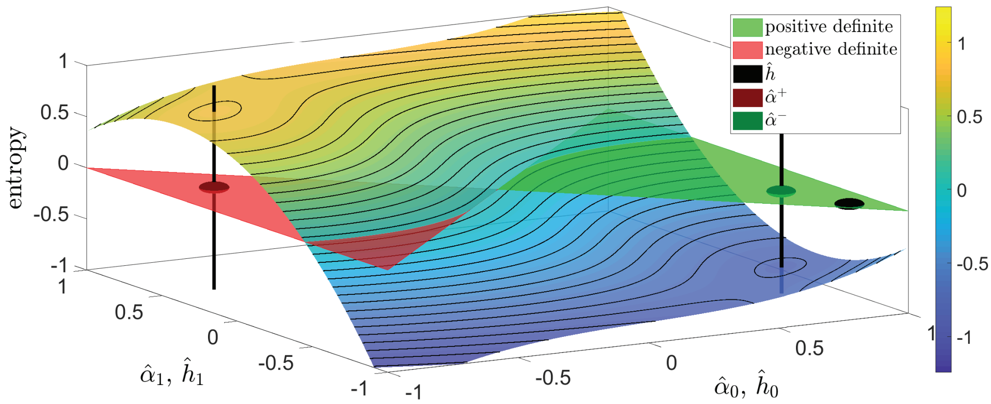

Figure 1: Contour plot of the function $\eta_{\hat{h}}(\hat{\alpha})$. Solutions $\hat{\alpha}^+ = (0.8,-0.5)^{\mathrm{T}}$ and $\hat{\alpha}^- = (-0.8,0.5)^{\mathrm{T}}$ for $\hat{h} = (0.89,-0.8)^{\mathrm{T}}$.

plotted in the third dimension. The local extrema, which are projected on the $(\hat{\alpha}_0,\hat{\alpha}_1)$-plane, are the square roots. Contours illustrate that extrema are unique on the sets $\mathbb{H}^{\pm}$ only. Note that we do not claim that a solution $\hat{\alpha} \in \mathbb{H}^+$ with $\hat{\alpha} * \hat{\alpha} = \hat{h}$ exists. Lemma 3.1 is only a *uniqueness* result and we refer the interested reader to [24, Sec. 4], where existence has been discussed.

Remark 3.1 (Burgers' Equation)**.** The stochastic Galerkin method applied to the **Burgers' equation** with flux function $f(\alpha) := \frac{\alpha^2}{2}$ leads to the projected system

$$\hat{\alpha}_t + \hat{f}(\hat{\alpha})_x = 0 \quad \text{for} \quad \hat{f}(\hat{\alpha}) := \frac{\hat{\alpha} * \hat{\alpha}}{2} \quad \text{and Jacobian} \quad \mathrm{D}_{\hat{\alpha}} \hat{f}(\hat{\alpha}) = \mathcal{P}(\hat{\alpha}). \tag{3.3}$$

Since the Jacobian is symmetric, the choice $\eta(\hat{\alpha}) = \|\hat{\alpha}\|_2^2$ gives an entropy [28, Ex. 3.2]. Furthermore, the choices

$$\Big(\eta_{\hat{h}}(\hat{\alpha}), \mu_{\hat{h}}(\hat{\alpha})\Big) := \Big(\tilde{\eta}(\hat{\alpha}), \tilde{\mu}(\hat{\alpha})\Big) - \Big(\hat{h}^{\mathrm{T}} \hat{\alpha}, \hat{h}^{\mathrm{T}} \frac{\hat{\alpha} * \hat{\alpha}}{2}\Big)$$
$$\text{with} \quad \Big(\tilde{\eta}(\hat{\alpha}), \tilde{\mu}(\hat{\alpha})\Big) := \left(\frac{\hat{\alpha}^{\mathrm{T}} \mathcal{P}(\hat{\alpha}) \hat{\alpha}}{3}, \frac{\hat{\alpha}^{\mathrm{T}} \mathcal{P}^2(\hat{\alpha}) \hat{\alpha}}{4}\right)$$

are entropy-entropy flux pairs on the convex set $\mathbb{H}^+$ for all $\hat{h} \in \mathbb{R}^{K+1}$. To see this, we calculate

$$\nabla_{\hat{\alpha}} \tilde{\eta}(\hat{\alpha}) = \hat{\alpha} * \hat{\alpha}, \quad \nabla_{\hat{\alpha}}^2 \eta_{\hat{h}}(\hat{\alpha}) = 2\mathcal{P}(\hat{\alpha}), \quad \nabla_{\hat{\alpha}} \tilde{\mu}(\hat{\alpha}) = \frac{1}{2} \nabla_{\hat{\alpha}} \big[\mathcal{P}(\hat{\alpha}) \hat{\alpha}\big] \mathcal{P}(\hat{\alpha}) \hat{\alpha} = \hat{\alpha}^{3*}.$$

The corresponding entropy flux satisfies the compatibility condition

$$\mathrm{D}_{\hat{\alpha}} \mu_{\hat{h}}(\hat{\alpha}) = \mathrm{D}_{\hat{\alpha}} \tilde{\eta}(\hat{\alpha}) \mathrm{D}_{\hat{\alpha}} \hat{f}(\hat{\alpha}) - \hat{h}^{\mathrm{T}} \mathcal{P}(\hat{\alpha}) = \mathrm{D}_{\hat{\alpha}} \eta_{\hat{h}}(\hat{\alpha}) \mathrm{D}_{\hat{\alpha}} \hat{f}(\hat{\alpha}).$$

Hence, the relation to Burgers' equation allows to interpret the stochastic Galerkin square root as a minimization of an energy.

4 Shallow water equations

We parameterize the shallow water equations by the germ ξ

$$\frac{\partial}{\partial t}\begin{pmatrix} h(t,x;\xi) \\ q(t,x;\xi)\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix} q(t,x;\xi) \\ \frac{q^2(t,x;\xi)}{h(t,x;\xi)}+\frac{1}{2}gh^2(t,x;\xi)\end{pmatrix}=0 \quad \mathbb{P}\text{-a.s.} \tag{$\mathcal{W}(\xi)$}$$

where the conserved variables are height h, momentum q and g denotes the gravitational constant. The system ($\mathcal{W}(\xi)$) can equivalently be represented with Roe variables, i.e.

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha^2(t,x;\xi) \\ (\alpha\beta)(t,x;\xi)\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix} (\alpha\beta)(t,x;\xi) \\ \beta^2(t,x;\xi)+\frac{1}{2}g\alpha^4(t,x;\xi)\end{pmatrix}=0 \quad \mathbb{P}\text{-a.s.} \tag{$\mathcal{R}(\xi)$}$$

We substitute the truncated gPC expansions into the systems ($\mathcal{W}(\xi)$) and ($\mathcal{R}(\xi)$). To obtain a system of equations for the gPC coefficients we make the residues be orthogonal to the basis functions, i.e.

$$\left\langle \frac{\partial}{\partial t}\begin{pmatrix} \mathcal{G}_K[h](t,x;\xi) \\ \mathcal{G}_K[q](t,x;\xi)\end{pmatrix} +\frac{\partial}{\partial x}\begin{pmatrix} \mathcal{G}_K[q](t,x;\xi) \\ \frac{\mathcal{G}_K^2[q](t,x;\xi)}{\mathcal{G}_K[h](t,x;\xi)}+\frac{1}{2}g\mathcal{G}_K^2[h](t,x;\xi)\end{pmatrix}, \quad \phi_k(\xi)\right\rangle=0, \tag{4.1}$$

$$\left\langle \frac{\partial}{\partial t}\begin{pmatrix} \hat{\mathcal{G}}_K[\alpha,\alpha](t,x;\xi) \\ \hat{\mathcal{G}}_K[\alpha,\beta](t,x;\xi)\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix} \hat{\mathcal{G}}_K[\alpha,\beta](t,x;\xi) \\ \hat{\mathcal{G}}_K[\beta,\beta](t,x;\xi)+\frac{1}{2}g\hat{\mathcal{G}}_K^{(4)}[\alpha](t,x;\xi)\end{pmatrix},\phi_k(\xi)\right\rangle=0. \tag{4.2}$$

It is shown in [21] that Eq. (4.1) leads to a non-hyperbolic system due to the term q^2/h. Similarly for isothermal Euler equations, a stochastic Galerkin method that is only based on conserved variables does not preserve hyperbolicity [24,40]. This issue can be circumvented by introducing Roe variables, which preserve the symmetry of the term β^2. The gPC modes of the formulation (4.2) are described by the system

$$\begin{pmatrix} \hat{\alpha}*\hat{\alpha} \\ \hat{\alpha}*\hat{\beta}\end{pmatrix}_t+\begin{pmatrix} \hat{\alpha}*\hat{\beta} \\ \hat{\beta}*\hat{\beta}+\frac{1}{2}g\,(\hat{\alpha}*\hat{\alpha})*(\hat{\alpha}*\hat{\alpha})\end{pmatrix}_x=0, \tag{$\hat{\mathcal{R}}_K$}$$

which we will endow with an entropy. We reformulate it in terms of the conserved variables $\hat{y}=\hat{\mathcal{Y}}(\hat{\omega})$ to obtain the conservative formulation $\hat{y}_t+\hat{f}(\hat{y})=0$ with flux function $\hat{f}(\hat{y}):=\hat{f}_1(\hat{y})+\hat{f}_2(\hat{y})$ for

$$\hat{f}_1(\hat{y}):=\begin{pmatrix} \hat{q} \\ \frac{1}{2}g\hat{h}*\hat{h}\end{pmatrix} \quad \text{and} \quad \hat{f}_2(\hat{y}):=\tilde{f}_2\Big(\hat{\mathcal{Y}}^{-1}(\hat{y})\Big):=\begin{pmatrix} 0 \\ \hat{\beta}*\hat{\beta}\end{pmatrix}. \tag{4.3}$$

Note that the first part $f_1(\hat{y})$ of the flux in Eq. (4.3) is expressed in terms of conserved variables alone, which motivates the choice of the 4-th moment (3.1). Furthermore, we need specific properties of a certain class of basis functions, which is introduced in Lemma 4.1.

Lemma 4.1. *The following properties are equivalent for the matrix $\mathcal{P}(\hat{\alpha})$, which is defined in Eq.* (3.2)*:*

(A1) *The precomputed matrices* $\mathcal{M}_\ell$ *and* $\mathcal{M}_k$ *commute for all* $\ell,k=0,\cdots,K$.

(A2) *The matrices* $\mathcal{P}(\hat{\alpha})$ *and* $\mathcal{P}(\hat{\beta})$ *commute for all* $\hat{\alpha},\hat{\beta}\in\mathbb{R}^{K+1}$.

(A3) *There is an eigenvalue decomposition* $\mathcal{P}(\hat{\alpha})=VD_{\mathcal{P}}(\hat{\alpha})V^{\mathrm{T}}$ *with constant eigenvectors.*

Proof. Properties (A2) and (A3) are equivalent, since symmetric matrices are simultaneously diagonalizable if and only if they are commutative [36]. If property (A1) holds, we have

$$\mathcal{P}(\hat{\alpha})\mathcal{P}(\hat{\beta})=\sum_{i,j=0}^{K}(\hat{\alpha}_i\mathcal{M}_i)(\hat{\beta}_j\mathcal{M}_j)=\sum_{j,i=0}^{K}(\hat{\beta}_j\mathcal{M}_j)(\hat{\alpha}_i\mathcal{M}_i)=\mathcal{P}(\hat{\beta})\mathcal{P}(\hat{\alpha}).$$

In turn, if property (A2) holds, the i,j-th unit vectors yield

$$\mathcal{M}_i\mathcal{M}_j=\mathcal{P}(e_i)\mathcal{P}(e_j)=\mathcal{P}(e_j)\mathcal{P}(e_i)=\mathcal{M}_j\mathcal{M}_i.$$

This completes the proof. □

Property (A1) is easily verified numerically, property (A2) is needed as technical assumption and property (A3) allows an efficient numerical implementation, since the eigenvalue decomposition of the matrix $\mathcal{P}(\hat{\alpha})$ is stable and cheap. In the following, we will

denote gPC bases that satisfy the properties (A1)–(A3) as $\boldsymbol{\mathcal{A}}$gPC bases.

