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Continuous Subsonic-Sonic Flows in Curved Convergent Nozzles with a Fixed Point

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Abstract. This paper concerns continuous subsonic-sonic potential flows in a two-dimensional, convergent nozzle, which is governed by a free boundary problem of a quasilinear degenerate elliptic equation. It is shown that for a given nozzle perturbed from a straight one, a given point on its wall where the curvature is zero, a given inlet which is a perturbation of an arc centered at the vertex, and a given incoming flow angle perturbed from the angle of the inner normal of the inlet, there exists uniquely a continuous subsonic-sonic flow whose velocity vector is along the normal direction at the sonic curve, which satisfies the slip conditions on the nozzle walls and whose sonic curve intersects the upper wall at the given point. Furthermore, the sonic curve of this flow is a free boundary, where the flow is singular in the sense that the speed is only $C^{1/2}$ Hölder continuous and the acceleration blows up. The perturbation problem is solved in the potential plane, where the flow is governed by a free boundary problem of a degenerate elliptic equation with three free boundaries and two nonlocal boundary conditions, and the equation is degenerate at one free boundary.

AMS subject classifications: 35R35, 76N10, 35J70

Key words: Degeneracy, free boundary, continuous subsonic-sonic flow, nonlocal boundary condition.

1 Introduction

This paper concerns subsonic-sonic flows in curved convergent nozzles. Such problems naturally arise in physical experiments and engineering designs, and

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there are many experiments and numerical simulations and rigorous theories involved in this field [2,6]. A two-dimensional steady isentropic inviscid compressible flow is governed by the following Euler system:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0, \tag{1.1}$$

$$\frac{\partial}{\partial x}(P+\rho u^2)+\frac{\partial}{\partial y}(\rho uv)=0, \tag{1.2}$$

$$\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(P+\rho v^2)=0, \tag{1.3}$$

where $(u,v),P$ and ρ represent the velocity, pressure and density of the flow, respectively, and $P=P(\rho)$ is a smooth function. In particular, for a polytropic gas with adiabatic exponent $\gamma>1$,

$$P(\rho)=\frac{1}{\gamma}\rho^{\gamma} \tag{1.4}$$

is the normalized pressure. Assume further that the flow is irrotational, i.e.

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}. \tag{1.5}$$

Then, the density ρ is expressed in terms of the speed q according to the Bernoulli law [2]

$$\rho(q^2)=\left(1-\frac{\gamma-1}{2}q^2\right)^{1/(\gamma-1)}, \quad q=\sqrt{u^2+v^2}, \quad 0<q<\sqrt{2/(\gamma-1)}. \tag{1.6}$$

The sound speed c is defined as $c^2=P'(\rho)$. At the sonic state, the sound speed is

$$c_*=\left(\frac{2}{\gamma+1}\right)^{1/2},$$

which is called the critical speed in the sense that the flow is subsonic when $q<c_*$, sonic when $q=c_*$, and supersonic when $q>c_*$. The system (1.1)-(1.6) can be transformed into the full potential equation

$$\operatorname{div}\Big(\rho\big(|\nabla\varphi|^2\big)\nabla\varphi\Big)=0, \tag{1.7}$$

where φ is the velocity potential with $\nabla\varphi=(u,v)$, and ρ is defined by (1.6). It is noted that (1.7) is elliptic in the subsonic region, degenerate at the sonic state, while hyperbolic in the supersonic region.

Subsonic and subsonic-sonic flows past a profile or in a nozzle have been studied for a long time. Frankl and Keldysh [12] first established the existence and uniqueness of the subsonic flows past a two-dimensional profile with sufficiently small Mach number at infinity. In the significant work, Bers [1] showed that there exists a unique subsonic flow past a two-dimensional profile if the free stream Mach number is less than a critical value; furthermore, the maximum flow speed tends to the sound speed as the freestream Mach number tends to the critical value. The same results for multi-dimensional cases were obtained in [7, 11]. In these works, the flow with the critical freestream Mach number was not considered. By the compensated compactness method, it was shown in [4] that the two-dimensional flows with sonic points past a profile may be realized as weak limits of sequences of strictly subsonic flows. However, their regularity and uniqueness remain unknown. A similar situation occurs for subsonic and subsonic-sonic flows in an infinitely long nozzle. It was proved in [29] that there exists a critical value such that a strictly subsonic flow exists uniquely as long as the incoming mass flux is less than the critical value, and subsonic-sonic flows can be obtained as the weak limits of strictly subsonic flows associated with the incoming mass fluxes increasing to the critical value. The multi-dimensional cases were dealt with in [5, 10]. There are also many studies on rotational subsonic flows, we refer readers to [3, 8, 9, 30, 31] and references therein.

For a straight convergent nozzle, there are symmetric continuous subsonic-sonic flows, whose potential depends only on the distance of the position to the vertex of the nozzle. A natural question is whether this radially symmetric subsonic-sonic flow is stable. The structural stability of such flows was first studied by Wang and Xin [23], where they considered the perturbation of the inlet. Precisely, for a given inlet, which is a perturbation of an arc centered at the vertex of the nozzle, and a given incoming mass flux, it was shown in [23] that there exists an open interval depending only on the adiabatic exponent and the length of the arc, such that a unique subsonic-sonic flow, which is a perturbation of the radially symmetric subsonic-sonic flow from the arc as the inlet with the same incoming mass flux, exists as long as the incoming mass flux belongs to this interval and the perturbation of the inlet is sufficiently small; furthermore, the sonic curve of this subsonic-sonic flow is a free boundary, where the flow is singular in the sense that the speed is only $C^{1/2}$ Hölder continuous and the acceleration blows up at the sonic state. There are also some studies on perturbational problems [13, 16–18, 20, 22, 28]. Recently, Wang and Xin [27] studied the Lipschitz continuous subsonic-sonic flow in general two-dimensional finitely long symmetric nozzles. Moreover, Wang and Xin [21, 24–26] studied smooth transonic flows of Meyer type in de Laval nozzles.

In the present paper, we investigate the structural stability of symmetric continuous subsonic-sonic flows when the inlet, the incoming flow angle and the nozzle wall are perturbed. It is shown from [23] that the existence of continuous subsonic-sonic flows depends on the geometry of the nozzle. The authors prescribed the incoming flow angle and the incoming mass flux, which are two physical quantities, to formulate the continuous subsonic-sonic flow problem in a straight nozzle; furthermore, the sonic curve of the continuous subsonic-sonic flow is a free boundary where the velocity vector is along the normal direction. It is obvious that, for a continuous subsonic-sonic flow whose velocity vector is along the normal direction at the sonic curve, the second derivative of the nozzle wall at the intersecting point with the sonic curve must be zero. Therefore, it is not suitable to prescribe the incoming flow angle and the incoming mass flux to formulate the continuous subsonic-sonic flow problem in a general nozzle. In this paper, we fix the intersecting point between the sonic curve and the upper wall to replace the incoming mass flux condition. As in [13, 16–18, 20, 22, 23, 27, 28], it is also assumed that the sonic curve of the continuous subsonic-sonic flow is a free boundary where the velocity vector is along the normal direction. It is clear that the curvature of the nozzle wall at the intersecting point with the sonic curve should be zero. Therefore, the considered problem is as follows: For a given nozzle which is a perturbation of an straight one, a given point on its wall where the curvature is zero, a given inlet which is a perturbation of an arc centered at the vertex, and a given incoming flow angle which is a perturbation of the angle of the inner normal of the inlet, we seek a continuous subsonic-sonic flow whose velocity vector is along the normal direction at the sonic curve, which satisfies the slip conditions on the nozzle walls and whose sonic curve intersects the upper wall at the fixed point. As shown in Fig. 1, it is assumed that that $P_0(x_0,-x_0\tan\vartheta_0)$ $(x_0<0, 0<\vartheta_0<\pi/2)$ is the fixed intersecting point at the upper wall with the sonic curve, and the upper and lower walls of the nozzle are given by

$$\Gamma_{\rm upw}: y=f(x)>0, \quad x_1\le x\le x_0 \quad \text{and} \quad \Gamma_{\rm lw}: y=x\tan\vartheta_0, \quad x_1\le x\le 0,$$

respectively, where $x_1<x_0$,

$$f(x_0)=-x_0\tan\vartheta_0, \quad f'(x_0)=-\tan\vartheta_0, \tag{1.8}$$

$$|f''(x)|\le\delta^2(x_0-x)^{1/2}, \quad x\in[x_1,x_0], \tag{1.9}$$

where $\delta>0$ be determined. The inlet is given by

$$\Gamma_{\rm in}: x=g(y), \quad x_1\tan\vartheta_0\le y\le f(x_1),$$

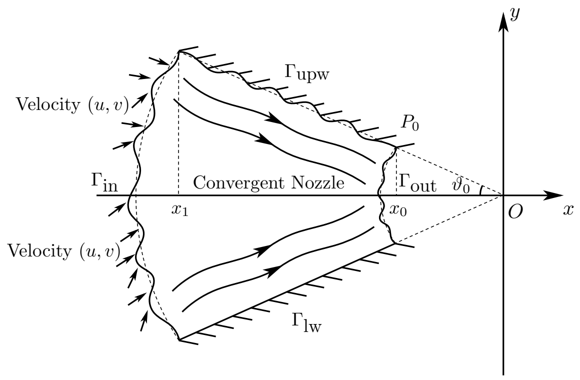

Figure 1: Continuous subsonic-sonic flows in a convergent nozzle with a fixed point.

which is orthogonal to the walls at $(x_1,f(x_1))$ and $(x_1,x_1\tan\vartheta_0)$ and is a small perturbation of the arc centered at the origin, i.e.

$$
\begin{aligned}
&g\big(f(x_1)\big)=g(x_1\tan\vartheta_0)=x_1,\\
&g'\big(f(x_1)\big)=-f'(x_1), \qquad\qquad (1.10)\\
&g'(x_1\tan\vartheta_0)=-\tan\vartheta_0,\\
&\|g-g_0\|_{1,1;(x_1\tan\vartheta_0,f(x_1))}\le\delta, \qquad\qquad (1.11)
\end{aligned}
$$

where

$$g_0(y)=-\sqrt{(x_1/\cos\vartheta_0)^2-y^2},\quad x_1\tan\vartheta_0\le y\le f(x_1).$$

Then, give an incoming flow angle $\Theta_{\rm in}\in C^1([x_1\tan\vartheta_0,f(x_1)])$ satisfying

$$\Theta_{\rm in}(x_1\tan\vartheta_0)=\vartheta_0,\quad \Theta_{\rm in}\big(f(x_1)\big)=\arctan f(x_1), \tag{1.12}$$

$$\|\Theta_{\rm in}+\arctan g'\|_{0,1;(x_1\tan\vartheta_0,f(x_1))}\le\delta. \tag{1.13}$$

The outlet is chosen to be the sonic curve of the flow, which is a free boundary from the fixed point P_0 at the upper wall to a point at the lower wall and is denoted by

$$\Gamma_{\rm out}:x=S(y),\quad x_2\tan\vartheta_0\le y\le -x_0\tan\vartheta_0,$$

where

$$S(-x_0\tan\vartheta_0)=x_0,\quad S(x_2\tan\vartheta_0)=x_2,\quad x_2\in(x_1,0).$$

Then, as in [13, 16–18, 20, 23, 28], the continuous subsonic-sonic problem is formulated as

$$\begin{aligned}
&\operatorname{div}\big(\rho(|\nabla\varphi|^2)\nabla\varphi\big)=0, && (x,y)\in\Omega, && (1.14)\\
&\frac{\partial\varphi}{\partial y}(x,x\tan\vartheta_0)-\tan\vartheta_0\frac{\partial\varphi}{\partial x}(x,x\tan\vartheta_0)=0, && x_1<x<x_2, && (1.15)\\
&\frac{\partial\varphi}{\partial y}\big(x,f(x)\big)-f'(x)\frac{\partial\varphi}{\partial x}\big(x,f(x)\big)=0, && x_1<x<x_0, && (1.16)\\
&\varphi\big(g(y),y\big)=\int_{f(x_1)}^{y}|\nabla\varphi(g(\tau),\tau)|\big(\sin\Theta_{\rm in}(\tau)+\cos\Theta_{\rm in}(\tau)g'(\tau)\big)d\tau, && && \\
& && x_1\tan\vartheta_0<y<f(x_1), && (1.17)\\
&|\nabla\varphi(S(y),y)|=c_*,\quad \varphi(S(y),y)=C_{\rm out}, && x_2\tan\vartheta_0<y<-x_0\tan\vartheta_0, && (1.18)\\
&|\nabla\varphi(x,y)|<c_*, && (x,y)\in\Omega, && (1.19)
\end{aligned}$$

where Ω is the domain bounded by $\Gamma_{\rm upw}$, $\Gamma_{\rm lw}$, $\Gamma_{\rm in}$ and $\Gamma_{\rm out}$, $(\varphi,S,C_{\rm out})$ is unknown. Generally speaking, there are three types of perturbations for the symmetric flow: the perturbation of the inlet, the perturbation of the incoming flow angle and the perturbation of the walls. Wang and Xin [23] studied the case in which only the inlet is perturbed. Nie and Wang [16] studied the case in which both the inlet and the incoming flow angle are perturbed. In [16, 23], the nozzle is straight and the incoming mass flux is prescribed. When the wall is perturbed, Gai *et al.* [13] considered the case in which the upper wall is perturbed away from the sonic state, and the sonic curve intersects the upper wall at a free point, but the inlet is an arc centered at the vertex of the nozzle. Later, based on this work, Nie *et al.* [18] considered the case in which the inlet is also perturbed. In [13,16,18,23], the incoming mass flux is prescribed. It can also consider the assumption that the sonic curve intersects the upper wall at a fixed point instead of a free point. Structural stability was proved for the case that both the inlet and the wall are perturbed in [17], whose sonic curve intersects the upper wall at the fixed point and the mass flux is free. Moreover, some subsonic-sonic flow problems in a convergent nozzle whose cross section changes slightly were considered in [20, 28], where the sonic curve is assumed to intersect the upper wall at a fixed point. This paper considers the case in which the inlet, the incoming flow angle and the wall are perturbed, and the sonic curve intersects the upper wall at a fixed point. As in [17], it is shown in this paper that if x_1 satisfies some conditions, the problem (1.14)-(1.19) admits a unique solution as long as the perturbation is sufficiently small. Moreover, the flow is singular in the sense that the speed is only $C^{1/2}$ Hölder continuous and the acceleration blows up.

