Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2022-0050

Vol. 32, No. 2, pp. 524-546
August 2022

A Holomorphic Operator Function Approach for the Transmission Eigenvalue Problem of Elastic Waves

Yingxia Xi[1] and Xia Ji[2,3,*]

[1] *School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, China.*
[2] *School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China.*
[3] *Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing 100081, China.*

Received 18 February 2022; Accepted (in revised version) 24 May 2022

Abstract. The paper presents a holomorphic operator function approach for the transmission eigenvalue problem of elastic waves using the discontinuous Galerkin method. To use the abstract approximation theory for holomorphic operator functions, we rewrite the elastic transmission eigenvalue problem as the eigenvalue problem of a holomorphic Fredholm operator function of index zero. The convergence for the discontinuous Galerkin method is proved following the abstract theory of the holomorphic Fredholm operator. The spectral indicator method is employed to compute the transmission eigenvalues. Extensive numerical examples are presented to validate the theory.

AMS subject classifications: 65N25, 65N30, 47B07

Key words: Discontinuous Galerkin method, transmission eigenvalue problem, elastic waves, Fredholm operator.

1 Introduction

The transmission eigenvalue problem is very important in inverse scattering theory. We can use it to obtain useful information on the physical properties of the scattering targets [8, 28]. The transmission eigenvalue problem can be not only applied to estimate the properties of the scattering objects [28], but also to the uniqueness and reconstruction in the inverse scattering theory.

*Corresponding author. *Email addresses:* `xiyingxia@njust.edu.cn` (Y. Xi), `jixia@bit.edu.cn` (X. Ji)

 ©2022 Global-Science Press

In this paper, we consider the numerical method for the transmission eigenvalue problem for the elastic wave. Similar to the cases of acoustic and electromagnetic transmission eigenvalue problems, the elasticity transmission eigenvalue (ETE) problem is crucial in the qualitative reconstruction methods for inhomogeneous media. The theoretical studies on the ETE problem [3,4,10,11] are limited. There exists a countable set of ETEs under suitable conditions on elastic tensors and mass densities [3].

The transmission eigenvalue problems are nonlinear and nonself-adjoint [30]. In general, it is highly non-trivial to develop finite element methods for these problems. The finite element discretization usually leads to non-Hermitian matrix eigenvalue problems. It is challenging to compute generalized eigenvalues for non-Hermitian matrices, especially when the size of matrices is large and there is no prior information of the spectrum. There are several numerical methods for the acoustic transmission eigenvalues and electromagnetic transmission eigenvalue problems [1,6,9,12,15,19,20,23–25,27,29,32,33].

To our knowledge, there are a few works on the numerical treatment of the ETE problem. In [18], the authors construct a nonlinear function whose values are generalized eigenvalues of a series of self-adjoint fourth order problems. The roots of the function are the transmission eigenvalues. In [31], the authors employ the C^0IP method for the transmission eigenvalue problems based on a mixed formulation, which can be used to estimate material property of the elastic body (see, e.g., [28]). The C^0IP method is a kind of discontinuous Galerkin (DG) method. The DG methods can be easily implemented on highly unstructured meshes because of relaxing the continuity of approximation functions across the finite element boundaries. The locality and flexibility also make the methods well suited for parallelization and applications of domain decomposition techniques. DG methods have not fit into the abstract convergence theory of Babuška and Osborn [2] due to the lack of the norm convergence of the solution operator. The convergence analysis follows the idea in [26] for the compact operators.

In this paper, we take a different method to prove the convergence of the ETE problem using DG method. We reformulate the ETE as an eigenvalue problem of a holomorphic operator function. Then DG method and the spectral projection are applied to compute the ETEs inside a region on the complex plane. The spectral projection method is based on eigenprojectors of compact operators. It can separate nearby eigenvalues, and does not require priori spectral information [16,17]. Using the fundamental properties of DG [31] for the ETE problem and the approximation results for the eigenvalues of holomorphic Fredholm operator functions [5,21,22], we prove the convergence of DG approximation.

The rest of this paper is organized as follows. In section 2, we introduce some preliminaries for the eigenvalue approximation theory of holomorphic Fredholm operator functions. Section 3 gives a brief introduction of the ETE problem. Section 4 presents the discontinuous Galerkin method for the ETE problem. Section 5 gives the error estimate of the ETE problem. The spectral indicator method is introduced in section 6. Numerical examples are presented in section 7. The last section is the concluding remark.