These properties, however, are in general *not* satisfied. Counterexamples are gPC bases with Legendre and Hermite polynomials. The assumption of constant eigenvectors is borrowed from [61]. There, it is shown for a *one-dimensional* germ, that the Wiener-Haar basis and piecewise linear multiwavelets form $\boldsymbol{\mathcal{A}}$gPC bases. Under these assumption we can prove the following Lemma. Its proof is moved to the appendix.

Lemma 4.2. *Define the variables* $\hat{u}(\hat{\omega}):=\mathcal{P}^{-1}(\hat{\alpha})\hat{\beta}$, $\widehat{u^2}(\hat{\omega}):=\mathcal{P}_2(\hat{\omega})\hat{\beta}$ *and the matrices* $\mathcal{P}_1(\hat{\omega}):=\mathcal{P}(\hat{\beta})\mathcal{P}^{-1}(\hat{\alpha})$, $\mathcal{P}_2(\hat{\omega}):=\mathcal{P}(\hat{\beta})\mathcal{P}^{-2}(\hat{\alpha})$. *Then,* $\boldsymbol{\mathcal{A}}$*gPC bases satisfy the equalities*

$$\hat{u}^{\mathrm{T}}(\hat{\omega})\mathcal{P}(\hat{h})=(\hat{\alpha}*\hat{\beta})^{\mathrm{T}}, \tag{4.4}$$

$$\mathrm{D}_{\hat{\omega}}\left[\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega})\hat{\beta}\right]=\Big(-\hat{\beta}^{\mathrm{T}}\mathcal{P}_1^2(\hat{\omega}),3\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega})\Big), \tag{4.5}$$

$$\mathrm{D}_{\hat{\omega}}\left[(\hat{\alpha}*\hat{\alpha})^{\mathrm{T}}(\hat{\alpha}*\hat{\beta})\right]=\Big(3\hat{\alpha}^{\mathrm{T}}\mathcal{P}(\hat{\alpha})\mathcal{P}(\hat{\beta}),\hat{\alpha}^{\mathrm{T}}\mathcal{P}^2(\hat{\alpha})\Big), \tag{4.6}$$

$$\mathrm{D}_{\hat{\omega}}\left[\hat{u}(\hat{\omega})\right]=\Big(-\mathcal{P}_2(\hat{\omega}),\mathcal{P}^{-1}(\hat{\alpha})\Big), \tag{4.7}$$

$$\mathrm{D}_{\hat{\omega}}\left[\mathcal{P}_2(\hat{\omega})\hat{\beta}\right]=\Big(-2\mathcal{P}_1(\hat{\omega})\mathcal{P}_2(\hat{\omega}),2\mathcal{P}_2(\hat{\omega})\Big). \tag{4.8}$$

Finally, we state an entropy-entropy flux pair for shallow water equations in our main theorem.

Theorem 4.1 (Shallow Water Equations)**.** *Let an $\boldsymbol{\mathcal{A}}$gPC basis be given, which satisfies the properties* (A1)–(A3)*, and let states in the open, admissible set*

$$\mathbb{H} := \Big\{ \hat{y} := (\hat{h},\hat{q})^{\mathrm{T}} \in \big(\mathbb{R}^+ \times \mathbb{R}^K\big) \times \mathbb{R}^{K+1} \;\Big|\; \hat{\alpha} \in \mathbb{H}^+ \text{ for } (\hat{\alpha},\hat{\beta})^{\mathrm{T}} = \hat{\mathcal{Y}}^{-1}(\hat{y}) \Big\}$$

be given. Then, the Jacobian of the flux function (4.3) *is*

$$\mathrm{D}_{\hat{y}} \hat{f}(\hat{y}) = \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ g\mathcal{P}(\hat{h}) - \mathcal{P}_1^2(\hat{\omega}) & 2\mathcal{P}_1(\hat{\omega}) \end{pmatrix}$$

for $\hat{\omega} = (\hat{\alpha},\hat{\beta})$ and $\mathcal{P}_1(\hat{\omega}) = \mathcal{P}(\hat{\beta})\mathcal{P}^{-1}(\hat{\alpha})$.

The eigenvalue decomposition $\mathrm{D}_{\hat{y}} \hat{f}(\hat{y}) = \big[\mathcal{V}\hat{T}(\hat{\omega})\big]\hat{\Lambda}(\hat{\omega})\big[\mathcal{V}\hat{T}(\hat{\omega})\big]^{-1}$ reads as

$$\begin{aligned} \hat{\Lambda}^{\pm}(\hat{\omega}) &:= D_{\mathcal{P}}(\hat{\beta}) D_{\mathcal{P}}^{-1}(\hat{\alpha}) \pm \sqrt{g D_{\mathcal{P}}(\hat{h})}, & \hat{\Lambda}(\hat{\omega}) &:= \mathrm{diag}\big\{\hat{\Lambda}^+(\hat{\omega}), \hat{\Lambda}^-(\hat{\omega})\big\}, \\ \hat{T}(\hat{\omega}) &:= \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \hat{\Lambda}^+(\hat{\omega}) & \hat{\Lambda}^-(\hat{\omega}) \end{pmatrix}, & \mathcal{V} &:= \mathrm{diag}\{V,V\}. \end{aligned}$$

An entropy-entropy flux pair is $(\eta,\mu)(\hat{y}) := (\eta_1+\eta_2, \mu_1+\mu_2)(\hat{y})$ with

$$\begin{aligned} \eta_1(\hat{y}) &:= \frac{g}{2}\|\hat{h}\|_2^2 \quad \text{and} \quad \eta_2(\hat{y}) := \tilde{\eta}_2\Big(\hat{\mathcal{Y}}^{-1}(\hat{y})\Big) := \frac{1}{2}\|\hat{\beta}\|_2^2, \\ \mu_1(\hat{y}) &:= g\hat{h}^{\mathrm{T}}\hat{q} \quad\quad \text{and} \quad \mu_2(\hat{y}) := \tilde{\mu}_2\Big(\hat{\mathcal{Y}}^{-1}(\hat{y})\Big) := \frac{1}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega})\hat{\beta}. \end{aligned}$$

Proof. The Jacobian of the flux function and the Jacobian of the entropy read as

$$\begin{aligned} \mathrm{D}_{\hat{y}} \hat{f}(\hat{y}) &= \mathrm{D}_{\hat{y}} \hat{f}_1(\hat{y}) + \mathrm{D}_{\hat{\omega}} \tilde{f}_2(\hat{\omega})[\mathrm{D}_{\hat{\omega}}\hat{\mathcal{Y}}]^{-1}(\hat{\omega}) \\ &= \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ g\mathcal{P}(\hat{h}) & \mathbb{O} \end{pmatrix} + \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & 2\mathcal{P}(\hat{\beta}) \end{pmatrix} \begin{pmatrix} \frac{1}{2}\mathcal{P}^{-1}(\hat{\alpha}) & \mathbb{O} \\ -\frac{1}{2}\mathcal{P}_2(\hat{\omega}) & \mathcal{P}^{-1}(\hat{\alpha}) \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ g\mathcal{P}(\hat{h}) - \mathcal{P}_1^2(\hat{\omega}) & 2\mathcal{P}_1(\hat{\omega}) \end{pmatrix}, \\ \mathrm{D}_{\hat{y}} \eta(\hat{y}) &= \mathrm{D}_{\hat{y}} \eta_1(\hat{y}) + \mathrm{D}_{\hat{\omega}} \tilde{\eta}_2(\hat{\omega})[\mathrm{D}_{\hat{\omega}}\hat{\mathcal{Y}}]^{-1}(\hat{\omega}) = (g\hat{h}^{\mathrm{T}},0) + (0,\hat{\beta}^{\mathrm{T}})[\mathrm{D}_{\hat{\omega}}\hat{\mathcal{Y}}]^{-1}(\hat{\omega}) \\ &= \Big(g\hat{h}^{\mathrm{T}} - \frac{1}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_2(\hat{\omega}), \hat{\beta}^{\mathrm{T}}\mathcal{P}^{-1}(\hat{\alpha})\Big). \end{aligned}$$

The compatibility condition (2.3) is equivalent to

$$\mathrm{D}_{\hat{y}} \eta(\hat{y}) \mathrm{D}_{\hat{y}} \hat{f}(\hat{y}) = \mathrm{D}_{\hat{y}} \mu(\hat{y}) = \mathrm{D}_{\hat{\omega}} \tilde{\mu}(\hat{\omega})[\mathrm{D}_{\hat{\omega}}\hat{\mathcal{Y}}]^{-1}(\hat{\omega}) \quad \text{for} \quad \tilde{\mu}(\hat{\omega}) := \mu\Big(\hat{\mathcal{Y}}(\hat{\omega})\Big).$$

This holds due to Lemma 4.2, which yields

$$\begin{aligned}
&\Big(\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})\Big)\hat{\mathcal{Y}}(\hat{\omega})\\
&=\Big(g\hat{u}^{\mathrm{T}}(\hat{\omega})\mathcal{P}(\hat{h})-\hat{u}^{\mathrm{T}}(\hat{\omega})\mathcal{P}_1^2(\hat{\omega}),\, g\hat{h}^{\mathrm{T}}+\frac{3}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_2(\hat{\omega})\Big)\hat{\mathcal{Y}}(\hat{\omega})\\
&=\Big(g(\hat{\alpha}*\hat{\beta})^{\mathrm{T}}-\hat{u}^{\mathrm{T}}(\hat{\omega})\mathcal{P}_1^2(\hat{\omega}),\qquad g\hat{h}^{\mathrm{T}}+\frac{3}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_2(\hat{\omega})\Big)\hat{\mathcal{Y}}(\hat{\omega})\\
&=\Big(3g(\hat{h}*\hat{\beta})^{\mathrm{T}}-\frac{1}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_1^2(\hat{\omega}),\qquad g(\hat{h}*\hat{\alpha})^{\mathrm{T}}+\frac{3}{2}\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega})\Big)\\
&=\mathrm{D}_{\hat{\omega}}\tilde{\mu}(\hat{\omega}).
\end{aligned}$$

We define the auxiliary variables $\widetilde{\nabla}\eta(\hat{\omega}):=\nabla_{\hat{y}}\eta\big(\hat{\mathcal{Y}}(\hat{\omega})\big)$ and $D_1(\hat{\omega}):=D_{\mathcal{P}}(\hat{\beta})D_{\mathcal{P}}^{-1}(\hat{\alpha})$. With Lemma 4.2 we obtain the eigenvalue decomposition of the Hessian

$$\begin{aligned}
\nabla_{\hat{y}}^2\eta\big(\hat{\mathcal{Y}}(\hat{y})\big)&=\mathrm{D}_{\hat{\omega}}\widetilde{\nabla}\eta(\hat{\omega})[\mathrm{D}_{\hat{\omega}}\hat{\mathcal{Y}}]^{-1}(\hat{\omega})=\begin{pmatrix} g\mathbb{1}+\mathcal{P}_2^2(\hat{\omega}) & -\mathcal{P}_1(\hat{\omega})\mathcal{P}^{-2}(\hat{\alpha})\\ -\mathcal{P}_1(\hat{\omega})\mathcal{P}^{-2}(\hat{\alpha}) & \mathcal{P}^{-2}(\hat{\alpha})\end{pmatrix}\\
&=T_\eta(\hat{\alpha})D_\eta(\hat{\omega})T_\eta^{-1}(\hat{\alpha})\\
\text{with}\quad T_\eta(\hat{\alpha}):=&\begin{pmatrix}\mathcal{P}^{-1}(\hat{\alpha})V & \\ & \mathcal{P}^{-1}(\hat{\alpha})V\end{pmatrix},\quad D_\eta(\hat{\omega}):=\begin{pmatrix} gD_{\mathcal{P}}^2(\hat{\alpha})+D_1^2(\hat{\omega}) & -D_1(\hat{\omega})\\ -D_1(\hat{\omega}) & \mathbb{1}\end{pmatrix}.
\end{aligned}$$

Due to the block diagonal structure of the similar and symmetric matrix $D_\eta(\hat{\omega})$, the real eigenvalues are strictly positive if and only if $\big|D_{\mathcal{P}}(\hat{\alpha})\big|\neq 0$ holds, since they read as

$$\sigma\{D_\eta(\hat{\omega})\}=\frac{1}{2}\Big(gD_{\mathcal{P}}^2(\hat{\alpha})+D_1^2(\hat{\omega})+1\Big)\pm\frac{1}{2}\left[\Big(gD_{\mathcal{P}}^2(\hat{\alpha})+D_1^2(\hat{\omega})+1\Big)^2-4gD_{\mathcal{P}}^2(\hat{\alpha})\right]^{1/2}.$$