The problem (1.14)-(1.19) is a free boundary problem of a degenerate elliptic equation with three free boundaries. Furthermore, the degeneracy occurs at one free boundary and the degeneracy is characteristic [19, 32]. We solve such a free boundary problem in the potential plane as in [13, 16–18, 20, 23, 28]. The reason lies in that for the sonic curve of such flows, both its form and its location are unknown in the physical plane, while its form is known although its location is unknown in the potential plane; furthermore, it is more convenient to estimate the speed of the flow in the potential plane than in the physical plane since it is a solution in the potential plane, while it is the absolute value of the gradient of a solution in the physical plane. In the potential plane, the problem (1.14)-(1.19) is transformed into a free boundary problem of a degenerate elliptic equation with three free boundaries and two nonlocal and unfamiliar boundary conditions. Moreover, the equation is degenerate at one free boundary. There are three perturbations in this paper: the perturbation of the inlet, the perturbation of the incoming flow angle and the perturbation of the nozzle wall, and the sonic curve is assumed to intersect the upper wall at a given point where the curvature is zero. For the problem (1.14)-(1.19) in the potential plane, the difference of the velocity potential between the sonic curve and the inlet is free, and the mass flux is free, and the boundary condition at the inlet and the upper wall are nonlocal and unfamiliar, and the inlet is also free and both its form and its location are unknown. Furthermore, the optimal estimate of the flow on the upper wall is needed. We have to overcome some new technique difficulty in this paper. We use the Schauder fixed-point theorem to prove the existence of subsonic-sonic flows: for a given speed at the inlet and the upper wall in a suitable space, we first formulate and solve a related free boundary problem with known boundary conditions and then show that the fixed-point theorem can be applied to find a solution to the problem (1.14)-(1.19). It is noted that the speed at the inlet is a perturbation of a constant, while the speed at the upper wall is a perturbation of a function. As mentioned in [23], we have to prescribe a Neumann boundary condition at the inlet instead of a Robin one since the problem with the latter boundary condition may be ill-posed. The key is precise estimates of solutions to boundary problems of a degenerate elliptic equation. Different from works in [13, 16–18, 20, 23, 28], we should overcome some difficulties of the new technique of the three perturbations and obtain more elaborate estimates than the ones in [13, 16–18, 20, 23, 28] to get the desired solution. For the uniqueness of the subsonic-sonic flow, as mentioned in [23], not only the degeneracy and free boundaries but also nonlocal terms bring essential difficulties. One can fix the free boundaries into fixed boundaries and transform the nonlocal boundary conditions into common boundary conditions by a suitable coordinate transformation. Then, the uniqueness of the continuous subsonic-sonic flow is proved by the method of energy estimate and a series of complicated calculations.

The paper is arranged as follows. We formulate the problem in the potential plane and state the main results (existence and uniqueness) of the paper in Section 2. Then we prove the existence and uniqueness results in the last two sections, respectively.

2 Formulation in the potential plane and main results

As shown in [23], if

$$f(x)=f_0(x)=-x\tan\vartheta_0,\quad x_1\le x\le x_0$$

and

$$g(y)=g_0(y),\quad \Theta_{\rm in}(y)=-\arctan g_0'(y),\quad x_1\tan\vartheta_0\le y\le f(x_1),$$

respectively, then there is uniquely a symmetric continuous subsonic-sonic flow in the nozzle such that it is sonic at P_0. Precisely, the speed of the symmetric continuous subsonic-sonic flow is given by

$$|\nabla\varphi(x,y)|=\hat{Q}\left(-\sqrt{x^2+y^2}\right),\quad x_0^2\le(x^2+y^2)\cos^2\vartheta_0\le x_1^2,\quad x\tan\vartheta_0\le y\le -x\tan\vartheta_0,$$

where

$$\hat{Q}\in C^\infty([x_1/\cos\vartheta_0,x_0/\cos\vartheta_0))\cap C^{1/2}([x_1/\cos\vartheta_0,x_0/\cos\vartheta_0])$$

solves

$$\begin{cases}\left(r\rho(\hat{Q}^2(r))\hat{Q}(r)\right)'=0,\quad x_1/\cos\vartheta_0<r<x_0/\cos\vartheta_0,\\ \hat{Q}(x_0/\cos\vartheta_0)=c_*.\end{cases}$$

In particular, on the upper wall, it is

$$\hat{Q}_{\rm up}(x)=|\nabla\varphi(x,-x\tan\vartheta_0)|=\hat{Q}(x/\cos\vartheta_0),\quad x_1\le x\le x_0. \tag{2.1}$$

For this symmetric continuous subsonic-sonic flow, the mass flux is

$$m_0=-\frac{2\vartheta_0x_0c_*\rho(c_*^2)}{\cos\vartheta_0}. \tag{2.2}$$

In this paper, we solve the problem (1.14)-(1.19) under the assumption that $f\in C^{2,\alpha}([x_1,x_0])$ satisfying (1.8) and (1.9), $g\in C^{2,\alpha}([x_1\tan\vartheta_0,f(x_1)])$ satisfying (1.10) and (1.11), and $\Theta_{\rm in}\in C^{1,\alpha}([x_1\tan\vartheta_0,f(x_1)])$ satisfying (1.12) and (1.13), where $\alpha\in(0,1)$ and $\delta>0$.

2.1 Formulation in the potential plane

In the potential-stream coordinates (φ,ψ), as in [23], the system (1.1)-(1.5) can be reduced to

$$\frac{\partial^2 A(q)}{\partial\varphi^2}+\frac{\partial^2 B(q)}{\partial\psi^2}=0,$$

where

$$A(q)=\int_{c_*}^{q}\frac{\rho(s^2)+2s^2\rho'(s^2)}{s\rho^2(s^2)}ds,\quad B(q)=\int_{c_*}^{q}\frac{\rho(s^2)}{s}ds,\qquad 0<q<\sqrt{2/(\gamma-1)},$$

furthermore, $A(\cdot)$ and $B(\cdot)$ are both the strictly increasing functions in $(0,c_*]$. Assume the speed of the flow at the inlet and at the upper wall to be denoted by

$$\begin{aligned}&q\big(g(y),y\big)=Q_{\rm in}(y),\quad x_1\tan\vartheta_0\le y\le f(x_1),\\&q\big(x,f(x)\big)=Q_{\rm up}(x),\quad x_1\le x\le x_0,\end{aligned}$$

respectively. Let the angle of the velocity at the upper wall be denoted by

$$\Theta_{\rm up}(x)=\arctan f'(x),\quad x_1\le x\le x_0.$$

And the incoming mass flux is given by

$$m_{\rm in}=\int_{x_1\tan\vartheta_0}^{f(x_1)}Q_{\rm in}(y)\rho\big(Q_{\rm in}^2(y)\big)\big(\cos\Theta_{\rm in}(y)-\sin\Theta_{\rm in}(y)g'(y)\big)dy>0. \tag{2.3}$$

At the inlet, the stream function is expressed as

$$\psi\big(g(y),y\big)=\Psi_{\rm in}(y),\quad x_1\tan\vartheta_0\le y\le f(x_1)$$

with

$$\begin{aligned}\Psi_{\rm in}(y)=&\int_{x_1\tan\vartheta_0}^{y}Q_{\rm in}(s)\rho(Q_{\rm in}^2(s))\big(\cos\Theta_{\rm in}(s)-\sin\Theta_{\rm in}(s)g'(s)\big)ds,\\&x_1\tan\vartheta_0\le y\le f(x_1).\end{aligned} \tag{2.4}$$

Based on the above assumptions, the potential at $(x_1,f(x_1))$ is zero, the potential at the inlet is given by

$$\varphi(g(y),y)=\Phi_{\rm in}(y),\quad x_1\tan\vartheta_0\le y\le f(x_1)$$

with

$$\Phi_{\rm in}(y)=\int_{f(x_1)}^{y}Q_{\rm in}(s)\big(\sin\Theta_{\rm in}(s)+\cos\Theta_{\rm in}(s)g'(s)\big)ds,\quad x_1\tan\vartheta_0\le y\le f(x_1). \tag{2.5}$$

It follows from (1.12) and (1.10) that

$$\Phi_{\rm in}'(x_1\tan\vartheta_0)=0,\quad \Phi_{\rm in}'\big(f(x_1)\big)=0. \tag{2.6}$$

The potential at the sonic curve is given by

$$\varphi_1=\int_{x_1}^{x_0}\frac{Q_{\rm up}(x)}{\cos\Theta_{\rm up}(x)}dx>0. \tag{2.7}$$

At the upper wall, the potential function expressed as

$$\varphi\big(x,f(x)\big)=\Phi_{\rm up}(x)=\int_{x_1}^{x}\frac{Q_{\rm up}(s)}{\cos\Theta_{\rm up}(s)}ds,\quad x_1\le x\le x_0. \tag{2.8}$$

The inverse functions of $\Psi_{\rm in}$ and $\Phi_{\rm up}$ are denoted by

$$\begin{aligned} &Y_{\rm in}(\psi)=\Psi_{\rm in}^{-1}(\psi), && 0\le\psi\le m_{\rm in}, && (2.9)\\ &X_{\rm up}(\varphi)=\Phi_{\rm up}^{-1}(\varphi), && 0=\mathscr{G}_{\rm in}(m_{\rm in})\le\varphi\le\varphi_1, && (2.10)\end{aligned}$$

respectively. Set

$$\mathscr{G}_{\rm in}(\psi)=\Phi_{\rm in}\big(Y_{\rm in}(\psi)\big),\quad 0\le\psi\le m_{\rm in}. \tag{2.11}$$

Owing to (2.6),

$$\mathscr{G}_{\rm in}(m_{\rm in})=0,\quad \mathscr{G}_{\rm in}'(0)=\mathscr{G}_{\rm in}'(m_{\rm in})=0. \tag{2.12}$$

Then, similar to [13, 16–18, 20, 23, 28], the subsonic-sonic flow problem in the potential plane can be formulated as follows:

$$\begin{aligned} &\frac{\partial^2 A(q)}{\partial\varphi^2}+\frac{\partial^2 B(q)}{\partial\psi^2}=0, && (\varphi,\psi)\in\mathcal{D}, && (2.13)\\ &\frac{\partial q}{\partial\psi}(\varphi,0)=0, && \varphi\in\big(\mathscr{G}_{\rm in}(0),\varphi_1\big), && (2.14)\\ &\frac{\partial B(q)}{\partial\psi}(\varphi,m_{\rm in})=\frac{\Theta_{\rm up}'(x)\cos\Theta_{\rm up}(x)}{Q_{\rm up}(x)}\bigg|_{x=X_{\rm up}(\varphi)}, && \varphi\in\big(\mathscr{G}_{\rm in}(m_{\rm in}),\varphi_1\big), && (2.15)\end{aligned}$$

$$\begin{aligned} &\frac{\partial A(q)}{\partial\varphi}\big(\mathscr{G}_{\rm in}(\psi),\psi\big)-\mathscr{G}_{\rm in}'(\psi)\frac{\partial B(q)}{\partial\psi}\big(\mathscr{G}_{\rm in}(\psi),\psi\big)\\ &=-\frac{\Theta_{\rm in}'(y)}{Q_{\rm in}(y)\rho(Q_{\rm in}^2(y))(\cos\Theta_{\rm in}(y)-\sin\Theta_{\rm in}(y)g'(y))}\bigg|_{y=Y_{\rm in}(\psi)},\quad 0\le\psi\le m_{\rm in}, \end{aligned} \tag{2.16}$$

$$q(\varphi_1,\psi)=c_*, \qquad \psi\in(0,m_{\rm in}), \tag{2.17}$$

$$Q_{\rm in}(y)=q(\mathscr{G}_{\rm in}(\psi),\psi)\big|_{\psi=\Psi_{\rm in}(y)}, \quad y\in[x_1\tan\vartheta_0,f(x_1)], \tag{2.18}$$

$$Q_{\rm up}(x)=q(\varphi,m_{\rm in})\big|_{\varphi=\Phi_{\rm up}(x)}, \qquad x\in[x_1,x_0], \tag{2.19}$$

where $(q,\varphi_1,m_{\rm in})$ is unknown,

$$\mathcal{D}=\{(\varphi,\psi)|0<\psi<m_{\rm in},\mathscr{G}_{in}(\psi)<\varphi<\varphi_1\}. \tag{2.20}$$

Remark 2.1. For the domain $\mathcal{D}$, the right boundary is free although its form is known (a segment parallel to ψ-axis), while the left boundary is also free. Furthermore, the left boundary depends on the solution and both its form and its location are unknown.

