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Low-Dissipation Central-Upwind Schemes for Elasticity in Heterogeneous Media

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Abstract. We develop new low-dissipation central-upwind (LDCU) schemes for nonlinear elasticity equations in heterogeneous media. In general, central-upwind schemes belong to the class of finite-volume Godunov-type schemes, which consist of three steps: reconstruction, evolution, and projection onto the original grid. In our new method, the evolution is performed in the standard way by integrating the system over the space-time control volumes. However, the reconstruction and projection are performed in a special manner. First, we take into account the fact that the conservative variables (strain and momentum) are discontinuous across the material interfaces, while the flux variables (velocity and strain) are continuous there: we therefore reconstruct the flux variables. Second, we use a special projection recently introduced in [A. Kurganov and R. Xin, J. Sci. Comput., 96, 2023] to complete the derivation of the LDCU scheme. Our numerical experiments demonstrate that the developed schemes are capable of accurately resolving waves with dispersive behavior that over a long period of time evolve into solitary waves.

AMS subject classifications: 65M08, 76M12, 35L65, 74B20

Key words: Hyperbolic systems of conservation laws, low-dissipation central-upwind schemes, nonlinear elasticity, heterogeneous media, waves with dispersive behavior, solitary waves.

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1 Introduction

Consider the one-dimensional (1-D) elasticity system

$$\begin{aligned}&\varepsilon_t-u_x=0,\\&\big(\rho(x)u\big)_t-\sigma_x\big(K(x);\varepsilon\big)=0,\end{aligned}\tag{1.1}$$

where $\varepsilon(x,t)$ is the strain, $u(x,t)$ is the velocity, $\rho(x)$ is the density, $K(x)$ is the bulk modulus of compressibility, and $\sigma(K(x);\varepsilon)$ is the stress. If $\rho(x)$ and $K(x)$ are both constants, then the medium is homogeneous. A nonconstant $\rho(x)$ and $K(x)$ correspond to a heterogeneous medium. We consider a layered medium consisting of two different materials of length ℓ with densities ρ_1 and ρ_2 and bulk moduli of compressibility K_1 and K_2 so that for all integer j,

$$\begin{aligned}&\rho(x)=\begin{cases}\rho_1, & \text{if} \quad 2j\ell<x<(2j+1)\ell,\\ \rho_2, & \text{otherwise,}\end{cases}\\&K(x)=\begin{cases}K_1, & \text{if} \quad 2j\ell<x<(2j+1)\ell,\\ K_2, & \text{otherwise.}\end{cases}\end{aligned}\tag{1.2}$$

The stress-strain relation in the linear case has the form $\sigma_i(\varepsilon)=K_i\varepsilon, i=1,2$. A more realistic model is obtained when a nonlinear stress-strain relation is considered. We take either

$$\sigma_i(\varepsilon)=K_i\varepsilon+\beta K_i^2\varepsilon^2, \quad i\in\{1,2\}, \quad \beta=\text{Const}\tag{1.3}$$

or

$$\sigma_i(\varepsilon)=e^{K_i\varepsilon}-1, \quad i\in\{1,2\}\tag{1.4}$$

as examples of such relations; see [6,12,18–20]. In a homogeneous medium with a nonlinear stress-strain relation, a generic solution of (1.1) and (1.3) will typically develop shock and rarefaction waves. But in a heterogeneous medium with a nonlinear stress-strain relation, the resulting waves will have dispersive behaviors that will lead to solitary waves instead of shock waves [6, 12, 17–20].

The elasticity system (1.1) can be put into the framework of conservation laws with space-dependent flux. To this end, we rewrite (1.1) as

$$\boldsymbol{U}_t+\boldsymbol{F}(\mathbf{C}(x);\boldsymbol{U})_x=\mathbf{0},\tag{1.5}$$

where

$$\boldsymbol{U}=(\varepsilon,m)^\top, \quad \boldsymbol{F}\big(\mathbf{C}(x);\boldsymbol{U}\big)=\Big(-\frac{m}{\rho(x)},-\sigma\big(K(x);\varepsilon\big)\Big)^\top,\tag{1.6}$$

$m=\rho u$ denotes the momentum, and $\mathbf{C}(x):=(\rho(x),K(x))^\top$.

Since (1.5) is a hyperbolic system of conservation laws, it is very natural to numerically solve it by a finite-volume Godunov-type scheme. Such schemes form a class of