We obtain the eigenvalue decomposition of the Jacobian by calculating

$$\begin{aligned}
\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})&=\mathcal{V}\begin{pmatrix}\mathbb{O} & \mathbb{1}\\ gD_{\mathcal{P}}(\hat{h})-D_{\mathcal{P}}^2(\hat{\beta})D_{\mathcal{P}}^{-2}(\hat{\alpha}) & 2D_{\mathcal{P}}(\hat{\beta})D_{\mathcal{P}}^{-1}(\hat{\alpha})\end{pmatrix}\mathcal{V}^{\mathrm{T}}\\
&=\mathcal{V}\Big[\hat{T}(\hat{\omega})\hat{\Lambda}(\hat{\omega})\hat{T}^{-1}(\hat{\omega})\Big]\mathcal{V}^{\mathrm{T}}=\big[\mathcal{V}\hat{T}(\hat{\omega})\big]\hat{\Lambda}(\hat{\omega})\big[\mathcal{V}\hat{T}(\hat{\omega})\big]^{-1}.
\end{aligned}$$

This completes the proof. □

Note that the entropy-entropy flux pair reduces to the physical entropy in the deterministic case, see e.g. [18] for

$$\eta_0(y):=\frac{1}{2}\frac{q^2}{h}+\frac{g}{2}h^2,\quad \mu_0(y):=\frac{1}{2}\frac{q^3}{h^2}+gqh. \tag{4.9}$$

Remark 4.1. Isothermal Euler equations describe the density of a gas ρ and read as

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho(t,x)\\ q(t,x)\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix} q(t,x)\\ \frac{q^2(t,x)}{\rho(t,x)}+a^2\rho(t,x)\end{pmatrix}=0$$

with the speed of sound $a>0$. An intrusive formulation is

$$\frac{\partial}{\partial t}\begin{pmatrix}\hat{\rho}(t,x)\\ \hat{q}(t,x)\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\hat{q}(t,x)\\ \hat{\beta}(t,x)*\hat{\beta}(t,x)+a^2\hat{\rho}(t,x)\end{pmatrix}=0. \tag{4.10}$$

It has been shown in [24] for *arbitrary* gPC bases that the eigenvalues of the system (4.10) are real and there is a full set of eigenvectors provided that $\hat{\alpha}\in\mathbb{H}^+$ holds. We cannot show symmetric hyperbolicity for arbitrary bases and we cannot state an entropy. At least for $\boldsymbol{\mathcal{A}}$gPC bases, however, the system remains symmetrizable: We define the matrix

$$H(\hat{y}):=\tilde{H}\left(\hat{\mathcal{Y}}^{-1}(\hat{y})\right):=\begin{pmatrix}\mathcal{P}^2(\hat{\beta})\mathcal{P}^{-4}(\hat{\alpha})+a^2\mathcal{P}^{-2}(\hat{\alpha}) & -\mathcal{P}(\hat{\beta})\mathcal{P}^{-3}(\hat{\alpha})\\ -\mathcal{P}(\hat{\beta})\mathcal{P}^{-3}(\hat{\alpha}) & \mathcal{P}^{-2}(\hat{\alpha})\end{pmatrix},$$

which reduces in the deterministic case to the Hessian matrix $\nabla_y^2\eta(y)=H(y)$ for the entropy $\eta(y)={}^{q^2}\!/_{2\rho}+a^2\rho\ln(\rho)$. Provided that $\hat{\alpha}\in\mathbb{H}^+$ holds, the matrix $H(\hat{y})$ is strictly positive definite and the product $H(\hat{y})\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})$ is symmetric.

5 Energy estimates

We summarize our findings and state the notion of hyperbolicity in more detail. Similar to [28,33] we call a system

weakly hyperbolic	if eigenvalues of the Jacobian are real,
strongly hyperbolic	if eigenvalues are real and there is a complete set of eigenvectors,
strictly hyperbolic	if eigenvalues of the Jacobian are real and distinct,
symmetric hyperbolic	if a symmetric, strictly positive definite matrix $H(\hat{y})$ exists so that the product $H(\hat{y})\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})$ is symmetric.

Note that weakly hyperbolic systems are not necessarily stable [33]. All systems in this paper are at least strongly hyperbolic and hence stable. As illustrated in Fig. 2, symmetric and strictly hyperbolic systems form important classes. Deterministic Burgers', Euler and shallow water equations are both symmetric and strictly hyperbolic. In general, stochastic Galerkin formulations fail to have distinct eigenvalues. As an example one may consider the state $\hat{\alpha}:=(\hat{\alpha}_0,0,\cdots,0)^{\mathrm{T}}$, where the Jacobian of Burgers' equation reads as $\mathcal{P}(\hat{\alpha})=\hat{\alpha}_0\mathbb{1}$. The presented formulations, however, remain symmetric hyperbolic, since they are endowed with entropy-entropy flux pairs [9,29]. This allows energy estimates, which do not hold for general strongly hyperbolic systems [33].

In particular, classical solutions are well-posed [18, Th. 5.3.1]. A *Lipschitz continuous* classical solution $\hat{y}^*(t,x)\in\mathbb{H}_c$ of the Cauchy problem (2.1) on a finite time domain $[0,T)$ with initial data $\hat{\mathcal{I}}^*(x)$ which takes values in a convex, compact subset $\mathbb{H}_c\subset\mathbb{H}$ and *any* η-admissible weak solution $\hat{y}(t,x)\in\mathbb{H}_c$ with initial values $\hat{\mathcal{I}}(x)$ are related as follows.

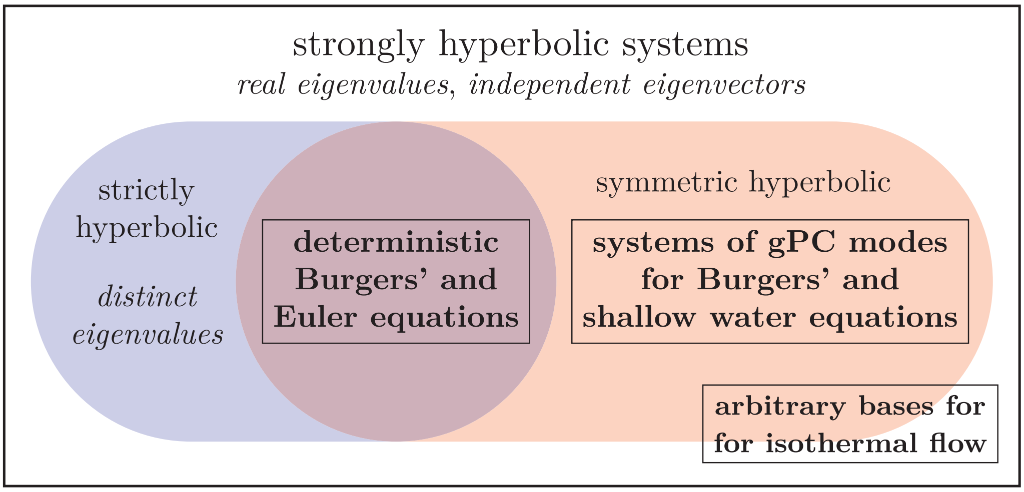

Figure 2: **Summary of considered hyperbolic systems**

(i) The classical solution exists up to some point in time $T>0$.

(ii) The classical solution $\hat{y}^*$ with initial values $\hat{\mathcal{I}}^*$ is the unique η-admissible weak entropy solution.

(iii) For any $r>0$, $t\in[0,T)$, there are positive constants $a,b,s\geq 0$ such that

$$\int_{|x|<r} \left\|\hat{y}^*(t,x)-\hat{y}(t,x)\right\| \mathrm{d}x \leq a e^{bt} \int_{|x|<r+st} \left\|\hat{\mathcal{I}}^*(x)-\hat{\mathcal{I}}(x)\right\| \mathrm{d}x.$$

The constant b depends on the Lipschitz continuity of the classical solution $\hat{y}^*$.

For numerical purposes, it is important to quantify the error between a (not known) reference solution $\hat{y}^*$ that satisfies the PDE $\hat{y}^*_t+\hat{f}(\hat{y}^*)_x=0$ and a perturbation $\hat{y}$, which may arise due to an adaptive gPC truncation. Similarly to the error estimates in [18,21,27], we define the residuum

$$\mathcal{R}(\hat{y}) := \hat{y}_t+\hat{f}(\hat{y})_x$$

as well as the relative entropy and the relative entropy flux as

$$\begin{aligned}
\eta(\hat{y}^*|\hat{y}) &:= \eta(\hat{y}^*)-\eta(\hat{y})-\mathrm{D}_{\hat{y}}\eta(\hat{y})\left(\hat{y}^*-\hat{y}\right),\\
\mu(\hat{y}^*|\hat{y}) &:= \mu(\hat{y}^*)-\mu(\hat{y})-\mathrm{D}_{\hat{y}}\eta(\hat{y})\left(f(\hat{y}^*)-f(\hat{y})\right).
\end{aligned}$$

Then, for general systems that are endowed with an entropy the following Lemma is proven in the appendix. It is similar to [18, Th. 5.2.1], [26, Lemma 2.7], [27, Th. 3.8].

Lemma 5.1. *Assume the approximation $\hat{y}$ is Lipschitz continuous in space $x\in\mathbb{R}$. Then, the following inequality holds:*

$$\begin{aligned}\int_{\mathbb{R}}\eta\Big(\hat{y}^*(t,x)\Big|\hat{y}(t,x)\Big)\,\mathrm{d}x\leq&\int_{\mathbb{R}}\eta\Big(\hat{\mathcal{I}}^*(x)\Big|\hat{\mathcal{I}}(x)\Big)\,\mathrm{d}x\\&-\int_0^t\int_{\mathbb{R}}\mathcal{R}^{\mathrm{T}}\Big(\hat{y}(s,x)\Big)\nabla^2_{\hat{y}}\eta\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x)-\hat{y}(s,x)\Big)\\&+\hat{y}_x^{\mathrm{T}}(s,x)\nabla^2_{\hat{y}}\eta\big(\hat{y}(s,x)\big)\Big[\mathrm{D}_{\hat{y}}\hat{f}\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x)-\hat{y}(s,x)\Big)\\&-\Big(\hat{f}\big(\hat{y}^*(s,x)\big)-\hat{f}\big(\hat{y}(s,x)\big)\Big)\Big]\,\mathrm{d}x\mathrm{d}s.\end{aligned}$$

The inner product $\langle\hat{y}^*,\hat{y}\rangle_{\nabla^2}:=\langle\hat{y}^*,\nabla^2_{\hat{y}}\eta(\hat{y})\hat{y}\rangle$ is well-defined for strictly convex entropies. Second-order Taylor approximations of the scalar entropy and the vector valued flux function yield the expressions

$$\begin{aligned}&\eta(\hat{y}^*|\hat{y})=\frac{1}{2}\|\hat{y}^*-\hat{y}\|^2_{\nabla^2}+\mathcal{O}\big(\|\hat{y}^*-\hat{y}\|_2^2\big),\\&\Big\|\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})(\hat{y}^*-\hat{y})-\big(\hat{f}(\hat{y}^*)-\hat{f}(\hat{y})\big)\Big\|\leq\frac{c_{\hat{f}}(\hat{y})}{2}\|\hat{y}^*-\hat{y}\|^2_{\nabla^2},\end{aligned}$$

where $c_{\hat{f}}(\hat{y})$ depends only on the flux function and on the approximated states. With this second-order approximation, Lemma 5.1, Cauchy-Schwarz and Young's inequality for products we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\|\hat{y}^*-\hat{y}\|^2_{\nabla^2}\mathrm{d}x-\int_{\mathbb{R}}\|\hat{\mathcal{I}}^*-\hat{\mathcal{I}}\|^2_{\nabla^2}\mathrm{d}x\\\leq&2\int_0^T\int_{\mathbb{R}}\Big|\Big\langle\hat{y}^*-\hat{y},\mathcal{R}(\hat{y})\Big\rangle_{\nabla^2}\Big|+\Big|\Big\langle\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})(\hat{y}^*-\hat{y})-\big(\hat{f}(\hat{y}^*)-\hat{f}(\hat{y})\big),\hat{y}_x\Big\rangle_{\nabla^2}\Big|\,\mathrm{d}x\mathrm{d}s\\\leq&\int_0^T\int_{\mathbb{R}}\|\hat{y}^*-\hat{y}\|^2_{\nabla^2}+\big\|\mathcal{R}(\hat{y})\big\|^2_{\nabla^2}+\|\hat{y}_x\|_{\nabla^2}c_{\hat{f}}(\hat{y})\|\hat{y}^*-\hat{y}\|^2_{\nabla^2}\,\mathrm{d}x\mathrm{d}s\\\leq&\int_0^T c(s;\hat{y})\int_{\mathbb{R}}\|\hat{y}^*-\hat{y}\|^2_{\nabla^2}\mathrm{d}x+\int_0^T\int_{\mathbb{R}}\big\|\mathcal{R}(\hat{y})\big\|^2_{\nabla^2}\mathrm{d}x\mathrm{d}s\end{aligned}$$

with the constant

$$c(s;\hat{y}):=\max_{x\in\mathbb{R}}\Big\{1+\big\|\hat{y}_x(s,x)\big\|_{\nabla^2}c_{\hat{f}}\big(\hat{y}(s,x)\big)\Big\}.$$