Remark 2.2. For the special case that $\Theta_{\rm in}$ is the angle of the inner normal of the inlet inclination to the x-axis, i.e. $\Theta_{\rm in}=-\arctan g'$, then

$$\mathscr{G}_{\rm in}(\psi)=0, \quad \psi\in[0,m_{\rm in}],$$

and the left boundary of $\mathcal{D}$ is fixed, which is just $\{0\}\times[0,m_{\rm in}]$.

Remark 2.3. A natural boundary condition at the inlet seems to be

$$\begin{aligned}&\frac{\partial A(q)}{\partial\varphi}(\mathscr{G}_{\rm in}(\psi),\psi)-\mathscr{G}'_{\rm in}(\psi)\frac{\partial B(q)}{\partial\psi}(\mathscr{G}_{\rm in}(\psi),\psi)\\ =&-\frac{\Theta'(y)}{q(\mathscr{G}_{\rm in}(\psi),\psi)\rho(q^2(\mathscr{G}_{\rm in}(\psi),\psi))(\cos\Theta(y)-\sin\Theta(y)g'(y))}\bigg|_{y=Y_{\rm in}(\psi)}\end{aligned} \tag{2.21}$$

instead of (2.16), where $0\le\psi\le m_{\rm in}$. However, since

$$\frac{d}{dq}\left(\frac{1}{q\rho(q^2)}\right)<0 \quad \text{in } q\in(0,c_*)$$

and $g''(y)/((1+(g'(y))^2)^{3/2})|_{y=Y_{\rm in}(\psi)}$ is a small perturbation of $-x_1/\cos\vartheta$, it seems difficult to obtain the uniqueness of the solution to the problem (2.13)-(2.15), (2.21), (2.17). So, similar to [13,16–18,20,23,28], we use (2.16) instead of (2.21).

Remark 2.4. In this paper, the flow is expected to be a small perturbation of the symmetric continuous subsonic-sonic flow. As discussed in [23], in the potential plane, the symmetric continuous subsonic-sonic flow is

$$q(\varphi,\psi)=\hat{q}(\varphi)=A^{-1}\left(\frac{2\vartheta_0(\varphi-\hat{\varphi}_1)}{m_0}\right), \quad 0\le\varphi\le\hat{\varphi}_1, \quad 0\le\psi\le m_0,$$

where

$$\hat{\varphi}_1 = -\frac{m_0 A(q_0)}{2\vartheta_0} \tag{2.22}$$

with $q_0 \in (0, c_*)$ satisfying

$$q_0 \rho(q_0^2) = -\frac{m_0 \cos\vartheta_0}{2x_1 \vartheta_0}. \tag{2.23}$$

Here we need to extend the function $\hat{q}$ to $(-\infty, \hat{\varphi}_1]$, i.e.

$$\hat{q}(\varphi) = A^{-1}\Big(A(q_0) + \frac{2\vartheta_0 \varphi}{m_0}\Big), \quad \varphi \le \hat{\varphi}_1. \tag{2.24}$$

Remark 2.5. As discussed in [17], the problem (2.13)-(2.19) admits a unique solution if the perturbation is sufficiently small and x_1 satisfies some conditions. For the existence,

$$\frac{c_* \rho(c_*^2)}{q_* \rho(q_*^2)} x_0 < x_1 < x_0, \quad L(x_1) \le \kappa_0, \tag{2.25}$$

where $q_* \in (0, c_*)$ satisfying $\rho(q_*^2)|A(q_*)| = 1$,

$$\begin{aligned} L(x_1) &= \int_0^{\hat{\varphi}_1} \frac{A'(q_0)}{A'(\hat{q}(\varphi))\hat{q}(\varphi)} \left(1 - \frac{(1+\rho(q_0^2)|A(q_0)|)\varphi}{2\hat{\varphi}_1}\right) d\varphi, \quad x_1 < x_0, \\ \kappa_0 &= \frac{-x_1(1-\rho(q_0^2)|A(q_0)|)(\rho(q_0^2) + 2q_0^2 \rho'(q_0^2))}{2\rho(q_0^2)\cos\vartheta_0}. \end{aligned} \tag{2.26}$$

For the uniqueness, these exists a constant $\hat{x}_1 < x_0$ such that when

$$\hat{x}_1 < x_1 < x_0, \tag{2.27}$$

there exist $\nu_1 > 0$ and $\nu_2 > 0$ such that

$$\begin{aligned} & q_0 \rho(q_0^2) \cos\vartheta_0 - \frac{\rho(q_0^2)\hat{\varphi}_1 \cos^2\vartheta_0}{-x_1} \\ & > \nu_1 + \frac{(x_0 - x_1)\vartheta_0 \cos^2\vartheta_0}{4\nu_1 x_1^2 \varsigma_1^2 q_0 \tan\vartheta_0} + \frac{2(x_0 - x_1)^{1/2} \vartheta_0^2 \cos^2\vartheta_0}{\nu_1 \nu_2 \varsigma_1^4 q_0^4}, \end{aligned} \tag{2.28}$$

$$\frac{\varsigma_2}{q_0 \rho(q_0^2)\cos\vartheta_0} > \frac{\nu_2 \cos^2\vartheta_0}{4\nu_1 x_1^2}, \tag{2.29}$$

where

$$\varsigma_1 = \frac{\sqrt{4\rho(c_*^2)+c_*^2\rho^2(c_*^2)|A(q_0)|}-c_*\rho(c_*^2)|A(q_0)|^{1/2}}{(1-(\gamma-1)q_0^2/2)^{\gamma/(\gamma-1)}}\sqrt{\frac{\cos\vartheta_0}{q_0\rho(q_0^2)(-x_1)}}, \tag{2.30}$$

$$\varsigma_2 = \frac{\rho^2(c_*^2)c_*^2}{2}\sqrt{\frac{q_0\rho(q_0^2)(-x_1)}{c_*}}. \tag{2.31}$$

Remark 2.6. The case that only the inlet is perturbed was studied in [23], where the incoming mass flux is given instead of the sonic curve intersecting the upper nozzle wall at P_0. In [23], the restrict on x_1 is the first formula of (2.25). Otherwise, the linearized problem of the symmetric continuous subsonic-sonic flow is unstable ([23, Proposition 2.1]).

2.2 Main results

One of the main results in this paper is the following existence theorem.

Theorem 2.1. *For given $\vartheta_0\in(0,\pi/2)$, $\alpha\in(0,1)$, $x_0<0$, and x_1 satisfying* (2.25)*, there exists $\delta_1>0$ depending only on γ, ϑ_0, x_0 and x_1, such that for $f\in C^{2,\alpha}([x_1,x_0])$ satisfying* (1.8) *and* (1.9)*, $g\in C^{2,\alpha}([x_1\tan\vartheta_0,f(x_1)])$ satisfying* (1.10) *and* (1.11)*, $\Theta_{\rm in}\in C^{1,\alpha}([x_1\tan\vartheta_0,f(x_1)])$ satisfying* (1.12) *and* (1.13)*, if $0<\delta\le\delta_1$, then the problem* (2.13)-(2.19) *admits at least one weak solution $q\in C^\infty(\mathcal{D})\cap C(\overline{\mathcal{D}})$ with $\varphi_1>\max_{[0,m_{\rm in}]}\mathscr{G}_{\rm in}(\psi)$, and $(q,\varphi_1,m_{\rm in})$ possesses the following properties:*

(i)

$$\begin{aligned}
&|q(\mathscr{G}_{\rm in}(\psi),\psi)-q_0|\le\sqrt{\delta},\quad \psi\in[0,m_{\rm in}],\\
&\left|\int_{x_1}^{x_0} q(\Phi_{\rm up}(x),m_{\rm in})\sqrt{1+(f'(x))^2}dx-\int_{x_1}^{x_0}\frac{\hat{Q}_{\rm up}(x)}{\cos\vartheta_0}dx\right|\le\kappa_0\sqrt{\delta},\\
&|\varphi_1-\hat{\varphi}_1|\le\kappa_0\sqrt{\delta},\quad |m_{\rm in}-m_0|\le\frac{-2x_1\vartheta_0q_0\rho^2(q_0^2)A'(q_0)}{\cos\vartheta_0}\sqrt{\delta}+\kappa_1\delta,
\end{aligned}$$

where q_0, $\Phi_{\rm up}$, $\hat{Q}_{\rm up}$, κ_0, $\hat{\varphi}_1$ and m_0 are given by (2.23), (2.8), (2.1), (2.26), (2.22) *and* (2.2)*, respectively, and $\kappa_1>0$ depends only on γ, ϑ_0, x_0 and x_1.*

(ii)

$$\begin{aligned}
&|q(\varphi,\psi)-\bar{q}(\varphi)|\le M_2\sqrt{\delta}(\varphi_1-\varphi)^{1/2}, && (\varphi,\psi)\in\overline{\mathcal{D}},\\
&\left|q(\varphi,\psi)-\hat{q}\left(\frac{\hat{\varphi}_1}{\varphi_1}\varphi\right)\right|\le M_2\sqrt{\delta}(\varphi_1-\varphi)^{1/2}, && (\varphi,\psi)\in\overline{\mathcal{D}},
\end{aligned}$$

$$M_1(\varphi_1-\varphi)^{1/2}\le c_*-q(\varphi,\psi)\le M_2(\varphi_1-\varphi)^{1/2},\quad (\varphi,\psi)\in\overline{\mathcal{D}},$$

where

$$\bar{q}(\varphi)=A^{-1}\left(\frac{(\varphi_1-\varphi)\cos\vartheta_0}{x_1q_0\rho(q_0^2)}\right),\quad \varphi\le\varphi_1,$$

where $\hat{q}$ *and* $\hat{\varphi}_1$ *are given by* (2.24) *and* (2.22)*, and* $0<M_1\le M_2$ *depending only on* γ*,* ϑ_0*,* x_0 *and* x_1.

(iii) $q\in C^\beta(\overline{\mathcal{D}})\cap C^{(1/2,1)}(\overline{\omega})$, $\partial A(q)/\partial\varphi,\partial q/\partial\psi\in L^\infty(\omega)$ *and*

$$\begin{aligned}
&[q]_{\beta;\overline{\mathcal{D}}}\le M,\\
&\left\|\frac{\partial A(q)}{\partial\varphi}\right\|_{L^\infty(\omega)}\le M,\\
&\left|\frac{\partial q}{\partial\psi}(\varphi,\psi)\right|\le M\sup_{(0,m_{\rm in})}\left|\frac{\partial q}{\partial\psi}(\tilde{\varphi},\cdot)\right|(\varphi_1-\varphi)^{1/2}, &&(\varphi,\psi)\in\omega,\\
&|q(\varphi',\psi')-q(\varphi'',\psi'')|\le M\Big(|\varphi'-\varphi''|^{1/2}+|\psi'-\psi''|\Big), &&(\varphi',\psi'),(\varphi'',\psi'')\in\omega,
\end{aligned}$$

where $0<\beta\le1/2$ *and* $M>0$ *both depending only on* γ*,* ϑ*,* x_0*,* x_1*,* α*,* $\|f\|_{2,\alpha;(x_1,x_0)}$*,* $\|g\|_{2,\alpha;(x_1\tan\vartheta,f(x_1))}$ *and* $\|\Theta_{\rm in}\|_{1,\alpha;(x_1\tan\vartheta,f(x_1))}$*,*

$$\omega=\big\{(\varphi,\psi)\,|\,0<\psi<m_{\rm in},\,\tilde{\varphi}<\varphi<\varphi_1\big\},\quad \tilde{\varphi}=\big(\max_{[0,m_{\rm in}]}\mathscr{G}_{\rm in}+\varphi_1\big)/2.$$

(iv) $q\in C^{1,\alpha}(\{(\varphi,\psi)|0\le\psi\le m_{\rm in},\,\mathscr{G}_{\rm in}(\psi)\le\varphi<\varphi_1\})$*, and for each* $0<\varepsilon<1-(\max_{[0,m_{\rm in}]}\mathscr{G}_{\rm in})/\varphi_1$*,*

$$\begin{aligned}
&\left\|q(\varphi,\psi)-\hat{q}\left(\frac{\hat{\varphi}_1}{\varphi_1}\varphi\right)\right\|_{1,\alpha;\overline{\mathcal{D}}_\varepsilon}\\
&\le M(\varepsilon)\Big(\sqrt{\delta}+\|g-g_0\|_{2,\alpha;(x_1\tan\vartheta_0,f(x_1))}+\|f(x)+x\tan\vartheta_0\|_{2,\alpha;(x_1,x_0)}\\
&\qquad+\|\Theta_{\rm in}+\arctan g'\|_{1,\alpha;(x_1\tan\vartheta_0,f(x_1))}\Big),
\end{aligned}$$

where

$$\mathcal{D}_\varepsilon=\big\{(\varphi,\psi)\,|\,0<\psi<m_{\rm in},\mathscr{G}_{\rm in}(\psi)<\varphi<(1-\varepsilon)\varphi_1\big\},$$

$M(\varepsilon)>0$ *depends only on* γ*,* α*,* x_0*,* x_1*,* ϑ_0 *and* ε.

Remark 2.7. The subsonic-sonic flow to the problem (2.13)-(2.19) is singular in the sense that the speed is only $C^{1/2}$ Hölder continuous and the acceleration blows up at the sonic state.

To establish the uniqueness theorem, we need restrict a suitable space of solutions for the boundary value problem of (2.13). Due to Theorem 2.1, for each $f\in C^{2,\alpha}([x_1,x_0])$ satisfying (1.8) and (1.9), each $g\in C^{2,\alpha}([x_1\tan\vartheta_0,f(x_1)])$ satisfying (1.10) and (1.11), and each $\Theta_{\rm in}\in C^{1,\alpha}([x_1\tan\vartheta_0,f(x_1)])$ satisfying (1.12) and (1.13), if x_1 satisfies (2.27), and δ_1 is sufficiently small, then the problem (2.13)-(2.19) admits a solution $(q,\varphi_1,m_{\rm in})$ with $q\in\mathscr{C}$, where

$$\begin{aligned}\mathscr{C}=\Big\{&q\in C^2(\mathcal{D})\cap C^1(\overline{\mathcal{D}}\setminus\{\varphi_1\}\times[0,m_{\rm in}])\cap C(\overline{\mathcal{D}})(\varphi_1>\max_{[0,m_{\rm in}]}\mathscr{G}_{\rm in}):\\ &\|q(\Phi_{\rm up}(x),m_{\rm in})-Q_0(x)\|_{L^\infty((x_1,x_0))}+\|q(\mathscr{G}_{\rm in}(\psi),\psi)-q_0\|_{L^\infty((0,m_{\rm in}))}\le\tau(\delta),\\ &\left\|q(\varphi,\psi)-\hat{q}\left(\frac{\hat{\varphi}_1}{\varphi_1}\varphi\right)\right\|_{0,1;\mathcal{D}_\varepsilon}\le\tau_\varepsilon(\delta)\quad\text{for any}\quad 0<\varepsilon<1-\frac{1}{\varphi_1}\max_{[0,m_{\rm in}]}\mathscr{G}_{\rm in},\\ &\left\|\frac{\partial A(q)}{\partial\varphi}\right\|_{L^\infty(\mathcal{D})}\le\overline{M},\ -\overline{M}(\varphi_1-\varphi)\le A(q)(\varphi,\psi)\le-\underline{M}(\varphi_1-\varphi),\\ &\text{and}\quad\left|\frac{\partial B(q)}{\partial\psi}(\varphi,\psi)\right|\le\tau(\delta)(\varphi_1-\varphi)^{1/2}\quad\text{for any}\quad(\varphi,\psi)\in\overline{\mathcal{D}},\\ &\text{where}\quad 0<\underline{M}\le\overline{M}\quad\text{and}\quad\lim_{\delta\to0^+}\tau(\delta)=\lim_{\delta\to0^+}\tau_\varepsilon(\delta)=0\Big\}.\end{aligned}$$

In fact, such a solution is also unique.

Theorem 2.2. *Assume that $\vartheta_0\in(0,\pi/2)$, $\alpha\in(0,1)$, $x_0<0$, and x_1 satisfies* (2.27). *There exists $\delta_2>0$ such that for any $f\in C^{2,\alpha}([x_1,x_0])$ satisfying* (1.8) *and* (1.9), *$g\in C^{2,\alpha}([x_1\tan\vartheta_0,f(x_1)])$ satisfying* (1.10) *and* (1.11), *and $\Theta_{\rm in}\in C^{1,\alpha}([x_1\tan\vartheta_0,f(x_1)])$ satisfying* (1.12) *and* (1.13), *if $0<\delta\le\delta_2$, then the problem* (2.13)-(2.19) *admits at most one solution $(q,\varphi_1,m_{\rm in})$ with $q\in\mathscr{C}$, where δ_2 depends only on γ, ϑ_0, x_0, x_1 and $\underline{M}$, $\overline{M}$, $\tau(\cdot)$, $\tau_\varepsilon(\cdot)$ in the definition of $\mathscr{C}$.*

In terms of the physical variables, the above two theorems can be transformed as

Theorem 2.3. *Assume that $\vartheta_0\in(0,\pi/2)$, $\alpha\in(0,1)$, $x_0<0$, and x_1 satisfies* (2.25). *There exists $\delta_1>0$ depending only on γ, ϑ_0, x_0 and x_1, such that for any $f\in C^{2,\alpha}([x_1,x_0])$ satisfying* (1.8) *and* (1.9), *$g\in C^{2,\alpha}([x_1\tan\vartheta_0,f(x_1)])$ satisfying* (1.10) *and* (1.11), *and $\Theta_{\rm in}\in C^{1,\alpha}([x_1\tan\vartheta_0,f(x_1)])$ satisfying* (1.12) *and* (1.13), *if $0<\delta\le\delta_1$, then the problem* (1.14)-(1.19) *admits at least one solution (φ,S,C_{out}) with $\varphi\in C^\infty(\Omega)\cap C^{2,\alpha}(\overline{\Omega}\setminus S)\cap C^{1,1/2}(\overline{\Omega})$. Moreover, as a function of (φ,ψ), $q=|\nabla\varphi|$ possesses the estimates in Theorem* 2.1.

Theorem 2.4. *Assume that $\vartheta_0 \in (0,\pi/2)$, $\alpha \in (0,1)$, $x_0 < 0$, and x_1 satisfies* (2.27). *There exists $\delta_2 > 0$ such that for any $f \in C^{2,\alpha}([x_1,x_0])$ satisfying* (1.8) *and* (1.9), *$g \in C^{2,\alpha}([x_1 \tan\vartheta_0, f(x_1)])$ satisfying* (1.10) *and* (1.11), *and $\Theta_{\rm in} \in C^{1,\alpha}([x_1 \tan\vartheta_0, f(x_1)])$ satisfying* (1.12) *and* (1.13), *if $0<\delta\le\delta_2$, then the problem* (1.14)-(1.19) *admits at most one solution (φ,S,C_{out}) with $\varphi\in C^3(\Omega)\cap C^2(\overline{\Omega}\setminus S)\cap C^1(\overline{\Omega})$ and $q=|\nabla\varphi|\in\mathscr{C}$ as a function of (φ,ψ), where δ_2 depends only on γ, ϑ_0, x_0, x_1 and $\underline{M}$, $\overline{M}$, $\tau(\cdot)$, $\tau_\varepsilon(\cdot)$ in the definition of $\mathscr{C}$.*

3 Proof of the existence theorem

In this section, we prove Theorem 2.1 by the Schauder fixed point theorem.

3.1 A fixed boundary problem

We study the following fixed boundary value problem:

$$\frac{\partial^2 A(q)}{\partial\varphi^2}+\frac{\partial^2 B(q)}{\partial\psi^2}=0, \qquad (\varphi,\psi)\in\mathcal{D}, \tag{3.1}$$

$$\frac{\partial B(q)}{\partial\psi}(\varphi,0)=0, \qquad \varphi\in\big(\mathscr{G}_{\rm in}(0),\varphi_1\big), \tag{3.2}$$

$$\frac{\partial B(q)}{\partial\psi}(\varphi,m_{\rm in})=h_{\rm up}(\varphi), \qquad \varphi\in\big(\mathscr{G}_{\rm in}(m_{\rm in}),\varphi_1\big), \tag{3.3}$$

$$\frac{\partial A(q)}{\partial\varphi}\big(\mathscr{G}_{\rm in}(\psi),\psi\big)-\mathscr{G}_{\rm in}'(\psi)\frac{\partial B(q)}{\partial\psi}\big(\mathscr{G}_{\rm in}(\psi),\psi\big)=h_{\rm in}(\psi), \quad \psi\in(0,m_{\rm in}), \tag{3.4}$$

$$q(\varphi_1,\psi)=c_*, \qquad \psi\in(0,m_{\rm in}), \tag{3.5}$$

where $m_{\rm in}>0$, $\mathscr{G}_{\rm in}\in C^1([0,m_{\rm in}])$, $\varphi_1>\max_{[0,m_{\rm in}]}\mathscr{G}_{\rm in}$, $0<h_{\rm in}\in C([0,m_{\rm in}])$, and $h_{\rm up}\in C([0,\varphi_1])$. More generally, (3.5) is replaced by

$$q(\varphi_1,\psi)=V(\psi), \quad \psi\in(0,m_{\rm in}), \tag{3.6}$$

where $V\in C^1([0,m_{\rm in}])$ satisfies

$$0<V(\psi)\le c_*, \quad 0\le\psi\le m_{\rm in}.$$

Solutions to the problem (3.1)-(3.4), (3.6) are defined as follows.

Definition 3.1. *A function* $q\in C(\overline{\mathcal{D}})$ $(\varphi_1>\max_{[0,m_{\rm in}]}\mathscr{G}_{\rm in})$ *is said to be a weak supersolution (subsolution, solution) the problem* (3.1)-(3.4), (3.6), *if*

$$0<\inf_{\mathcal{D}} q\le\sup_{\mathcal{D}} q\le c_*, \tag{3.7}$$

and

$$\begin{aligned}
&\int_0^{m_{\rm in}}\int_{\mathscr{G}_{\rm in}(\psi)}^{\varphi_1}\left(A\big(q(\varphi,\psi)\big)\frac{\partial^2\zeta}{\partial\varphi^2}(\varphi,\psi)+B\big(q(\varphi,\psi)\big)\frac{\partial^2\zeta}{\partial\psi^2}(\varphi,\psi)\right)d\varphi d\psi\\
&\quad+\int_0^{m_{\rm in}}\frac{\Theta_{\rm in}'(y)}{Q_{\rm in}(y)\rho(Q_{\rm in}^2(y))(\cos\Theta_{\rm in}(y)-\sin\Theta_{\rm in}(y)g'(y))}\bigg|_{y=Y_{\rm in}(\psi)}\zeta(\mathscr{G}_{\rm in}(\psi),\psi)d\psi\\
&\quad-\int_0^{m_{\rm in}}A\big(V(\psi)\big)\frac{\partial\zeta}{\partial\varphi}(\varphi_1,\psi)d\psi+\int_0^{\varphi_1}\frac{1}{Q_{\rm up}(x)}\frac{d}{dx}\big(\sin\Theta_{\rm up}(x)\big)\bigg|_{x=X_{\rm up}(\varphi)}\zeta(\varphi,m_{\rm in})d\varphi\\
&\quad+\int_0^{m_{\rm in}}A\big(q(\mathscr{G}_{\rm in}(\psi),\psi)\big)\frac{\partial\zeta}{\partial\varphi}\big(\mathscr{G}_{\rm in}(\psi),\psi\big)d\psi\\
&\quad-\int_0^{m_{\rm in}}B\big(q(\mathscr{G}_{\rm in}(\psi),\psi)\big)\mathscr{G}_{\rm in}'(\psi)\frac{\partial\zeta}{\partial\psi}\big(\mathscr{G}_{\rm in}(\psi),\psi\big)d\psi\le(\ge,=)0
\end{aligned}\tag{3.8}$$

for each nonnegative $\zeta\in C^2(\overline{\mathcal{D}})$ *with*

$$\frac{\partial\zeta}{\partial\psi}(\cdot,0)\bigg|_{(\mathscr{G}_{\rm in}(0),\varphi_1)}=\frac{\partial\zeta}{\partial\psi}(\cdot,m_{\rm in})\bigg|_{(\mathscr{G}_{\rm in}(m_{\rm in}),\varphi_1)}=0,\quad \zeta(\varphi_1,\cdot)\bigg|_{(0,m_{\rm in})}=0.$$

The following comparison principle follows from [23, Proposition 3.2].

Lemma 3.1. *Assume that* q_+ *and* q_- *are a weak supersolution and a weak subsolution to the problem* (3.1)-(3.4), (3.6), *respectively. Then*

$$q_+(\varphi,\psi)\ge q_-(\varphi,\psi),\quad (\varphi,\psi)\in\mathcal{D}.$$

Below, we establish the well-posedness of the problem (3.1)-(3.5).