Gronwall's inequality yields the **a posteriori estimate**

$$\begin{aligned}\int_{\mathbb{R}}\Big\|\hat{y}^*(t,x)-\hat{y}(t,x)\Big\|^2_{\nabla^2}\mathrm{d}x\leq&\Bigg[\int_{\mathbb{R}}\Big\|\hat{\mathcal{I}}^*(x)-\hat{\mathcal{I}}(x)\Big\|^2_{\nabla^2}\mathrm{d}x\\&+\int_0^t\int_{\mathbb{R}}\Big\|\mathcal{R}\big(\hat{y}(s,x)\big)\Big\|^2_{\nabla^2}\mathrm{d}x\mathrm{d}s\Bigg]\exp\Big(\int_0^t c(s;\hat{y})\,\mathrm{d}s\Big).\end{aligned}\tag{5.1}$$

Once an estimate of the form (5.1), *without* second-order approximation, is derived, the convergence of the gPC expansion in the PDE can be quantified – at least for *smooth* solutions. We will derive an estimate for shallow water equations *without* neglecting the term $\mathcal{O}(\|\hat{y}^*-\hat{y}\|_2^2)$. Similar estimates for Burgers' equation are given in [21,27].

Theorem 5.1 (A Posteriori Estimate for Shallow Water Equations)**.** *Define the auxiliary functions*

$$\hat{V}\big(\hat{\omega}(t,x)\big):=\mathcal{P}^{-1}\big(\hat{\alpha}(t,x)\big)\frac{\partial}{\partial x}\hat{\beta}(t,x)-\mathcal{P}^{-2}\big(\hat{\alpha}(t,x)\big)\mathcal{P}\big(\hat{\beta}(t,x)\big)\frac{\partial}{\partial x}\hat{\alpha}(t,x),$$
$$\mathcal{P}_0\big(\hat{\alpha}^*(t,x),\hat{\alpha}(t,x)\big):=\mathcal{P}^{-1}\big(\hat{\alpha}(t,x)\big)\mathcal{P}\big(\hat{\alpha}^*(t,x)\big)$$

and assume the approximation $\hat{y}$ is Lipschitz continuous. For states $\hat{y}^*,\hat{y}\in\mathbb{H}_c\subset\mathbb{H}$ there is a constant $c\in[1,\infty)$ such that the spectral radius $\sigma_{\max}\big\{\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\big\}<\sqrt{c}$ is bounded and there is the estimate

$$\begin{aligned}&\int_{\mathbb{R}}\eta\Big(\hat{y}^*(t,x)\Big|\hat{y}(t,x)\Big)\,\mathrm{d}x\\
\leq&\Big[\int_{\mathbb{R}}\eta\Big(\hat{\mathcal{I}}^*(x)\Big|\hat{\mathcal{I}}(x)\Big)\,\mathrm{d}x\\
&+\frac{c}{2}\int_0^t\int_{\mathbb{R}}\Big\|\mathcal{R}\big(\hat{y}(s,x)\big)\Big\|_2^2\mathrm{d}x\,\mathrm{d}s\Big]\exp\left(\int_0^t 1+2\max_{x\in\mathbb{R}}\Big\{\Big\|\mathcal{P}\Big(\hat{V}\big(\hat{\omega}(s,x)\big)\Big)\Big\|_2\Big\}\mathrm{d}s\right).\end{aligned}$$

Proof. Due to the equality

$$\begin{aligned}\mathrm{D}_{\hat{y}}\eta(\hat{y})(\hat{y}^*-\hat{y})&=\Big[g\hat{h}-\frac{1}{2}\mathcal{P}_2(\hat{\omega})\hat{\beta},\mathcal{P}^{-1}(\hat{\alpha})\hat{\beta}\Big]^{\mathrm{T}}(\hat{y}^*-\hat{y})\\
&=g\hat{h}^{\mathrm{T}}(\hat{h}^*-\hat{h})-\frac{1}{2}\|\hat{\beta}\|_2^2-\frac{1}{2}\big\|\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}\big\|_2^2+\hat{\beta}^{\mathrm{T}}\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}^*,\end{aligned}$$

we obtain the relative entropy

$$\begin{aligned}\eta(\hat{y}^*|\hat{y})&=\Big[\frac{g}{2}\|\hat{h}^*\|_2^2+\frac{1}{2}\|\hat{\beta}^*\|_2^2\Big]\\
&\quad-\Big[\frac{g}{2}\|\hat{h}\|_2^2+\frac{1}{2}\|\hat{\beta}\|_2^2\Big]-\Big[g\hat{h}-\frac{1}{2}\mathcal{P}_2(\hat{\omega})\hat{\beta},\mathcal{P}^{-1}(\hat{\alpha})\hat{\beta}\Big]^{\mathrm{T}}(\hat{y}^*-\hat{y})\\
&=\frac{g}{2}\|\hat{h}^*-\hat{h}\|_2^2+\frac{1}{2}\big\|\hat{\beta}^*-\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}\big\|_2^2.\end{aligned}$$

By definition we have $\hat{V}(\hat{\omega})=(0,\mathbb{1})\nabla_{\hat{y}}^2\eta(\hat{y})\hat{y}_x$ and we calculate

$$\begin{aligned}&\left|\Big(\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})(\hat{y}^*-\hat{y})-\big(\hat{f}(\hat{y}^*)-\hat{f}(\hat{y})\big)\Big)^{\mathrm{T}}\nabla_{\hat{y}}^2\eta(\hat{y})\hat{y}_x\right|\\
=&\left|\begin{pmatrix}0\\ \frac{g}{2}\big(\hat{h}^*-\hat{h}\big)^{2*}+\big(\hat{\beta}^*-\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}\big)^{2*}\end{pmatrix}^{\mathrm{T}}\nabla_{\hat{y}}^2\eta(\hat{y})\hat{y}_x\right|\end{aligned}$$

$$
\begin{aligned}
&=\left|\left(\frac{g}{2}\left(\hat{h}^*-\hat{h}\right)^{2*}+\left(\hat{\beta}^*-\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}\right)^{2*}\right)^{\mathrm{T}}\hat{V}(\hat{\omega})\right|\\
&\leq\left|\frac{g}{2}\left(\hat{h}^*-\hat{h}\right)^{\mathrm{T}}\mathcal{P}\left(\hat{V}(\hat{\omega})\right)\left(\hat{h}^*-\hat{h}\right)\right|\\
&\quad+\left|\left(\hat{\beta}^*-\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}\right)^{\mathrm{T}}\mathcal{P}\left(\hat{V}(\hat{\omega})\right)\left(\hat{\beta}^*-\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}\right)\right|\\
&\leq 2\left\|\mathcal{P}\left(\hat{V}(\hat{\omega})\right)\right\|_2\eta(\hat{y}^*|\hat{y}).
\end{aligned}\tag{5.2}
$$

For states $\hat{y}^*,\hat{y}\in\mathbb{H}_c$ and $\hat{\alpha}^*,\hat{\alpha}\in\mathbb{H}^+$ the constant $c:=\max\left\{1,c_1^2\right\}\in[1,\infty)$ with

$$
c_1:=\max_{\substack{t\in[0,T],\\ x\in\mathbb{R}}}\left\{\sigma_{\min}^{-1}\left\{\mathcal{P}\left(\hat{\alpha}(t,x)\right)\right\}\sigma_{\max}\left\{\mathcal{P}\left(\hat{\alpha}^*(t,x)\right)\right\}\right\}\geq\sigma_{\max}\left\{\mathcal{P}_0\left(\hat{\alpha}^*(t,x),\hat{\alpha}(t,x)\right)\right\}
$$

is bounded. Then, we obtain the estimate

$$
\begin{aligned}
\|\hat{y}^*-\hat{y}\|_{\nabla^2}^2&=\begin{pmatrix}\hat{h}^*-\hat{h}\\ \hat{q}^*-\hat{q}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}g(\hat{h}^*-\hat{h})-\mathcal{P}^{-2}(\hat{\alpha})\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\mathcal{P}(\hat{\beta})\left[\hat{\beta}^*-\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}\right]\\ \mathcal{P}^{-2}(\hat{\alpha})\mathcal{P}(\hat{\alpha}^*)\left[\hat{\beta}^*-\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}\right]\end{pmatrix}\\
&=g\|\hat{h}^*-\hat{h}\|_2^2+\left[\hat{\beta}^*-\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}\right]^{\mathrm{T}}\mathcal{P}_0^2(\hat{\alpha}^*,\hat{\alpha})\left[\hat{\beta}^*-\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}\right]\\
&\leq g\|\hat{h}^*-\hat{h}\|_2^2+c_1^2\left\|\hat{\beta}^*-\mathcal{P}_0(\hat{\alpha}^*,\hat{\alpha})\hat{\beta}\right\|_2^2\\
&\leq 2\max\{1,c_1^2\}\eta(\hat{y}^*|\hat{y}).
\end{aligned}\tag{5.3}
$$

Estimate (5.3), Cauchy-Schwarz and Young's inequality for products imply

$$
\begin{aligned}
\left|\left\langle\mathcal{R}(\hat{y}),\hat{y}^*-\hat{y}\right\rangle_{\nabla^2}\right|&\leq\left\|\mathcal{R}(\hat{y})\right\|_{\nabla^2}\left\|\hat{y}^*-\hat{y}\right\|_{\nabla^2}\leq\left\|\mathcal{R}(\hat{y})\right\|_{\nabla^2}\sqrt{2}\max\{1,c_1\}\eta(\hat{y}^*|\hat{y})^{1/2}\\
&\leq\frac{\max\{1,c_1^2\}}{2}\left\|\mathcal{R}(\hat{y})\right\|_{\nabla^2}^2+\eta(\hat{y}^*|\hat{y}).
\end{aligned}\tag{5.4}
$$

Lemma 5.1 with the estimates (5.2) and (5.4) yield

$$
\begin{aligned}
&\int_{\mathbb{R}}\eta(\hat{y}^*|\hat{y})\,\mathrm{d}x\\
&\leq\int_{\mathbb{R}}\eta(\hat{\mathcal{I}}^*|\hat{\mathcal{I}})\,\mathrm{d}x+\int_0^t\int_{\mathbb{R}}\frac{c}{2}\left\|\mathcal{R}(\hat{y})\right\|_{\nabla^2}^2+\eta(\hat{y}^*|\hat{y})+2\left\|\mathcal{P}\left(\hat{V}(\hat{\omega})\right)\right\|_2\eta(\hat{y}^*|\hat{y})\,\mathrm{d}x\mathrm{d}s\\
&\leq\int_{\mathbb{R}}\eta(\hat{\mathcal{I}}^*|\hat{\mathcal{I}})\,\mathrm{d}x+\int_0^t\int_{\mathbb{R}}\frac{c}{2}\left\|\mathcal{R}(\hat{y})\right\|_{\nabla^2}^2\mathrm{d}x\mathrm{d}s\\
&\quad+\int_0^t\left(1+2\max_{x\in\mathbb{R}}\left\{\left\|\mathcal{P}\left(\hat{V}(\hat{\omega})\right)\right\|_2\right\}\right)\int_{\mathbb{R}}\eta(\hat{y}^*|\hat{y})\,\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

The claim follows from Gronwall's inequality. □

6 Numerical results

First, we illustrate theoretical results. To this end, we show the solutions and the entropies for truncated Wiener-Haar expansions. In particular, we highlight their smoothness properties and state statistics of interest. Then, we illustrate the decay of entropies, which mimics the stability of the system. Finally, we show an application to balance laws.