Proposition 3.1. *Assume that*

$$\begin{aligned}
&l_1\le h_{\rm in}(\psi)\le l_2,\quad |\mathscr{G}_{\rm in}(\psi)|\le l_3\delta,\quad |\mathscr{G}_{\rm in}'(\psi)|\le l_3\delta,\quad 0<\psi<m_{\rm in},\\
&|h_{\rm up}|\le l_3\delta^2(\varphi_1-\varphi)^{1/2},\quad 0<\varphi<\varphi_1,\quad l_4\le\varphi_1\le l_5,\quad l_6\le m_{\rm in}\le l_7,
\end{aligned}$$

where $0<\delta<\min\{1,l_1/2,l_4/(2l_3)\}$, $l_i>0$ $(1\le i\le 7)$ *satisfy* $l_1\le l_2$, $l_4\le l_5$, $l_6\le l_7$. *There exists* $\tilde{\delta}_0>0$ *depending only on* γ *and* $l_i>0$ $(1\le i\le 7)$, *such that if* $0<\delta<\tilde{\delta}_0$, *then the problem* (3.1)-(3.5) *admits uniquely a weak solution. Furthermore, the solution satisfies* *(i)* $q\in C^\infty(\mathcal{D})\cap C(\overline{\mathcal{D}})$ *satisfies*

$$q_-(\varphi,\psi)\le q(\varphi,\psi)\le q_+(\varphi,\psi),\qquad (\varphi,\psi)\in\overline{\mathcal{D}}, \tag{3.9}$$

$$M_1(\varphi_1-\varphi)^{1/2}\le c_*-q(\varphi,\psi)\le M_2(\varphi_1-\varphi)^{1/2},\quad (\varphi,\psi)\in\mathcal{D}, \tag{3.10}$$

where $0<M_1\le M_2$ *depend only on* γ *and* $l_i>0$ $(1\le i\le 7)$,

$$q_+(\varphi,\psi)=A^{-1}\left((\varphi-\varphi_1)\big(\inf_{(0,m_{\rm in})}h_{\rm in}-\delta\big)-\frac{\delta}{2}(\varphi^2-\varphi_1^2)+\frac{\delta^{3/2}l_3\psi^2}{2m_{\rm in}}(\varphi_1-\varphi)\right),$$

$$q_-(\varphi,\psi)=A^{-1}\left((\varphi-\varphi_1)\big(\sup_{(0,m_{\rm in})}h_{\rm in}+\delta\big)+\frac{\delta}{2}(\varphi^2-\varphi_1^2)-\frac{\delta^{3/2}l_3\psi^2}{2m_{\rm in}}(\varphi_1-\varphi)\right),$$

here $(\varphi,\psi)\in\overline{\mathcal{D}}$.

(ii) There exist $\beta\in(0,1/2]$ *and* $M>0$ *depending only on* γ *and* $l_i>0$ $(1\le i\le 7)$, *such that* $q\in C^\beta(\overline{\mathcal{D}})\cap C^{(1/2,1)}(\overline{\omega})$, *and* $\partial A(q)/\partial\varphi,\partial q/\partial\psi\in L^\infty(\omega)$,

$$[q]_{\beta;\mathcal{D}}\le M, \tag{3.11}$$

$$\left\|\frac{\partial A(q)}{\partial\varphi}\right\|_{L^\infty(\omega)}\le M, \tag{3.12}$$

$$\left|\frac{\partial q}{\partial\psi}(\varphi,\psi)\right|\le M\sup_{(0,m_{\rm in})}\left|\frac{\partial q}{\partial\psi}(\tilde\varphi,\cdot)\right|(\varphi_1-\varphi)^{1/2},\quad (\varphi,\psi)\in\omega, \tag{3.13}$$

$$\begin{aligned}&|q(\varphi',\psi')-q(\varphi'',\psi'')|\\ \le{}& M\Big(|\varphi'-\varphi''|^{1/2}+|\psi'-\psi''|\Big),\qquad (\varphi',\psi'),(\varphi'',\psi'')\in\omega,\end{aligned} \tag{3.14}$$

where

$$\omega=\big\{(\varphi,\psi)\,|\,0<\psi<m_{\rm in},\tilde\varphi<\varphi<\varphi_1\big\},\quad \tilde\varphi=\big(\max_{[0,m_{\rm in}]}\mathscr{G}_{\rm in}+\varphi_1\big)/2.$$

Proof. The uniqueness follows directly from the comparison principle. We establish the existence result by using a method of elliptic regularization. For each positive integer n, consider the following regularized problem:

$$\frac{\partial^2A(q_n)}{\partial\varphi^2}+\frac{\partial^2B(q_n)}{\partial\psi^2}=0,\qquad (\varphi,\psi)\in\mathcal{D}, \tag{3.15}$$

$$\frac{\partial q_n}{\partial\psi}(\varphi,0)=0,\qquad \varphi\in\big(\mathscr{G}_{\rm in}(0),\varphi_1\big), \tag{3.16}$$

$$\frac{\partial B(q_n)}{\partial\psi}(\varphi,m_{\rm in})=h_{\rm up}(\varphi),\quad \varphi\in\big(\mathscr{G}_{\rm in}(m_{\rm in}),\varphi_1\big), \tag{3.17}$$

$$
\begin{aligned}
&\frac{\partial A(q_n)}{\partial \varphi}(\mathscr{G}_{\rm in}(\psi),\psi)-\mathscr{G}'_{\rm in}(\psi)\frac{\partial B(q_n)}{\partial \psi}(\mathscr{G}_{\rm in}(\psi),\psi)\\
=&h_{\rm in}(\psi), && \psi\in(0,m_{\rm in}), && (3.18)\\
&q_n(\varphi_1,\psi)=\frac{nc_*}{n+1}, && \psi\in(0,m_{\rm in}). && (3.19)
\end{aligned}
$$

Note that (3.15) is a uniformly elliptic equation provided that $0<c_1\le q_n\le c_2<c_*$ for some c_1 and c_2. Therefore, to get the existence of the weak solution to the problem (3.15)-(3.19), it suffices to construct suitable supersolutions and subsolutions. For $(\varphi,\psi)\in\overline{\mathcal{D}}$, define

$$
\begin{aligned}
q_{n,+}(\varphi,\psi)&=A^{-1}\left(A\left(\frac{nc_*}{n+1}\right)+\left(\inf_{(0,m_{\rm in})}h_{\rm in}-\frac{\delta^{3/2}l_3\psi^2}{2m_{\rm in}}\right)(\varphi-\varphi_1)-\frac{\delta}{2}(\varphi^2-\varphi_1^2)\right),\\
q_{n,-}(\varphi,\psi)&=A^{-1}\left(A\left(\frac{nc_*}{n+1}\right)+\left(\sup_{(0,m_{\rm in})}h_{\rm in}+\frac{\delta^{3/2}l_3\psi^2}{2m_{\rm in}}\right)(\varphi-\varphi_1)+\frac{\delta}{2}(\varphi^2-\varphi_1^2)\right).
\end{aligned}
$$

Similar to the discussion in [16,17], there exists $\tilde{\delta}_0>0$ depending only on γ and $l_i>0$ $(1\le i\le 7)$, such that if $0<\delta<\tilde{\delta}_0$, $q_{n,+}$ and $q_{n,-}$ are super and sub solutions to the problem (3.15)-(3.19). Due to Lemma 3.1 and the classical theory [14], the problem (3.15)-(3.19) admits a unique solution $q_n\in C^\infty(\mathcal{D})\cap C(\overline{\mathcal{D}})$ satisfying

$$
0<q_{n,-}(\varphi,\psi)\le q_n(\varphi,\psi)\le q_{n,+}(\varphi,\psi)\le q_{n,+}(\varphi_1,\psi)=\frac{nc_*}{n+1},\quad (\varphi,\psi)\in\overline{\mathcal{D}}. \quad (3.20)
$$

Lemma 3.1 yields that

$$
q_{n_1,\pm}(\varphi,\psi)\le q_{n_2,\pm}(\varphi,\psi),\quad q_{n_1}(\varphi,\psi)\le q_{n_2}(\varphi,\psi),\quad (\varphi,\psi)\in\overline{\mathcal{D}},\quad n_1\le n_2.
$$

Set

$$
q_\pm(\varphi,\psi)=\lim_{n\to+\infty}q_{n,\pm}(\varphi,\psi),\quad q(\varphi,\psi)=\lim_{n\to+\infty}q_n(\varphi,\psi),\quad (\varphi,\psi)\in\overline{\mathcal{D}}.
$$

Thanks to the classical theory on elliptic equations and (3.20), one can get that $q\in C^\infty(\mathcal{D})\cap C(\overline{\mathcal{D}})$ satisfying (3.9) and (3.10) is a weak solution to the problem (3.1)-(3.5). Due to (3.9), (3.1) is uniformly elliptic away from the right boundary $\{\varphi_1\}\times[0,m_{\rm in}]$. Then, the Harnack inequality established in [15] shows that $q\in C^\beta(\overline{\omega})$ satisfying (3.11). Finally, $\partial A(q)/\partial\varphi,\partial q/\partial\psi\in L^\infty(\omega)$, $q\in C^{(1/2,1)}(\overline{\omega})$ with (3.12)-(3.14) can be proved by the same argument in [23]. □

3.2 Proof of Theorem 2.1

In order to prove the existence theorem (Theorem 2.1) by a similar fixed point argument as in [23], one chooses $Q_{\rm in}\in C([x_1\tan\vartheta_0,f(x_1)])$ satisfying

$$\|Q_{\rm in}-q_0\|_{L^\infty(x_1\tan\vartheta_0,f(x_1))}\le\sigma, \tag{3.21}$$

and $Q_{\rm up}\in C([x_1,x_0])$ satisfying

$$\left|\int_{x_1}^{x_0}Q_{\rm up}(x)\sqrt{1+(f'(x))^2}dx-\int_{x_1}^{x_0}\frac{\hat{Q}_{\rm up}(x)}{\cos\vartheta_0}dx\right|\le\kappa_0\sigma, \tag{3.22}$$

where $0<\sigma<\min\{1,q_0/(2\kappa_0),q_0/2\}$, while q_0, $\hat{Q}_{\rm up}$ and κ_0 are given by (2.23), (2.1) and (2.26). Define

$$\begin{aligned}\mathscr{S}_\sigma=\Big\{&(Q_{\rm in},Q_{\rm up})\in C([x_1\tan\vartheta_0,f(x_1)])\times C([x_1,x_0]):\\ &Q_{\rm in}\ \text{ satisfies }\ (3.21)\ \text{ and }\ Q_{\rm up}\ \text{ satisfies }\ (3.22)\Big\}\end{aligned}$$

with the norm

$$\|(Q_{\rm in},Q_{\rm up})\|_{\mathscr{S}_\sigma}=\max\Big\{\|Q_{\rm in}\|_{L^\infty(x_1\tan\vartheta_0,f(x_1))},\|Q_{\rm up}\|_{L^\infty(x_1,x_0)}\Big\},\quad (Q_{\rm in},Q_{\rm up})\in\mathscr{S}_\sigma.$$

For $0<\delta<1$ and $0<\alpha<1$, define

$$\begin{aligned}\mathcal{H}_\delta=\Big\{&(f,g,\Theta_{\rm in})\in C^{2,\alpha}([x_1,x_0])\times C^{2,\alpha}([x_1\tan\vartheta_0,f(x_1)])\times C^{1,\alpha}([x_1\tan\vartheta_0,f(x_1)]):\\ &f\ \text{ satisfies (1.8) and (1.9)},\ g\ \text{ satisfies (1.10) and (1.11)},\\ &\Theta_{\rm in}\ \text{ satisfies (1.12) and (1.13)}\Big\}.\end{aligned}$$