To this end, an interval $[0,x_{\text{end}}]$ is divided into N cells by a space discretization $\Delta x > 0$ with $\Delta x N = x_{\text{end}}$. The centers are $x_j := (j+\frac{1}{2})\Delta x$ and the edges are $x_{j+1/2} := j\Delta x$. The evolution of cell averages

$$\hat{\mathbf{y}}_j(t) := \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \hat{y}(t,x)\,\mathrm{d}x$$

of a balance law $\hat{y}_t + \hat{f}(\hat{y})_x = -\hat{S}(\hat{y};x)$ is described by the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{y}}_j(t) = -\frac{1}{\Delta x}\left[\hat{f}\Big(\hat{y}(t,x_{j+1/2})\Big) - \hat{f}\Big(\hat{y}(t,x_{j-1/2})\Big)\right] - \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \hat{S}\Big(\hat{y}(t,x);x\Big)\,\mathrm{d}x.$$

To obtain a semi-discretization in space, we use the local Lax-Friedrichs flux

$$\hat{\mathbf{f}}(\hat{\mathbf{y}}_\ell,\hat{\mathbf{y}}_r) := \frac{1}{2}\Big(\hat{f}(\hat{\mathbf{y}}_\ell) + \hat{f}(\hat{\mathbf{y}}_r)\Big) - \frac{1}{2}\max_{j=\ell,r}\Big\{\sigma\Big\{\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})\big|_{\hat{y}=\hat{\mathbf{y}}_j}\Big\}\Big\}(\hat{\mathbf{y}}_r - \hat{\mathbf{y}}_\ell),$$

where the spectrum $\sigma\{\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})\}$ is given in Theorem 4.1. Furthermore, the central, weighted, essentially non oscillatory (CWENO) reconstruction from [16] is applied. A third-order reconstruction is of the form

$$\text{CWENO}: \big[\hat{\mathbf{y}}_{j-1},\hat{\mathbf{y}}_j,\hat{\mathbf{y}}_{j+1}\big] \mapsto \mathit{Poly3}(x),$$

where $\mathit{Poly3}(x)$ denotes a third-order reconstruction polynomial. We denote the reconstruction at the right side of a cell interface by $\hat{\mathbf{y}}^+_{j-1/2}(t) := \mathit{Poly3}(x_{j-1/2})$, at the left side by $\hat{\mathbf{y}}^-_{j+1/2}(t) := \mathit{Poly3}(x_{j+1/2})$ and at the cell center by $\hat{\mathbf{y}}^c_j(t) := \mathit{Poly3}(x_j)$. The source term is discretized by the Gauss-Lobatto rule with three quadrature nodes. Then, the resulting semi-discretization with the local Lax-Friedrichs flux reads as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{y}}_j(t) = &-\frac{1}{\Delta x}\left[\hat{\mathbf{f}}\Big(\hat{\mathbf{y}}^-_{j+1/2}(t),\hat{\mathbf{y}}^+_{j+1/2}(t)\Big) - \hat{\mathbf{f}}\Big(\hat{\mathbf{y}}^-_{j-1/2}(t),\hat{\mathbf{y}}^+_{j-1/2}(t)\Big)\right] \\ &-\frac{1}{6}\Big[\hat{S}\big(\hat{\mathbf{y}}^+_{j-1/2}(t);x_{j-1/2}\big) + 4\hat{S}\big(\hat{\mathbf{y}}^c_j(t);x_j\big) + \hat{S}\big(\hat{\mathbf{y}}^-_{j+1/2}(t);x_{j+1/2}\big)\Big] + \mathcal{O}\big(\Delta x^3\big).\end{aligned}$$

It is approximated in time with a strong stability preserving (SSP) Runge-Kutta method with three stages [39]. All simulations are done with `Matlab`. The CWENO reconstruction is borrowed from the authors of [16] and compiled in `Matlab` as `C`-implementation.

6.1 Illustration of the theoretical results

We illustrate the analysis using a **dam break problem** [49]. The solution consists of a rarefaction wave, moving with negative speed, and a shock wave with positive speed. Both waves are connected by an intermediate state y_{m}. For given states $\bar{\mathbf{y}}_\ell=(\bar{\mathbf{h}}_\ell,\bar{\mathbf{q}}_\ell)^{\mathrm{T}}$ and $\bar{\mathbf{y}}_r=(\bar{\mathbf{h}}_r,\bar{\mathbf{q}}_r)^{\mathrm{T}}$ with $\bar{\mathbf{h}}_\ell\geq\bar{\mathbf{h}}_r>0$ and $\bar{\mathbf{q}}_\ell=\bar{\mathbf{q}}_r=0$, the dam break problem with initial states $y(0,x_\ell)=\bar{\mathbf{y}}_\ell$ and $y(0,x_r)=\bar{\mathbf{y}}_r$ for $x_\ell<0<x_r$ is solved by

$$y(t,x)=\begin{cases}\bar{\mathbf{y}}_\ell & \text{if } x<t\lambda^-(\bar{\mathbf{y}}_\ell),\\ y_{\mathrm{rf}}(t,x;\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r) & \text{if } t\lambda^-(\bar{\mathbf{y}}_\ell)\leq x<t\lambda^-\big(y_{\mathrm{m}}(\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r)\big),\\ y_{\mathrm{m}}(\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r) & \text{if } t\lambda^-\big(y_{\mathrm{m}}(\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r)\big)\leq x<ts(\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r),\\ \bar{\mathbf{y}}_r & \text{if } ts(\bar{\mathbf{y}}_\ell,\bar{\mathbf{y}}_r)<x.\end{cases}\tag{6.1}$$

The expressions for the rarefaction wave y_{rf}, intermediate state y_{m} and shock speed s are found in [49]. We consider uniformly distributed left initial values $\bar{\mathbf{h}}_\ell(\xi)\sim U(3,4)$ and the deterministic right state $\bar{\mathbf{h}}_r=1$.

The Haar sequence [34, 53, 62] with level $J\in\mathbb{N}_0$ generates a gPC basis $\mathcal{S}_K$ with $K+1=2^{J+1}$ elements by

$$\mathcal{S}_K:=\big\{1,\psi(\xi),\psi_{j,k}(\xi)\;\big|\;k=0,\cdots,2^j-1,\;j=1,\cdots,J\big\}$$
$$\text{for}\quad \psi_{j,k}(\xi):=2^{j/2}\psi\big(2^j\xi-k\big)\quad\text{and}\quad \psi(\xi):=\begin{cases}1 & \text{if } 0\leq\xi<1/2,\\ -1 & \text{if } 1/2\leq\xi<1,\\ 0 & \text{else.}\end{cases}$$

Using a lexicographical order we identify the relation $\phi_1=\psi$, $\phi_2=\psi_{1,0}$ and $\phi_3=\psi_{1,1}$ with the supports $\mathrm{supp}\{\psi\}=[0,1)$, $\mathrm{supp}\{\psi_{1,0}\}=[0,1/2)$ and $\mathrm{supp}\{\psi_{1,1}\}=[1/2,1)$.

For the theoretical results, we use the equidistant space discretization $\Delta x=10^{-3}$, statistics are determined by a standard Monte-Carlo method with 10^5 samples. In Fig. 3 we consider a dam break problem, where the left initial state is uniformly distributed, i.e. $\bar{\mathbf{y}}_\ell(\xi)=3+\xi\sim U(3,4)$. The Wiener-Haar expansion in the intrusive formulation with level $J=1$ approximates the continuous function $\bar{\mathbf{y}}_\ell(\xi)$ as step function (blue, left subplot). While the first element ϕ_0 yields the mean (green), the remaining functions give the details (black). A zoom in the area of the shock, which are the diagrams in the lower left corners, reveals that it is described mostly by the mean ϕ_0 and the first detail function ϕ_1. In fact, we have $\hat{h}_2(t,x)=0$ close to the right half of the shock, since the corresponding basis element ϕ_2 describes low initial heights, which result in slow shock speeds. Furthermore, the 1.0-confidence region and realisations, corresponding to the jumps in the approximated input distribution, are shown. The entropy according to Theorem 4.1 is plotted with respect to the right y-axis in red. For initial states $\bar{\mathbf{y}}=(\bar{\mathbf{h}},0)^{\mathrm{T}}$, when the momentum is zero, we have the relation

$$\mathbb{E}\Big[\eta_0\Big(\mathcal{G}_K\big[\bar{\mathbf{y}}(\xi)\big]\Big)\Big]=\mathbb{E}\Big[\frac{g}{2}\mathcal{G}_K\big[\bar{\mathbf{h}}(\xi)\big]^2\Big]=\frac{g}{2}\|\hat{\mathbf{h}}\|_2^2=\eta(\hat{\mathbf{y}}),\tag{6.2}$$

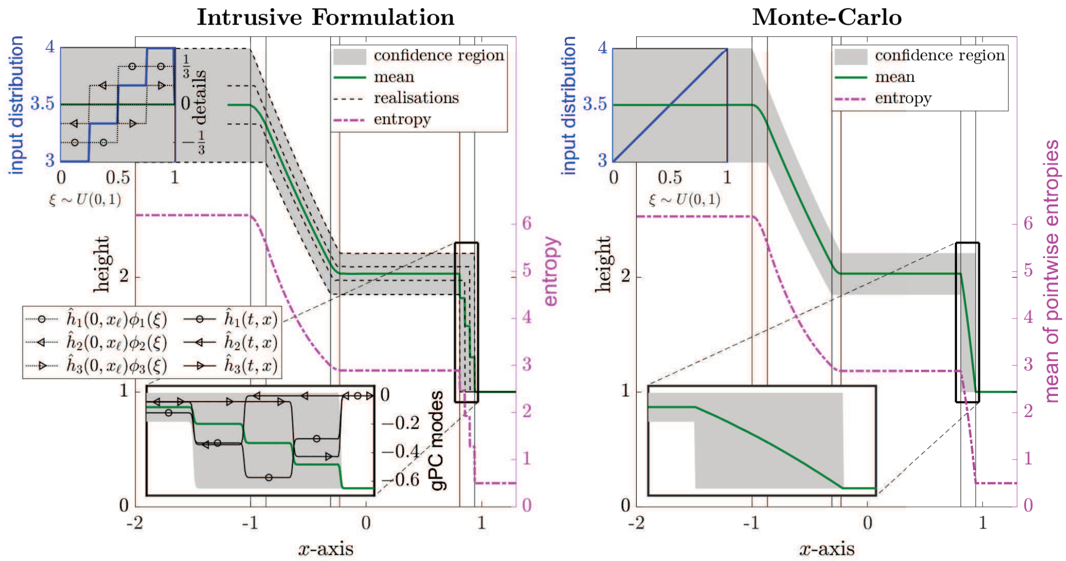

Figure 3: Solution of the dam break problem for the intrusive formulation of Theorem 4.1 compared to a Monte-Carlo simulation in $t=0.5$.

where η_0 is the pointwise entropy (4.9). This motivates the choice of the mean of pointwise entropies as a quantitative comparison. This choice is completely independent from our new results and only based on a Monte-Carlo simulation. Although the entropies of the intrusive formulation converge to the mean for the initial values, we do not claim that there is a convergence also for $t>0$.

Apart from the shock, good agreement is observed for both the entropy and the presented statistics of interest. The main difference is that there is no longer a smooth expectation of the shock. This issue has been observed also for continuous input distributions [21,24,60].

Fig. 4 consists of the amplifications of shocks and visualizes the regularity of truncated gPC expansions in more detail. For states satisfying $\hat{\Lambda}_i^-(\hat{\omega})<0<\hat{\Lambda}_j^+(\hat{\omega})$ for all $i,j=0,\cdots,K$ the initial discontinuity splits into at most $K+1$ distinct waves that have positive speed. Here, $K+1$ waves move at slightly different speeds to the right. We choose as reference solution the level $J=4$ with $K+1=32$ basis elements. The gPC modes are determined in a semi-intrusive way as $\hat{y}_k^{\text{ref}}:=\mathbb{E}\left[y^{\text{ref}}(t,x;\xi)\phi_k(\xi)\right]$, where $y^{\text{ref}}:=(h^{\text{ref}},q^{\text{ref}})^{\text{T}}$ denotes the reference solution (6.1).