Proposition 3.2. *Assume that* $\vartheta_0\in(0,\pi/2)$, $0<\alpha<1$, $x_0<0$, *and* x_1 *satisfies* (2.25). *There exists* $\hat{\delta}_0\in(0,1)$ *depending only on* γ, ϑ_0, x_0 *and* x_1, *such that for each* $(f,g,\Theta_{\rm in})\in\mathcal{H}_\delta$ *with* $0<\delta\le\hat{\delta}_0$ *and each* $(Q_{\rm in},Q_{\rm up})\in\mathscr{S}_\sigma$ *with* $\sigma=\sqrt{\delta}$, *the problem* (2.13)-(2.17) *admits a unique weak solution* $q\in C^\infty(\mathcal{D})\cap C^\beta(\overline{\mathcal{D}})\cap C^{(1/2,1)}(\overline{\omega})$ *with* $\varphi_1>\max_{[0,m_{\rm in}]}\mathscr{G}_{in}$, φ_1 *and* $m_{\rm in}$ *given by* (2.7) *and* (2.3), *and*

$$\begin{aligned}&|\varphi_1-\hat{\varphi}_1|\le\kappa_0\sqrt{\delta},\quad |m_{\rm in}-m_0|\le\frac{-2x_1\vartheta_0q_0\rho^2(q_0^2)A'(q_0)}{\cos\vartheta_0}\sqrt{\delta}+\kappa_1\delta,\\ &|q(\mathscr{G}_{\rm in}(\psi),\psi)-q_0|\le\sqrt{\delta},\quad \psi\in[0,m_{\rm in}],\end{aligned}$$

$$
\begin{aligned}
&\left|\int_{x_1}^{x_0} q(\Phi_{\rm up}(x),m_{\rm in})\sqrt{1+(f'(x))^2}dx-\int_{x_1}^{x_0}\frac{\hat{Q}_{\rm up}(x)}{\cos\vartheta_0}dx\right|\le\kappa_0\sqrt{\delta},\\
&\left|q(\varphi,\psi)-\hat{q}\left(\frac{\hat{\varphi}_1}{\varphi_1}\varphi\right)\right|\le M_2\sqrt{\delta}(\varphi_1-\varphi)^{1/2}, && (\varphi,\psi)\in\mathcal{D},\\
&M_1(\varphi_1-\varphi)^{1/2}\le c_*-q(\varphi,\psi)\le M_2(\varphi_1-\varphi)^{1/2}, && (\varphi,\psi)\in\overline{\mathcal{D}},\\
&[q]_{\beta;\mathcal{D}}\le M_2,\quad \left\|\frac{\partial A(q)}{\partial\varphi}\right\|_{L^\infty(\omega)}\le M_2,\\
&\left|\frac{\partial q}{\partial\psi}(\varphi,\psi)\right|\le M_2\sup_{(0,m_{\rm in})}\left|\frac{\partial q}{\partial\psi}(\tilde{\varphi},\cdot)\right|(\varphi_1-\varphi)^{1/2}, && (\varphi,\psi)\in\omega,\\
&|q(\varphi',\psi')-q(\varphi'',\psi'')|\le M_2\left(|\varphi'-\varphi''|^{1/2}+|\psi'-\psi''|\right), && (\varphi',\psi'),(\varphi'',\psi'')\in\omega,
\end{aligned}
$$

where $\hat{\varphi}_1$, m_0, q_0, $\Phi_{\rm up}$, $\hat{Q}_{\rm up}$, κ_0 *and* $\hat{q}$ *are given by* (2.22), (2.2), (2.23), (2.8), (2.1), (2.26) *and* (2.24), *respectively, and* $0<\beta\le 1/2$, $\kappa_1>0$ *and* $0<M_1\le M_2$ *depend only on* γ, ϑ_0, x_0 *and* x_1,

$$
\omega=\left\{(\varphi,\psi)\,|\,0<\psi<m_{\rm in},\,\tilde{\varphi}<\varphi<\varphi_1\right\},\quad \tilde{\varphi}=\big(\max_{[0,m_{\rm in}]}\mathscr{G}_{\rm in}+\varphi_1\big)/2.
$$

Proof. For $(f,g,\Theta_{\rm in})\in\mathcal{H}_\delta$ and $(Q_{\rm in},Q_{\rm up})\in\mathscr{S}_\sigma$, it is clear that

$$
\begin{aligned}
&\left\|\frac{\Theta_{\rm in}'(y)}{\cos\Theta_{\rm in}(y)-\sin\Theta_{\rm in}(y)g'(y)}+\frac{\cos\vartheta_0}{x_1}\right\|_{L^\infty(x_1\tan\vartheta_0,f(x_1))}\le N_0\delta,\\
&\left|\frac{f''(x)}{(1+(f'(x))^2)^{3/2}Q_{\rm up}(x)}\right|\le N_0\delta^2\left(\int_x^{x_0}Q_{\rm up}(s)\sqrt{1+(f'(s))^2}ds\right)^{1/2}, && x_1\le x\le x_0,\\
&\|\mathscr{G}_{\rm in}(\cdot)\|_{0,1;(0,m_{\rm in})}\le N_0\delta,\quad \left|\arctan f'(x)+\vartheta_0\right|\le N_0\delta^2, && x_1\le x\le x_0,
\end{aligned}
$$

where $N_0>0$ depends only on γ, ϑ_0, x_0 and x_1. From (3.21) and (3.22), there exists $\hat{\delta}_1>0$ depending only on γ, ϑ_0, x_0 and x_1, such that for $0<\delta\le\hat{\delta}_1$ with $\delta=\sigma^2$,

$$
\begin{aligned}
&|\varphi_1-\hat{\varphi}_1|\le\kappa_0\sigma,\quad |m_{\rm in}-m_0|\le\frac{-2x_1\vartheta_0 q_0\rho^2(q_0^2)A'(q_0)}{\cos\vartheta_0}\sigma+\kappa_1\sigma^2,\\
&\delta<\min\left\{1,-\cos\vartheta_0/(2x_1N_0)\right\},\\
&\sigma<\min\left\{q_0/2,\hat{\varphi}_1/(2\kappa_0),m_0/(4\kappa_1),(m_0/(4\kappa_1))^{1/2}\right\},
\end{aligned}
$$

where $\kappa_1>0$ depends only on γ, ϑ_0, x_0 and x_1. Set

$$
q_{1,+}(\varphi,\psi)=A^{-1}\left(\frac{1}{(q_0+\sigma)\rho((q_0+\sigma)^2)}\left(\frac{\cos\vartheta_0}{x_1}+N_0\delta\right)(\varphi_1-\varphi),\right.
$$

$$-\frac{\delta}{2}(\varphi^2-\varphi_1^2)+\frac{N_0\delta^{3/2}\psi^2}{2m_{\rm in}}(\varphi_1-\varphi)\Big),\quad (\varphi,\psi)\in\overline{\mathcal{D}},$$

$$q_{2,-}(\varphi,\psi)=A^{-1}\left(\frac{1}{(q_0-\sigma)\rho((q_0-\sigma)^2)}\left(\frac{\cos\vartheta_0}{x_1}-N_0\delta\right)(\varphi_1-\varphi)\right.$$
$$\left.+\frac{\delta}{2}(\varphi^2-\varphi_1^2)-\frac{N_0\delta^{3/2}\psi^2}{2m_{\rm in}}(\varphi_1-\varphi)\right),\quad (\varphi,\psi)\in\overline{\mathcal{D}}.$$

It follows from Proposition 3.1 that there exists $\hat{\delta}_2\in(0,\hat{\delta}_1]$ depending only on γ, ϑ_0, x_0 and x_1, such that for each $0<\delta\le\hat{\delta}_2$ and $\sigma=\sqrt{\delta}$, $q_{1,+}$ and $q_{2,-}$ are a supersolution and a subsolution to the problem (2.13)-(2.17), respectively. Note that (2.22) and (2.23) yield

$$A(q_0)=\frac{\cos\vartheta_0}{x_1q_0\rho(q_0^2)}\hat{\varphi}_1.$$

Therefore,

$$q_{1,+}(\mathscr{G}_{\rm in}(\psi),\psi)\le q_0+\frac{1+\rho(q_0^2)|A(q_0)|}{2}\sigma+O(\sigma^2)+O(\delta)+O(\delta^2),$$
$$q_{2,-}(\mathscr{G}_{\rm in}(\psi),\psi)\ge q_0-\frac{1+\rho(q_0^2)|A(q_0)|}{2}\sigma+O(\sigma^2)+O(\delta)+O(\delta^2),$$

uniformly on $[0,m_{\rm in}]$ as $\sigma\to0^+$ and $\delta\to0^+$. In the proof of this proposition, $O(\cdot)$ always depends only on γ, ϑ_0, x_0 and x_1. Owing to $0<\rho(q_0^2)|A(q_0)|<1$ from (2.25), there exists $\hat{\delta}_3\in(0,\hat{\delta}_2]$ depending only on γ, ϑ_0, x_0 and x_1, such that for each $0<\delta\le\hat{\delta}_3$ and $\sigma=\sqrt{\delta}$,

$$q_0-\sqrt{\delta}<q_{2,-}(\mathscr{G}_{\rm in}(\psi),\psi)\le q_{1,+}(\mathscr{G}_{\rm in}(\psi),\psi)<q_0+\sqrt{\delta},\quad \psi\in[0,m_{\rm in}]. \tag{3.23}$$

For $x\in[x_1,x_0]$, one gets from (2.8) that

$$\begin{aligned}
&A\big(q_{1,+}(\Phi_{\rm up}(x),m_{\rm in})\big)\\
&=\int_x^{x_0}Q_{\rm up}(s)\sqrt{1+(f'(s))^2}ds\\
&\quad\times\left[\frac{1}{(q_0+\sigma)\rho((q_0+\sigma)^2)}\left(\frac{\cos\vartheta_0}{x_1}+N_0\delta\right)\right.\\
&\qquad+\frac{\delta}{2}\left(\int_{x_1}^{x}Q_{\rm up}(s)\sqrt{1+(f'(s))^2}ds+\int_{x_1}^{x_0}Q_{\rm up}(s)\sqrt{1+(f'(s))^2}ds\right)\\
&\qquad\left.+\frac{N_0\delta^{3/2}m_{\rm in}}{2}\right],
\end{aligned}$$

$$
\begin{aligned}
&A\big(q_{2,-}(\Phi_{\rm up}(x),m_{\rm in})\big)\\
=&\int_x^{x_0} Q_{\rm up}(s)\sqrt{1+(f'(s))^2}ds\\
&\times\Bigg[\frac{1}{(q_0-\sigma)\rho((q_0-\sigma)^2)}\left(\frac{\cos\vartheta_0}{x_1}-N_0\delta\right)\\
&\quad-\frac{\delta}{2}\left(\int_{x_1}^{x} Q_{\rm up}(s)\sqrt{1+(f'(s))^2}ds+\int_{x_1}^{x_0} Q_{\rm up}(s)\sqrt{1+(f'(s))^2}ds\right)\\
&\quad-\frac{N_0\delta^{3/2}m_{\rm in}}{2}\Bigg].
\end{aligned}
$$

It is noted that

$$
A\big(\hat{Q}_{\rm up}(x)\big)=\frac{\cos\vartheta_0}{x_1q_0\rho(q_0^2)}\int_x^{x_0}\frac{\hat{Q}_{\rm up}(s)}{\cos\vartheta_0}ds,\quad x\in[x_1,x_0].
$$

Therefore,

$$
\begin{aligned}
&\int_{x_1}^{x_0} q_{1,+}(\Phi_{\rm up}(x),m_{\rm in})\sqrt{1+(f'(x))^2}dx-\int_{x_1}^{x_0}\frac{\hat{Q}_{\rm up}(x)}{\cos\vartheta_0}dx\\
\le&\,\sigma\int_{x_1}^{x_0}\frac{1}{A'(\hat{Q}(x))\cos\vartheta_0}\left(\frac{(1+\rho(q_0^2)|A(q_0)|)A'(q_0)}{2\hat{\varphi}_1}\int_x^{x_0}\frac{\hat{Q}_{\rm up}(x)}{\cos\vartheta_0}dx-\frac{A(q_0)\kappa_0}{\hat{\varphi}_1}\right)dx\\
&+O(\sigma^2)+O(\delta)\\
\le&\,\sigma\int_0^{\hat{\varphi}_1}\frac{A'(q_0)}{A'(\hat{q}(\varphi))\hat{q}(\varphi)}\left(\frac{(1+\rho(q_0^2)|A(q_0)|)(\hat{\varphi}_1-\varphi)}{2\hat{\varphi}_1}+\frac{1-\rho(q_0^2)|A(q_0)|}{2}\right)d\varphi\\
&+O(\sigma^2)+O(\delta),
\end{aligned}\tag{3.24}
$$

$$
\begin{aligned}
&\int_{x_1}^{x_0} q_{2,-}\big(\Phi_{\rm up}(x),m_{\rm in}\big)\sqrt{1+(f'(x))^2}dx-\int_{x_1}^{x_0}\frac{\hat{Q}_{\rm up}(x)}{\cos\vartheta_0}dx\\
\ge&\,\sigma\int_{x_1}^{x_0}\frac{-1}{A'(\hat{Q}(x))\cos\vartheta_0}\left(\frac{(1+\rho(q_0^2)|A(q_0)|)A'(q_0)}{2\hat{\varphi}_1}\int_x^{x_0}\frac{\hat{Q}_{\rm up}(x)}{\cos\vartheta_0}dx-\frac{A(q_0)\kappa_0}{\hat{\varphi}_1}\right)dx\\
&+O(\sigma^2)+O(\delta)\\
\ge&\,\sigma\int_0^{\hat{\varphi}_1}\frac{-A'(q_0)}{A'(\hat{q}(\varphi))\hat{q}(\varphi)}\left(\frac{(1+\rho(q_0^2)|A(q_0)|)(\hat{\varphi}_1-\varphi)}{2\hat{\varphi}_1}+\frac{1-\rho(q_0^2)|A(q_0)|}{2}\right)d\varphi\\
&+O(\sigma^2)+O(\delta)
\end{aligned}\tag{3.25}
$$

uniformly on $[x_1,x_0]$ as $\sigma\to 0^+$ and $\delta\to 0^+$. Similarly, one can get from (2.25) that there exists $\hat{\delta}_0\in(0,\hat{\delta}_3]$ depending only on γ, ϑ_0, x_0 and x_1, such that for each $0<\delta\le\hat{\delta}_0$ and $\sigma=\sqrt{\delta}$,

$$\begin{aligned}
&\int_{x_1}^{x_0}\frac{\hat{Q}_{\rm up}(x)}{\cos\vartheta_0}dx-\kappa_0\sqrt{\delta}\\
\le&\int_{x_1}^{x_0}q_{2,-}\big(\Phi_{\rm up}(x),m\big)\sqrt{1+(f'(x))^2}dx\\
\le&\int_{x_1}^{x_0}q_{1,+}\big(\Phi_{\rm up}(x),m\big)\sqrt{1+(f'(x))^2}dx\\
\le&\int_{x_1}^{x_0}\frac{\hat{Q}_{\rm up}(x)}{\cos\vartheta_0}dx+\kappa_0\sqrt{\delta}.
\end{aligned}\tag{3.26}$$

Then, the result of the proposition follows from (3.23)-(3.26) and some similar arguments in [13, 16–18, 20, 23, 28]. □

Below, we prove Theorem 2.1 by using the Schauder fixed point theorem.