Table 1 reports on numerical errors for the mean and the variance

$$E_K^{(\mathbb{E})}(t,x):=\left|\mathbb{E}\left[h^{\text{ref}}(t,x;\xi)\right]-\hat{h}_0(t,x)\right|,$$

$$E_K^{(\mathbb{V})}(t,x):=\left|\text{Var}\left[h^{\text{ref}}(t,x;\xi)\right]-\sum_{k=1}^{K}\hat{h}_k^2(t,x)\right|.$$

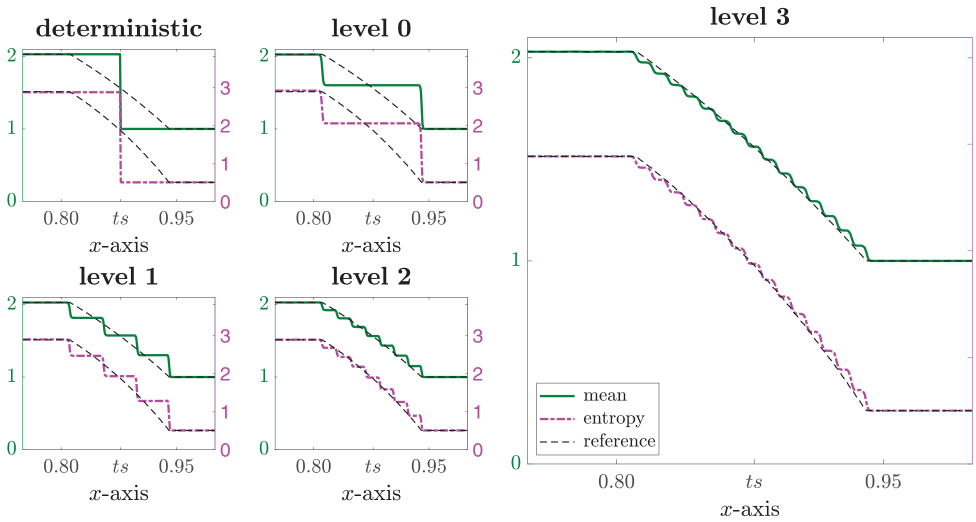

Figure 4: Zoom on shock; semi-intrusive reference solution (dashed) with $K+1=32$.

Table 1: Observed numerical errors for the dam break problem.

L^∞**-error:**	**rarefaction wave**				**shock**				**units**
level J	0	1	2	3	0	1	2	3	
$\hat{E}_K^{(\mathbb{E})}$	4.43	1.63	0.86	0.56	56.93	28.05	13.03	5.44	$[10^{-2}]$
$\hat{E}_K^{(\mathbb{V})}$	16.69	5.57	2.39	1.12	32.40	24.77	13.52	5.44	$[10^{-2}]$
L^1**-error:**	**rarefaction wave**				**shock**				**units**
level J	0	1	2	3	0	1	2	3	
$\hat{E}_K^{(\mathbb{E})}$	6.76	3.06	2.01	1.65	35.31	13.16	5.84	2.68	$[10^{-3}]$
$\hat{E}_K^{(\mathbb{V})}$	137.89	45.97	19.68	9.16	36.77	14.55	6.47	2.89	$[10^{-3}]$
	relative entropy for Cauchy problem								**units**
level J	0		1		2		3		
$\eta(\hat{y}^*\|\hat{y})$	108.05		27.60		9.89		5.66		$[10^{-3}]$

Table 1 is divided into the rarefaction wave for $x\in[-1.5,0]$ and the shock for $x\in[0,1.5]$. Then, for each level $J=0,\cdots,3$ with corresponding gPC order $K+1=2,4,8,16$ the L^1-and L^∞-norms $\int|\cdot|\mathrm{d}x$ and $\sup_x|\cdot|$ are stated. Indeed, we observe a convergence for the mean and the variance. Furthermore, we show the relative entropy and use again the semi-intrusively computed gPC modes $\hat{y}_k^{\text{ref}}\in\mathbb{R}^{64}$ as reference solution. We observe the expected decay also for this error measure. To verify the compatibility condition (2.3), we consider the L^2-error

$$E_K^{(C)}(t,x):=\left\|\mathrm{D}_{\hat{y}}\mu(\hat{y})-\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})\right\|_2(t,x)$$

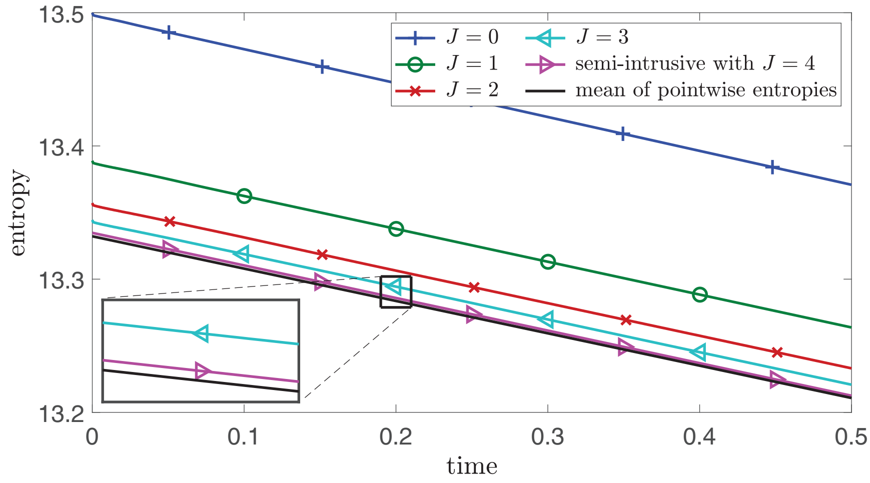

Figure 5: Time evolution of entropies according to inequality (6.3).

and we expect it to be close to zero for smooth solutions. The compatibility condition is fulfilled up to numerical errors, which are two powers smaller than the spatial discretization Δx.

The entropy inequality (2.2) guarantees a decaying entropy, since spatial integration yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \eta\big(\hat{y}(t,x)\big)\,\mathrm{d}x \le 0 \quad\Longrightarrow\quad \int_{\mathbb{R}} \eta\big(\hat{y}(t,x)\big)\,\mathrm{d}x \le \int_{\mathbb{R}} \eta\big(\hat{\mathcal{I}}(x)\big)\,\mathrm{d}x. \tag{6.3}$$

Fig. 5 aims to show this decay over time. We choose both the semi-intrusively computed gPC modes $\hat{y}_k^{\mathrm{ref}} \in \mathbb{R}^{64}$ and the mean of the pointwise entropies (6.2) as reference solution. Indeed, all computed entropies are decreasing. For a high refinement level J the entropies are close to the reference solutions.

6.2 Applications to balance laws with multiple sources of uncertainty

Common choices for multidimensional bases, see e.g. [53,81], are the

$$\begin{aligned}
&\text{tensor basis} \quad \mathbb{K}_{\mathrm{T}} := \big\{ \boldsymbol{k} \in \mathbb{N}_0^M \;\big|\; \|\boldsymbol{k}\|_0 \le K_{\mathrm{PC}} \big\} \quad \text{with} \quad |\mathbb{K}_{\mathrm{T}}| = (K_{\mathrm{PC}}+1)^M,\\
&\text{sparse basis} \quad \mathbb{K}_{\mathrm{S}} := \big\{ \boldsymbol{k} \in \mathbb{N}_0^M \;\big|\; \|\boldsymbol{k}\|_1 \le K_{\mathrm{PC}} \big\} \quad \text{with} \quad |\mathbb{K}_{\mathrm{S}}| = (M+K_{\mathrm{PC}})!(M!K_{\mathrm{PC}}!)^{-1},
\end{aligned}$$

where $K_{\mathrm{PC}} \in \mathbb{N}_0$ denotes the one-dimensional gPC truncation. It has already been observed for *linearized* shallow water equations that the sparse basis leads to a loss of hyperbolicity [64, Sec. 4.2]. Due to $\mathbb{K}_{\mathrm{S}} \subseteq \mathbb{K}_{\mathrm{T}}$, it is argued to enlarge the gPC basis to obtain a hyperbolic system. Also our theoretical results for the *nonlinear* shallow water equations cannot be directly applied to a Wiener-Haar expansion as sparse multidimensional basis, since property (A1) in Lemma 4.1 is violated. The choice $K_{\mathrm{PC}} = 1$, $M = 2$ serves

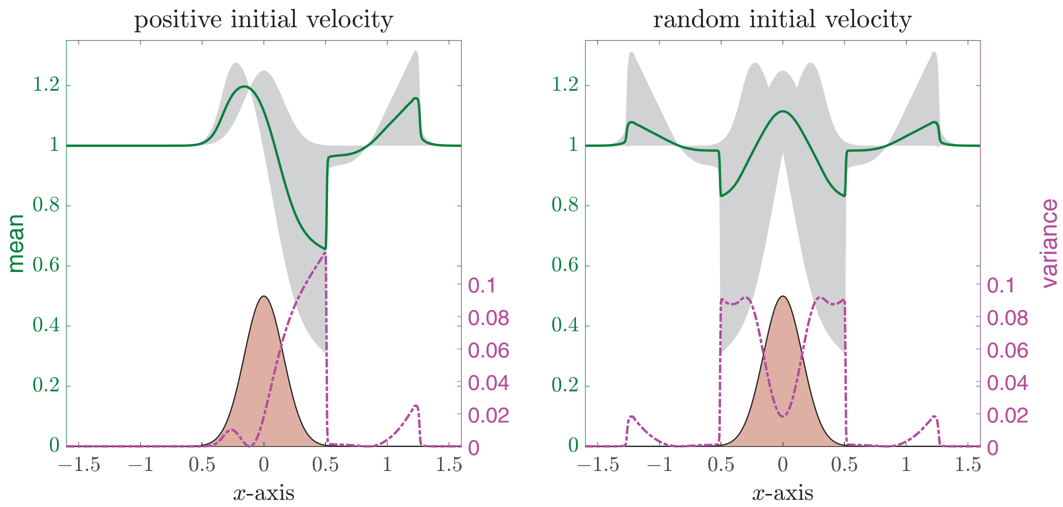

Figure 6: Shallow water equations with uncertain bottom topography, $\Delta x = 0.005$, $K_{\mathrm{PC}} = 7$.

as a counter example. For the full basis, however, we can verify property (A1) for the following problem.

We assume as initial values a constant free-water surface $w(0,x) := h(0,x;\xi) + B(x,\xi_1) = 1$. The source term $S(y;x,\xi_1) := (0, gS_q(h;x,\xi_1))^{\mathrm{T}}$ with

$$S_q(h;x,\xi_1) := hB'(x,\xi_1) \quad \text{and} \quad B(x,\xi_1) := \begin{cases} \frac{1}{2}\exp(-20x^2) & \text{with probability } 0.5, \\ 0 & \text{with probability } 0.5 \end{cases}$$

models the random space-varying bottom topography $B(x,\xi_1)$. We define the gPC modes $\hat{B}(x) := \frac{1}{4}\exp(-20x^2)(1,1,0,\cdots,0)^{\mathrm{T}}$ to represent the bottom topography exactly in a Wiener-Haar expansion as $B(x,\xi_1) = \hat{B}_0(x) + \hat{B}_1(x)\psi(\xi_1)$. Then, the projected source term reads as $\widehat{S_q}\big(\hat{h}(t,x)\big) = \hat{h}(t,x) * \hat{B}'(x)$. The left subplot in Fig. 6 shows the mean and the confidence region of the free-water surface with initial momentum $q(0,x) = 1$. In the second subplot the initial momentum with $\mathbb{P}\big[q(0,x;\xi_2) = 1\big] = \mathbb{P}\big[q(0,x;\xi_2) = -1\big]$ is an additional source of uncertainty.