Proof of Theorem 2.1. Choose

$$\delta_1=\min\big\{\hat{\delta}_0,q_0^2,(c_*-q_0)^2\big\},$$

where $\hat{\delta}_0$ is determined in Proposition 3.2. Let

$$0<\delta\le\delta_1,\quad \sigma=\sqrt{\delta}.$$

Give $(f,g,\Theta_{\rm in})\in\mathcal{H}_\delta$. From Proposition 3.2, for each $(Q_{\rm in},Q_{\rm up})\in\mathscr{S}_\sigma$, the problem (2.13)-(2.17) admits a unique weak solution $(q,\varphi_1,m_{\rm in})$ with $q\in C^\infty(\mathcal{D})\cap C^\beta(\overline{\mathcal{D}})\cap C(\overline{\mathcal{D}})$. Set

$$\begin{aligned}
&\hat{Q}_{\rm in}(y)=q\big(\Phi_{\rm in}(y),\Psi_{\rm in}(y)\big),\quad x_1\tan\vartheta_0\le y\le f(x_1),\\
&\hat{Q}_{\rm up}(x)=q\big(\Phi_{\rm up}(x),m_{\rm in}\big),\qquad x_1\le x\le x_0.
\end{aligned}$$

Then, it can be verified from the estimates in Proposition 3.2 that

$$\hat{Q}_{\rm in}\in C\big([x_1\tan\vartheta_0,f(x_1)]\big)\ \text{ satisfies (3.21) and }\ \hat{Q}_{\rm up}\in C\big([x_1,x_0]\big)\ \text{ satisfies (3.22).}$$

Therefore, one can define a mapping J from $\mathscr{S}_\sigma$ to itself as follows:

$$J\big((Q_{\rm in},Q_{\rm up})\big)=(\hat{Q}_{\rm in},\hat{Q}_{\rm up}),\quad (Q_{\rm in},Q_{\rm up})\in\mathscr{S}_\sigma.$$

Similar to the proof of [23, Theorem 4.1], one can show from Proposition 3.2 and Lemma 3.1 that J is compact and continuous. The Schauder fixed point theorem yields that J admits a fixed point $(Q_{\rm in}^*,Q_{\rm up}^*)\in\mathscr{S}_\sigma$. Hence, the weak solution $(q,\varphi_1,m_{\rm in})$ to the problem (2.13)-(2.17) with $(Q_{\rm in},Q_{\rm up})=(Q_{\rm in}^*,Q_{\rm up}^*)$ is just a weak solution to the problem (2.13)-(2.19). The properties of $(q,\varphi_1,m_{\rm in})$ follow from Proposition 3.2 and a similar discussion as [23, Theorem 2.1]. □

4 Proof of the uniqueness

As mentioned in [23], not only the degeneracy and the free boundaries but also the nonlocal terms in (2.15) and (2.16) bring essential difficulties for the uniqueness theorem. Although these nonlocal terms arise from the coordinates transformation from the Cartesian coordinates to the potential-stream coordinates, it is not convenient to prove the uniqueness theorem in the Cartesian coordinates since the inlet, the upper wall and the outlet of the problem (1.14)-(1.19) are free whose form and location are unknown.

In order to prove Theorem 2.2, we first introduce a suitable coordinates as in [17,20] transformation to transform (2.15) and (2.16) into usual boundary conditions and to fix free boundaries into fixed ones. Let $(q,\varphi_1,m_{\rm in})$, which is a small perturbation of the symmetric continuous subsonic-sonic flow $(\hat q,\hat\varphi_1,m_0)$, be a solution to the problem (2.13)-(2.19). Introduce the coordinates transformation

$$\begin{cases} x=X_{\rm up}\left(\dfrac{\varphi_1(\varphi-\mathscr{G}_{\rm in}(\psi))}{\varphi_1-\mathscr{G}_{\rm in}(\psi)}\right), \\ y=Y_{\rm in}(\psi), \end{cases}\qquad (\varphi,\psi)\in\overline{\mathcal{D}}$$

and

$$\begin{cases} \varphi=\Phi_{\rm up}(x)+\left(1-\dfrac{\Phi_{\rm up}(x)}{\Phi_{\rm up}(x_0)}\right)\Phi_{\rm in}(y), \\ \psi=\Psi_{\rm in}(y), \end{cases}\qquad (x,y)\in[x_1,x_0]\times[x_1\tan\vartheta_0,f(x_1)],$$

where $\Psi_{\rm in}$, $\Phi_{\rm in}$, $\Phi_{\rm up}$, $Y_{\rm in}$, $X_{\rm up}$, $\mathscr{G}_{\rm in}$ and $\mathcal{D}$ are given by (2.4), (2.5), (2.8)-(2.11) and (2.20), respectively. Define

$$w(x,y)=A\left(q\left(\Phi_{\rm up}(x)+\left(1-\frac{\Phi_{\rm up}(x)}{\Phi_{\rm up}(x_0)}\right)\Phi_{\rm in}(y),\Psi_{\rm in}(y)\right)\right),$$
$$(x,y)\in[x_1,x_0]\times[x_1\tan\vartheta_0,f(x_1)].$$

Then, w solves the problem

$$\begin{aligned}
&\left(\frac{\Phi_{\rm up}(x_0)l(x)}{(\Phi_{\rm up}(x_0)-\Phi_{\rm in}(y))h(y)}\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{(\Phi_{\rm up}(x_0)-\Phi_{\rm in}(y))h(y)}{\Phi_{\rm up}(x_0)l(x)}\frac{\partial K(w)}{\partial y}\right)\\
&\quad+\frac{\partial}{\partial x}\left(\frac{((\Phi_{\rm up}(x_0)-\Phi_{\rm up}(x))\Phi_{\rm in}'(y))^2h(y)l(x)}{\Phi_{\rm up}(x_0)(\Phi_{\rm up}(x_0)-\Phi_{\rm in}(y))}\frac{\partial K(w)}{\partial x}\right)\\
&\quad-\frac{\partial}{\partial y}\left(\frac{(\Phi_{\rm up}(x_0)-\Phi_{\rm up}(x))\Phi_{\rm in}'(y)h(y)}{\Phi_{\rm up}(x_0)}\frac{\partial K(w)}{\partial x}\right)\\
&\quad-\frac{\partial}{\partial x}\left(\frac{(\Phi_{\rm up}(x_0)-\Phi_{\rm up}(x))\Phi_{\rm in}'(y)h(y)}{\Phi_{\rm up}(x_0)}\frac{\partial K(w)}{\partial y}\right)=0,\\
&\qquad\qquad\qquad\qquad (x,y)\in(x_1,x_0)\times(x_1\tan\vartheta_0,f(x_1)), \qquad (4.1)\\
&\frac{\partial w}{\partial y}(x,x_1\tan\vartheta_0)=0, \qquad x\in(x_1,x_0), \qquad (4.2)\\
&\frac{\partial K(w)}{\partial y}(x,f(x_1))=\frac{\Theta_{\rm up}'(x)l(x)}{h(f(x_1))}, \quad x\in(x_1,x_0), \qquad (4.3)\\
&\frac{\Phi_{\rm up}(x_0)l(x_1)(1+(\Phi_{\rm in}'(y)h(y))^2K'(w)(x_1,y))}{(\Phi_{\rm up}(x_0)-\Phi_{\rm in}(y))h(y)}\frac{\partial w}{\partial x}(x_1,y)\\
&\quad-\Phi_{\rm in}'(y)h(y)\frac{\partial(w)}{\partial y}(x_1,y)\\
&=-\Theta_{\rm in}'(y), \qquad y\in(x_1\tan\vartheta_0,f(x_1)), \qquad (4.4)\\
&w(x_0,y)=0, \qquad y\in(x_1\tan\vartheta_0,f(x_1)), \qquad (4.5)
\end{aligned}$$

where

$$\begin{aligned}
&K(s)=B\big(A^{-1}(s)\big), \qquad s<0,\\
&h(y)=\frac{1}{A^{-1}(w(x_1,y))\rho\big((A^{-1}(w(x_1,y)))^2\big)\big(\cos\Theta_{\rm in}(y)-\sin\Theta_{\rm in}(y)g'(y)\big)},\\
&\qquad\qquad\qquad\qquad\qquad\qquad y\in[x_1\tan\vartheta_0,f(x_1)],\\
&l(x)=\frac{\cos\Theta_{\rm up}(x)}{A^{-1}((w(x,f(x_1)))}=\frac{1}{A^{-1}((w(x,f(x_1)))\sqrt{1+(f'(x))^2}}, \qquad x\in[x_1,x_0].
\end{aligned}$$

The problem (4.1)-(4.5) has common boundary conditions in a fixed domain.

Next following from the method of energy estimate and a series of complicated calculations, we prove Theorem 2.2.

Proof of Theorem 2.2. In the proof, we always assume that $O(\cdot)$ depend only on γ, ϑ_0, x_0, x_1, $\underline{M}$ and $\overline{M}$, where $\underline{M}$ and $\overline{M}$ are given in the definition of $\mathscr{C}$. Moreover, it is assumed that $\tau(\delta)\geq\delta$ without loss of generality. Let $(q_k,\varphi_{1,k},m_{\text{in},k})$ $(k=1,2)$ be two solutions to the problem (2.13)-(2.19). For $k=1,2$, denote $\Psi_{\text{in,k}}$, $\Phi_{\text{in,k}}$, $\Phi_{\text{up,k}}$, $Y_{\text{in,k}}$, $X_{\text{up,k}}$ and $\mathscr{G}_{\text{in,k}}$ to be the associated functions by (2.4), (2.5), (2.8), (2.9), (2.10) and (2.11) and set

$$
\begin{aligned}
&w_k(x,y)=A\left(q\left(\Phi_{\text{up,k}}(x)+\left(1-\frac{\Phi_{\text{up,k}}(x)}{\Phi_{\text{up,k}}(x_0)}\right)\Phi_{\text{in,k}}(y),\Psi_{\text{in,k}}(y)\right)\right),\\
&h_k(y)=\frac{1}{A^{-1}(w_k(x_1,y))\rho((A^{-1}(w_k(x_1,y)))^2)(\cos\Theta_{\text{in}}(y)-\sin\Theta_{\text{in}}(y)g'(y))},\\
&l_k(x)=\frac{\cos\Theta_{\text{up}}(x))}{A^{-1}((w_k(x,f(x_1)))},
\end{aligned}
$$

where $x\in[x_1,x_0]$, $y\in[x_1\tan\vartheta_0,f(x_1)]$. Let

$$
w(x,y)=w_1(x,y)-w_2(x,y),\quad (x,y)\in[x_1,x_0]\times[x_1\tan\vartheta_0,f(x_1)].
$$