Finally, we give an example for the sparse basis $\mathbb{K}_{\mathrm{S}}$. Since the properties (A1)–(A3) in Lemma 4.1 are not satisfied, we consider isothermal Euler equations. It has been proven in [24] that the formulation (4.10) is strongly hyperbolic for arbitrary gPC bases. However, it is not necessarily endowed with an entropy, as discussed in Remark 4.1. In extension to the results in [24] we use the space-varying source term $S(y;x) := (0, \frac{f(x)}{2D} S_q(y))^{\mathrm{T}}$ with $S_q(y) := \frac{q|q|}{\rho}$ to model friction. It is convenient to express the nonlinear term $S_q(y)$ in Roe variables, i.e. $S_q(y) = |\beta|\beta$, to obtain the Galerkin formulation $\widehat{S_q}(\hat{y}) = |\mathcal{P}(\hat{\beta})|\hat{\beta}$. For illustration, we consider the constant diameter of the pipe $D = 1$ and the random, space-varying friction factor $fr(x,\xi_1) \sim U(0, 20\exp(-20x^2))$. It is exactly represented in normalized Legendre polynomials $fr(x,\xi_1) := \widehat{fr}_0(x) + \widehat{fr}_1(x)\phi_1(\xi_1)$, where the gPC modes

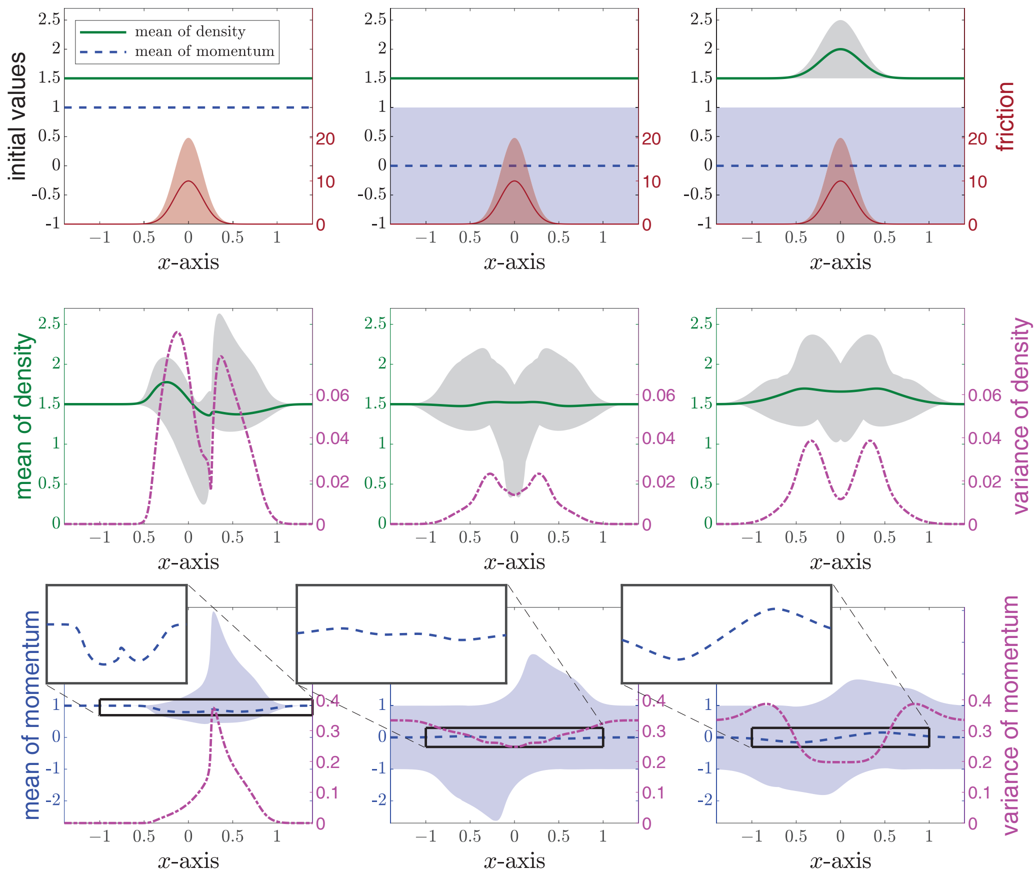

Figure 7: Isothermal Euler equations with uncertain friction factor, $\Delta x = 0.005$, $K_{\mathrm{PC}} = 6$.

read as $\widehat{fr}(x) := 10\exp(-20x^2)(1,1,0,\cdots,0)^{\mathrm{T}}$. The left subplots in Fig. 7 are based on deterministic initial values $y(0,x) = (1.5,1)$. Then, a second source of uncertainty is introduced as random initial momentum $q(0,x;\xi_2) \sim U(-1,1)$. The right subplots show three perturbations. There, also the initial density $\rho(0,x;\xi_3) \sim U(1.5,1.5+\exp(-20x^2))$ is uniformly distributed.

7 Summary

For certain basis functions we have analyzed a stochastic Galerkin formulation of shallow water equations that is based on the Roe variable transform [61]. This transform involves the stochastic Galerkin square root, which we have defined uniquely as a minimization problem. An important consequence is the bijective mapping between conserved and Roe variables. For the hyperbolic Galerkin formulation a generalization of the physical entropy to the stochastic Galerkin formulation has been presented. As important conclusions wellposedness of classical solutions as well as energy estimates have been stated. Our analysis covers also the case of multiple random dimensions. Sparse basis, however, may violate properties that guarantee hyperbolicity. This confirms findings for random linearized shallow water equations in previous works [64]. Numerical experiments have confirmed the theoretical findings. While symmetric hyperbolic systems in multiple spatial dimensions can also be expanded in entropy variables [20], similar results for Roe variables are not yet known.

Acknowledgments

This work is supported by DFG HE5386/14,15, BMBF 05M18PAA, DFG-GRK 2326.

We would like to offer special thanks to Aleksey Sikstel and Giuseppe Visconti.

Appendix

For the proof of Lemma 4.2 we recall $\mathrm{D}_{\hat{\alpha}_k}\mathcal{P}^{-1}(\hat{\alpha}) = -\mathcal{P}^{-1}(\hat{\alpha})\mathrm{D}_{\hat{\alpha}_k}\mathcal{P}(\hat{\alpha})\mathcal{P}^{-1}(\hat{\alpha})$.

Proof of Lemma 4.2. Eq. (4.4) follows from

$$\hat{u}^{\mathrm{T}}(\hat{\omega})\mathcal{P}(\hat{h}) = \hat{\beta}^{\mathrm{T}}\mathcal{P}(\hat{h})\mathcal{P}^{-1}(\hat{\alpha}) = \hat{\alpha}^{\mathrm{T}}\mathcal{P}(\hat{\alpha})\mathcal{P}(\hat{\beta})\mathcal{P}^{-1}(\hat{\alpha}) = (\hat{\alpha}*\hat{\beta})^{\mathrm{T}},$$

where we have used symmetry and the commutativity property (A2). We calculate

$$\begin{aligned}
&\mathcal{P}(\hat{\beta}) = \Big[\Big(\sum_{j=0}^{K}\hat{\beta}_j\langle\phi_i\phi_j,\phi_0\rangle\Big)\Big|\cdots\Big|\Big(\sum_{j=0}^{K}\hat{\beta}_j\langle\phi_i\phi_j,\phi_K\rangle\Big)\Big]_{i=0,\cdots,K} = \Big[\mathcal{M}_0\hat{\beta}\Big|\cdots\Big|\mathcal{M}_K\hat{\beta}\Big],\\
&\mathcal{P}\big(\hat{u}(\hat{\omega})\big)\hat{\gamma} = \mathcal{P}(\hat{\gamma})\mathcal{P}^{-1}(\hat{\alpha})\hat{\beta} = \mathcal{P}_1(\hat{\omega})\hat{\gamma},\\
&\mathcal{P}\big(\widehat{u^2}(\hat{\omega})\big)\hat{\gamma} = \mathcal{P}(\hat{\gamma})\mathcal{P}^{-2}(\hat{\alpha})\mathcal{P}(\hat{\beta})\hat{\beta} = \mathcal{P}_1^2(\hat{\omega})\hat{\gamma}
\end{aligned}$$

to obtain Eqs. (4.5) and (4.6) as

$$\begin{aligned}
\mathrm{D}_{\hat{\alpha}}\Big[\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega})\hat{\beta}\Big] &= -\Big[\hat{\beta}^{\mathrm{T}}\mathcal{P}(\hat{\beta})\mathcal{P}^{-1}(\hat{\alpha})\mathcal{M}_k\mathcal{P}^{-1}(\hat{\alpha})\hat{\beta}\Big]_{k=0,\cdots,K}\\
&= -\hat{\beta}^{\mathrm{T}}\mathcal{P}_1^2(\hat{\omega}),\\
\mathrm{D}_{\hat{\beta}}\Big[\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega})\hat{\beta}\Big] &= \mathrm{D}_{\bar{\beta}}\Big[\bar{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega})\bar{\beta}\Big]\Big|_{\bar{\beta}=\hat{\beta}} + \mathrm{D}_{\bar{\beta}}\Big[\hat{\beta}^{\mathrm{T}}\mathcal{P}(\bar{\beta})\hat{u}(\hat{\omega})\Big]\Big|_{\bar{\beta}=\hat{\beta}}\\
&= 2\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega}) + \hat{\beta}^{\mathrm{T}}\mathcal{P}\big(\hat{u}(\hat{\omega})\big)\\
&= 3\hat{\beta}^{\mathrm{T}}\mathcal{P}_1(\hat{\omega}),
\end{aligned}$$

$$
\begin{aligned}
\mathrm{D}_{\hat\alpha}\Big[(\hat\alpha*\hat\alpha)^{\mathrm T}(\hat\alpha*\hat\beta)\Big] &= \mathrm{D}_{\bar\alpha}\Big[\bar\alpha^{\mathrm T}\mathcal{P}(\hat\alpha)\mathcal{P}(\hat\beta)\bar\alpha\Big]\Big|_{\bar\alpha=\hat\alpha} + \mathrm{D}_{\bar\alpha}\Big[\hat\alpha^{\mathrm T}\mathcal{P}(\bar\alpha)\mathcal{P}(\hat\beta)\hat\alpha\Big]\Big|_{\bar\alpha=\hat\alpha}\\
&= 2\hat\alpha^{\mathrm T}\mathcal{P}(\hat\alpha)\mathcal{P}(\hat\beta) + \Big[\hat\alpha^{\mathrm T}\mathcal{M}_0\mathcal{P}(\hat\beta)\hat\alpha\Big|\cdots\Big|\hat\alpha^{\mathrm T}\mathcal{M}_K\mathcal{P}(\hat\beta)\hat\alpha\Big]\\
&= 3\hat\alpha^{\mathrm T}\mathcal{P}(\hat\alpha)\mathcal{P}(\hat\beta),\\
\mathrm{D}_{\hat\beta}\Big[(\hat\alpha*\hat\alpha)^{\mathrm T}(\hat\alpha*\hat\beta)\Big] &= \mathrm{D}_{\hat\beta}\Big[\hat\alpha^{\mathrm T}\mathcal{P}^2(\hat\alpha)\hat\beta\Big] = \hat\alpha^{\mathrm T}\mathcal{P}^2(\hat\alpha).
\end{aligned}
$$

The matrices (4.7) and (4.8) follow from

$$
\begin{aligned}
\mathrm{D}_{\hat\alpha}\Big[\hat u(\hat\omega)\Big] &= -\Big[\mathcal{P}^{-1}(\hat\alpha)\mathcal{M}_0\mathcal{P}^{-1}(\hat\alpha)\hat\beta\Big|\cdots\Big|\mathcal{P}^{-1}(\hat\alpha)\mathcal{M}_K\mathcal{P}^{-1}(\hat\alpha)\hat\beta\Big]\\
&= -\mathcal{P}_2(\hat\omega),\\
\mathrm{D}_{\hat\alpha}\Big[\mathcal{P}_2(\hat\omega)\hat\beta\Big] &= 2\mathrm{D}_{\bar\alpha}\Big[\mathcal{P}_1(\hat\omega)\mathcal{P}^{-1}(\bar\alpha)\hat\beta\Big]\Big|_{\bar\alpha=\hat\alpha}\\
&= -2\mathcal{P}_1(\hat\omega)\Big[\mathcal{P}^{-1}(\hat\alpha)\mathcal{M}_0\mathcal{P}^{-1}(\hat\alpha)\hat\beta\Big|\cdots\Big|\mathcal{P}^{-1}(\hat\alpha)\mathcal{M}_K\mathcal{P}^{-1}(\hat\alpha)\hat\beta\Big]\\
&= -2\mathcal{P}_1(\hat\omega)\mathcal{P}_2(\hat\omega),\\
\mathrm{D}_{\hat\beta}\Big[\mathcal{P}_2(\hat\omega)\hat\beta\Big] &= \mathrm{D}_{\hat\beta}\Big[\mathcal{P}^{-2}(\hat\alpha)\hat\beta*\hat\beta\Big] = 2\mathcal{P}_2(\hat\omega).
\end{aligned}
$$

This completes the proof. □

The proof of Lemma 5.1 is similar to [18, 26, 27, 45]. It is a slight adaptation which follows from exploiting the fact that systems endowed with entropies are symmetrizable, see e.g. [28, Th. 3.1]. Then, the matrix $\nabla^2_{\hat y}\eta(\hat y)\mathrm{D}_{\hat y}\hat f(\hat y)$ is symmetric.

Proof of Lemma 5.1. Due to the compatibility condition (2.3) and due to the symmetry of the matrix $\nabla^2_{\hat y}\eta(\hat y)\mathrm{D}_{\hat y}\hat f(\hat y)$, we obtain

$$
\mathrm{D}_{\hat y}\eta(\hat y)\mathcal{R}(\hat y) = \mathrm{D}_{\hat y}\eta(\hat y)\hat y_t + \mathrm{D}_{\hat y}\mu(\hat y)\hat y_x = \eta(\hat y)_t + \mu(\hat y)_x, \tag{A.1}
$$

$$
\begin{aligned}
\hat f(\hat y)_x^{\mathrm T}\nabla^2_{\hat y}\eta(\hat y)(\hat y^* - \hat y) &= \hat y_x^{\mathrm T}\mathrm{D}_{\hat y}\hat f(\hat y)^{\mathrm T}\nabla^2_{\hat y}\eta(\hat y)(\hat y^* - \hat y)\\
&= \hat y_x^{\mathrm T}\nabla^2_{\hat y}\eta(\hat y)\mathrm{D}_{\hat y}\hat f(\hat y)(\hat y^* - \hat y).
\end{aligned} \tag{A.2}
$$

For every non-negative C^1-function φ with compact support Rademacher's theorem yields that the approximation $\hat y$ and hence the auxiliary function $\tilde\varphi := \nabla_{\hat y}\eta(\hat y)\varphi$ are differentiable almost everywhere. We obtain in the distributional sense

$$
\tilde\varphi_t = \nabla^2_{\hat y}\eta(\hat y)\hat y_t\varphi + \nabla_{\hat y}\eta(\hat y)\varphi_t = \nabla^2_{\hat y}\eta(\hat y)\big(\mathcal{R}(\hat y) - \hat f(\hat y)_x\big)\varphi + \nabla_{\hat y}\eta(\hat y)\varphi_t, \tag{A.3}
$$

$$
\tilde\varphi_x = \nabla^2_{\hat y}\eta(\hat y)\hat y_x\varphi + \nabla_{\hat y}\eta(\hat y)\varphi_x. \tag{A.4}
$$

With Eqs. (A.3) and (A.4) we conclude

$$
0 = \tilde\varphi^{\mathrm T}\mathcal{R}(\hat y) + \tilde\varphi^{\mathrm T}\big(\hat y^* - \hat y\big)_t + \tilde\varphi^{\mathrm T}\Big(\hat f(\hat y^*) - \hat f(\hat y)\Big)_x,
$$

$$
\begin{aligned}
0=&\int_0^T\int_{\mathbb{R}}\tilde{\varphi}_s^{\mathrm{T}}\big(\hat{y}^*-\hat{y}\big)+\tilde{\varphi}_x^{\mathrm{T}}\Big(\hat{f}(\hat{y}^*)-\hat{f}(\hat{y})\Big)-\tilde{\varphi}^{\mathrm{T}}\mathcal{R}(\hat{y})\,\mathrm{d}x\,\mathrm{d}s+\int_{\mathbb{R}}\tilde{\varphi}_0^{\mathrm{T}}\big(\hat{\mathcal{I}}^*-\hat{\mathcal{I}}\big)\,\mathrm{d}x\\
=&\int_0^T\int_{\mathbb{R}}\Big[\varphi\big(\mathcal{R}(\hat{y})-\hat{f}(\hat{y})_x\big)^{\mathrm{T}}\nabla_{\hat{y}}^2\eta(\hat{y})+\varphi_s\mathrm{D}_{\hat{y}}\eta(\hat{y})\Big]\big(\hat{y}^*-\hat{y}\big)\\
&+\Big[\varphi\hat{y}_x^{\mathrm{T}}\nabla_{\hat{y}}^2\eta(\hat{y})+\varphi_x\mathrm{D}_{\hat{y}}\eta(\hat{y})\Big]\Big(\hat{f}(\hat{y}^*)-\hat{f}(\hat{y})\Big)-\varphi\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y})\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_{\mathbb{R}}\varphi_0\mathrm{D}_{\hat{\mathcal{I}}}\eta(\hat{\mathcal{I}})\big(\hat{\mathcal{I}}^*-\hat{\mathcal{I}}\big)\,\mathrm{d}x.
\end{aligned}
$$

We rearrange these terms and use Eq. (A.2) to obtain

$$
\begin{aligned}
&\int_0^{\mathrm{T}}\int_{\mathbb{R}}\mathrm{D}_{\hat{y}}\eta(\hat{y})\big(\hat{y}^*-\hat{y}\big)\varphi_s+\mathrm{D}_{\hat{y}}\eta(\hat{y})\Big(\hat{f}(\hat{y}^*)-\hat{f}(\hat{y})\Big)\varphi_x\,\mathrm{d}x\,\mathrm{d}s\\
=&-\int_0^T\int_{\mathbb{R}}\Big(\mathcal{R}(\hat{y})-\hat{f}(\hat{y})_x\Big)^{\mathrm{T}}\nabla_{\hat{y}}^2\eta(\hat{y})\big(\hat{y}^*-\hat{y}\big)\varphi-\hat{y}_x^{\mathrm{T}}\nabla_{\hat{y}}^2\eta(\hat{y})\Big(\hat{f}(\hat{y}^*)-\hat{f}(\hat{y})\Big)\varphi\\
&+\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y})\varphi\,\mathrm{d}x\,\mathrm{d}s-\int_{\mathbb{R}}\mathrm{D}_{\hat{\mathcal{I}}}\eta(\hat{\mathcal{I}})\big(\hat{\mathcal{I}}^*-\hat{\mathcal{I}}\big)\varphi_0\,\mathrm{d}x\\
=&-\int_0^T\int_{\mathbb{R}}\mathcal{R}^{\mathrm{T}}(\hat{y})\nabla_{\hat{y}}^2\eta(\hat{y})(\hat{y}^*-\hat{y})\varphi\\
&-\hat{y}_x^{\mathrm{T}}\nabla_{\hat{y}}^2\eta(\hat{y})\Big[\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})(\hat{y}^*-\hat{y})-\big(\hat{f}(\hat{y}^*)-\hat{f}(\hat{y})\big)\Big]\varphi\,\mathrm{d}x\,\mathrm{d}s\\
&+\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y})\varphi\,\mathrm{d}x\,\mathrm{d}s-\int_{\mathbb{R}}\mathrm{D}_{\hat{\mathcal{I}}}\eta(\hat{\mathcal{I}})\big(\hat{\mathcal{I}}^*-\hat{\mathcal{I}}\big)\varphi_0\,\mathrm{d}x.
\end{aligned}
\tag{A.5}
$$

The entropy inequality $\eta(\hat{y}^*)_t+\mu(\hat{y}^*)_x\le 0$ and Eq. (A.1) imply in the distributional sense

$$
\big(\eta(\hat{y}^*)-\eta(\hat{y})\big)_t+\big(\mu(\hat{y}^*)-\mu(\hat{y})\big)_x+\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y})\le 0
$$

which reads with Eq. (A.5) as

$$
\begin{aligned}
0\le&\int_0^T\int_{\mathbb{R}}\big(\eta(\hat{y}^*)-\eta(\hat{y})\big)\varphi_s+\big(\mu(\hat{y}^*)-\mu(\hat{y})\big)\varphi_x-\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y})\varphi\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_{\mathbb{R}}\big(\eta(\hat{\mathcal{I}}^*)-\eta(\hat{\mathcal{I}})\big)\varphi_0\,\mathrm{d}x\\
=&\int_0^T\int_{\mathbb{R}}\eta(\hat{y}^*|\hat{y})\varphi_s+\mu(\hat{y}^*|\hat{y})\varphi_x-\mathrm{D}_{\hat{y}}\eta(\hat{y})\mathcal{R}(\hat{y})\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^{\mathrm{T}}\int_{\mathbb{R}}\mathrm{D}_{\hat{y}}\eta(\hat{y})\big(\hat{y}^*-\hat{y}\big)\varphi_s+\mathrm{D}_{\hat{y}}\eta(\hat{y})\Big(\hat{f}(\hat{y}^*)-\hat{f}(\hat{y})\Big)\varphi_x\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_{\mathbb{R}}\big(\eta(\hat{\mathcal{I}}^*)-\eta(\hat{\mathcal{I}})\big)\varphi_0\,\mathrm{d}x\\
=&\int_0^T\int_{\mathbb{R}}\eta(\hat{y}^*|\hat{y})\varphi_s+\mu(\hat{y}^*|\hat{y})\varphi_x\,\mathrm{d}x\,\mathrm{d}s+\int_0^T\int_{\mathbb{R}}-\mathcal{R}^{\mathrm{T}}(\hat{y})\nabla_{\hat{y}}^2\eta(\hat{y})(\hat{y}^*-\hat{y})\varphi\\
&+\hat{y}_x^{\mathrm{T}}\nabla_{\hat{y}}^2\eta(\hat{y})\Big[\mathrm{D}_{\hat{y}}\hat{f}(\hat{y})(\hat{y}^*-\hat{y})-\big(\hat{f}(\hat{y}^*)-\hat{f}(\hat{y})\big)\Big]\varphi\,\mathrm{d}x\,\mathrm{d}s+\int_{\mathbb{R}}\eta(\hat{\mathcal{I}}^*|\hat{\mathcal{I}})\varphi_0\,\mathrm{d}x.
\end{aligned}
$$

For all Lebesgue points $t\in[0,T)$ and

$$\varphi_\varepsilon(s,x;t):=\begin{cases}1 & \text{if } s<t,\\ 1-\frac{s-t}{\varepsilon} & \text{if } t<s<t+\varepsilon,\\ 0 & \text{if } t+\varepsilon<s,\end{cases}$$

we obtain

$$\begin{aligned}
0\le&\int_{\mathbb{R}}-\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\eta\Big(\hat{y}^*(s,x)\Big|\hat{y}(s,x)\Big)\mathrm{d}x+\int_{\mathbb{R}}\eta\Big(\hat{\mathcal{I}}^*(x)\Big|\hat{\mathcal{I}}(x)\Big)\mathrm{d}x\\
&-\int_0^T\int_{\mathbb{R}}\mathcal{R}^{\mathrm{T}}\Big(\hat{y}(s,x)\Big)\nabla^2_{\hat{y}}\eta\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x)-\hat{y}(s,x)\Big)\varphi_\varepsilon(s,x;t)\\
&\qquad+\hat{y}_x^{\mathrm{T}}(s,x)\nabla^2_{\hat{y}}\eta\Big(\hat{y}(s,x)\Big)\Big[\mathrm{D}_{\hat{y}}\hat{f}\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x)-\hat{y}(s,x)\Big)\\
&\qquad-\Big(\hat{f}\big(\hat{y}^*(s,x)\big)-\hat{f}\big(\hat{y}(s,x)\big)\Big)\Big]\varphi_\varepsilon(s,x;t)\,\mathrm{d}x\mathrm{d}s\\
\xrightarrow{\varepsilon\to 0}&-\int_{\mathbb{R}}\eta\Big(\hat{y}^*(t,x)\Big|\hat{y}(t,x)\Big)\mathrm{d}x+\int_{\mathbb{R}}\eta\Big(\hat{\mathcal{I}}^*(x)\Big|\hat{\mathcal{I}}(x)\Big)\mathrm{d}x\\
&-\int_0^t\int_{\mathbb{R}}\mathcal{R}^{\mathrm{T}}\Big(\hat{y}(s,x)\Big)\nabla^2_{\hat{y}}\eta\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x)-\hat{y}(s,x)\Big)\\
&\qquad+\hat{y}_x^{T}(s,x)\nabla^2_{\hat{y}}\eta\Big(\hat{y}(s,x)\Big)\Big[\mathrm{D}_{\hat{y}}\hat{f}\Big(\hat{y}(s,x)\Big)\Big(\hat{y}^*(s,x)-\hat{y}(s,x)\Big)\\
&\qquad-\Big(\hat{f}\big(\hat{y}^*(s,x)\big)-\hat{f}\big(\hat{y}(s,x)\big)\Big)\Big]\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

This completes the proof. □

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