Owing to (4.1)-(4.5) for w_1 and w_2, w satisfies

$$
\begin{aligned}
&\left(\frac{\Phi_{\text{up},1}(x_0)l_1(x)}{(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))h_1(y)}\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))h_1(y)}{\Phi_{\text{up},1}(x_0)l_1(x)}K'(w_1)\frac{\partial w}{\partial y}\right)\\
&\quad+\frac{\partial}{\partial x}\left(\frac{((\Phi_{\text{up},1}(x_0)-\Phi_{\text{up},1}(x))\Phi'_{\text{in},1}(y))^2h_1(y)l_1(x)}{\Phi_{\text{up},1}(x_0)(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))}K'(w_1)\frac{\partial w}{\partial x}\right)\\
&\quad-\frac{\partial}{\partial x}\left(\frac{\Phi_{\text{up},2}(x_0)l_2(x)}{(\Phi_{\text{up},2}(x_0)-\Phi_{\text{in},2}(y))h_2(y)}-\frac{\Phi_{\text{up},1}(x_0)l_1(x)}{(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))h_1(y)}\right)\frac{\partial^2 w_2}{\partial x^2}\\
&\quad+\frac{\partial E_1}{\partial x}(x,y)+\frac{\partial E_2}{\partial y}(x,y)=0,\qquad (x,y)\in(x_1,x_0)\times(x_1\tan\vartheta_0,f(x_1)),
\end{aligned}
\tag{4.6}
$$

and

$$
\frac{\partial w}{\partial y}(x,x_1\tan\vartheta_0)=0,\qquad x\in(x_1,x_0), \tag{4.7}
$$

$$
\begin{aligned}
&\frac{h_1(f(x_1))}{l_1(x)}\frac{\partial K(w_1)}{\partial y}(x,f(x_1))\\
=&\frac{h_2(f(x_1))}{l_2(x)}\frac{\partial K(w_2)}{\partial y}(x,f(x_1))=\Theta'_{\text{up}}(x),\qquad x\in(x_1,x_0),
\end{aligned}
\tag{4.8}
$$

$$
\frac{\Phi_{\text{up},1}(x_0)l_1(x_1)(1+(\Phi'_{\text{in},1}(y)h_1(y))^2K'(w_1)(x_1,y))}{(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))h_1(y)}\frac{\partial w_1}{\partial x}(x_1,y)
$$

$$
\begin{aligned}
&\quad -\Phi'_{\mathrm{in},1}(y)h_1(y)\frac{\partial K(w_1)}{\partial y}(x_1,y)\\
&=\frac{\Phi_{\mathrm{up},2}(x_0)l_2(x_1)(1+(\Phi'_{\mathrm{in},2}(y)h_2(y))^2K'(w_2)(x_1,y))}{(\Phi_{\mathrm{up},2}(x_0)-\Phi_{\mathrm{in},2}(y))h_2(y)}\frac{\partial w_2}{\partial x}(x_1,y)\\
&\quad -\Phi'_{\mathrm{in},2}(y)h_2(y)\frac{\partial K(w_2)}{\partial y}(x_1,y)\\
&=-\Theta'_{\mathrm{in}}(y), \qquad y\in(x_1\tan\vartheta_0,f(x_1)), \qquad (4.9)\\
&w(x_0,y)=0, \qquad y\in(x_1\tan\vartheta_0,f(x_1)), \qquad (4.10)
\end{aligned}
$$

where

$$
\begin{aligned}
E_1(x,y)=&\frac{((\Phi_{\mathrm{up},1}(x_0)-\Phi_{\mathrm{up},1}(x))\Phi'_{\mathrm{in},1}(y))^2h_1(y)l_1(x)}{\Phi_{\mathrm{up},1}(x_0)(\Phi_{\mathrm{up},1}(x_0)-\Phi_{\mathrm{in},1}(y))}p_1(x,y)w(x,y)\\
&+\left(\frac{((\Phi_{\mathrm{up},1}(x_0)-\Phi_{\mathrm{up},1}(x))\Phi'_{\mathrm{in},1}(y))^2}{\Phi_{\mathrm{up},1}(x_0)(\Phi_{\mathrm{up},1}(x_0)-\Phi_{\mathrm{in},1}(y))}\right.\\
&\qquad\left.-\frac{((\Phi_{\mathrm{up},2}(x_0)-\Phi_{\mathrm{up},2}(x))\Phi'_{\mathrm{in},2}(y))^2}{\Phi_{\mathrm{up},2}(x_0)(\Phi_{\mathrm{up},2}(x_0)-\Phi_{\mathrm{in},2}(y))}\right)\\
&\times h_1(y)l_1(x)\frac{\partial K(w_2)}{\partial x}(x,y)\\
&+\frac{((\Phi_{\mathrm{up},2}(x_0)-\Phi_{\mathrm{up},2}(x))\Phi'_{\mathrm{in},2}(y))^2}{\Phi_{\mathrm{up},2}(x_0)(\Phi_{\mathrm{up},2}(x_0)-\Phi_{\mathrm{in},2}(y))}\\
&\times\big(h_1(y)l_1(x)-h_2(y)l_2(x)\big)\frac{\partial K(w_2)}{\partial x}(x,y)\\
&-\frac{(\Phi_{\mathrm{up},1}(x_0)-\Phi_{\mathrm{up},1}(x))\Phi'_{\mathrm{in},1}(y)h_1(y)}{\Phi_{\mathrm{up},1}(x_0)}K'(w_1)(x,y)\frac{\partial w}{\partial y}(x,y)\\
&-\frac{(\Phi_{\mathrm{up},1}(x_0)-\Phi_{\mathrm{up},1}(x))\Phi'_{\mathrm{in},1}(y)h_1(y)}{\Phi_{\mathrm{up},1}(x_0)}p_2(x,y)w(x,y)\\
&-\frac{\Phi_{\mathrm{up},1}(x_0)\Phi_{\mathrm{up},2}(x)-\Phi_{\mathrm{up},2}(x_0)\Phi_{\mathrm{up},1}(x)}{\Phi_{\mathrm{up},1}(x_0)\Phi_{\mathrm{up},2}(x_0)}\Phi'_{\mathrm{in},1}(y)h_1(y)\frac{\partial K(w_2)}{\partial y}(x,y)\\
&-\frac{\big(\Phi_{\mathrm{up},2}(x_0)-\Phi_{\mathrm{up},2}(x)\big)}{\Phi_{\mathrm{up},2}(x_0)}\big(\Phi'_{\mathrm{in},1}(y)h_1(y)-\Phi'_{\mathrm{in},2}(y)h_2(y)\big)\frac{\partial K(w_2)}{\partial y}(x,y),
\end{aligned}
$$

$$
E_2(x,y)=\frac{(\Phi_{\mathrm{up},1}(x_0)-\Phi_{\mathrm{in},1}(y))h_1(y)}{\Phi_{\mathrm{up},1}(x_0)l_1(x)}p_2(x,y)w(x,y)
$$

$$
\begin{aligned}
&+\left(\frac{\Phi_{\mathrm{up},1}(x_0)-\Phi_{\mathrm{in},1}(y)}{\Phi_{\mathrm{up},1}(x_0)}-\frac{\Phi_{\mathrm{up},2}(x_0)-\Phi_{\mathrm{in},2}(y)}{\Phi_{\mathrm{up},2}(x_0)}\right)\frac{h_1(y)}{l_1(x)}\frac{\partial K(w_2)}{\partial y}(x,y)\\
&+\frac{\Phi_{\mathrm{up},2}(x_0)-\Phi_{\mathrm{in},2}(y)}{\Phi_{\mathrm{up},2}(x_0)}\left(\frac{h_1(y)}{l_1(x)}-\frac{h_2(y)}{l_2(x)}\right)\frac{\partial K(w_2)}{\partial x}(x,y)\\
&-\frac{(\Phi_{\mathrm{up},1}(x_0)-\Phi_{\mathrm{up},1}(x))\Phi'_{\mathrm{in},1}(y)h_1(y)}{\Phi_{\mathrm{up},1}(x_0)}K'(w_1)(x,y)\frac{\partial w}{\partial x}(x,y)\\
&-\frac{(\Phi_{\mathrm{up},1}(x_0)-\Phi_{\mathrm{up},1}(x))\Phi'_{\mathrm{in},1}(y)h_1(y)}{\Phi_{\mathrm{up},1}(x_0)}p_1(x,y)w(x,y)\\
&-\frac{\Phi_{\mathrm{up},1}(x_0)\Phi_{\mathrm{up},2}(x)-\Phi_{\mathrm{up},2}(x_0)\Phi_{\mathrm{up},1}(x)}{\Phi_{\mathrm{up},1}(x_0)\Phi_{\mathrm{up},2}(x_0)}\Phi'_{\mathrm{in},1}(y)h_1(y)\frac{\partial K(w_2)}{\partial x}(x,y)\\
&-\frac{(\Phi_{\mathrm{up},2}(x_0)-\Phi_{\mathrm{up},2}(x))}{\Phi_{\mathrm{up},2}(x_0)}\left(\Phi'_{\mathrm{in},1}(y)h_1(y)-\Phi'_{\mathrm{in},2}(y)h_2(y)\right)\frac{\partial K(w_2)}{\partial x}(x,y),
\end{aligned}
$$

$$
p_1(x,y)=\frac{\partial w_2}{\partial x}(x,y)\int_0^1 K''\big(\lambda w_1(x,y)+(1-\lambda)w_2(x,y)\big)d\lambda,
$$

$$
p_2(x,y)=\frac{\partial w_2}{\partial y}(x,y)\int_0^1 K''\big(\lambda w_1(x,y)+(1-\lambda)w_2(x,y)\big)d\lambda
$$

with $(x,y)\in[x_1,x_0]\times[x_1\tan\vartheta_0,f(x_1)]$. From the definition of $\mathscr{C}$, direct calculations show that for $x\in[x_1,x_0]$, $y\in[x_1\tan\vartheta_0,f(x_1)]$,

$$
A^{-1}(w_k(x_1,y))=q_0+O\big(\tau(\delta)\big),\qquad k=1,2, \tag{4.11}
$$

$$
A^{-1}\big(w_k(x,f(x_1))\big)=\hat{Q}_{\mathrm{up}}(x)+O\big(\tau(\delta)\big),\qquad k=1,2, \tag{4.12}
$$

$$
\cos\Theta_{\mathrm{in}}(y)-\sin\Theta_{\mathrm{in}}(y)g'(y)=\frac{-x_1}{\sqrt{x_1^2-y^2\cos^2\vartheta_0}}+O\big(\tau(\delta)\big), \tag{4.13}
$$

$$
|\Theta_{\mathrm{up}}(x)+\vartheta_0|\le O\big(\tau(\delta)\big),\quad |\Theta'_{\mathrm{up}}(x)|\le O\big(\tau(\delta)\big)(x_0-x)^{1/2}, \tag{4.14}
$$

$$
h_k(y)=\frac{1}{q_0\rho(q_0^2)(\cos\Theta_{\mathrm{in}}(y)-\sin\Theta_{\mathrm{in}}(y)g'(y))}+O\big(\tau(\delta)\big),\qquad k=1,2, \tag{4.15}
$$

$$
h_2(y)-h_1(y)=\left(\frac{1}{q_0(\cos\Theta_{\mathrm{in}}(y)-\sin\Theta_{\mathrm{in}}(y)g'(y))}+O\big(\tau(\delta)\big)\right)w(x_1,y), \tag{4.16}
$$

$$
l_k(x)=\frac{\cos\Theta_{\mathrm{up}}}{\hat{Q}_{\mathrm{up}}(x)}+O\big(\tau(\delta)\big),\qquad k=1,2, \tag{4.17}
$$

$$
l_2(x)-l_1(x)=\left(\frac{\cos\Theta_{\mathrm{up}}}{\hat{Q}_{\mathrm{up}}^2(x)A'(\hat{Q}_{\mathrm{up}}(x))}+O\big(\tau(\delta)\big)\right)w\big(x,f(x_1)\big), \tag{4.18}
$$

$$A'\big(\hat{Q}_{\rm up}(x)\big) \geq \varsigma_1 \left(\int_x^{x_0} \frac{\hat{Q}_{\rm up}(s)}{\cos\vartheta_0} ds \right)^{1/2}, \tag{4.19}$$

$$\Phi_{\rm in,k}(y) = O\big(\tau(\delta)\big), \quad \Phi'_{\rm in,k}(y) = O\big(\tau(\delta)\big), \qquad k=1,2, \tag{4.20}$$

$$|\Phi_{\rm in,1}(y) - \Phi_{\rm in,2}(y)| \leq O\big(\tau(\delta)\big) \int_{x_1\tan\vartheta_0}^{f(x_1)} |w(x_1,y)| dy, \tag{4.21}$$

$$|\Phi'_{\rm in,1}(y) - \Phi'_{\rm in,2}(y)| \leq O\big(\tau(\delta)\big) |w(x_1,y)|, \tag{4.22}$$

$$\begin{aligned} &|\Phi_{\rm up,1}(x) - \Phi_{\rm up,2}(x)| \\ \leq & \int_x^{x_0} \left(\frac{1}{\cos\vartheta_0 A'(\hat{Q}_{\rm up}(x))} + O\big(\tau(\delta)\big) \right) |w(s,f(x_1))| ds, \end{aligned} \tag{4.23}$$

$$K'\big(w_1(x,y)\big) \geq \varsigma_2 (x_0 - x)^{-1/2} + O\big(\tau(\delta)\big), \tag{4.24}$$

$$\left| \frac{\partial w_2}{\partial x}(x,y) \right| \leq \frac{O(1)\hat{Q}_{\rm up}(x)}{\cos\vartheta_0}, \qquad \left| \frac{\partial K(w_2)}{\partial y}(x,y) \right| \leq O\big(\tau(\delta)\big), \tag{4.25}$$

$$|p_1(x,y)| \leq O(1)(x_0 - x)^{-3/2}, \quad |p_2(x,y)| \leq O\big(\tau(\delta)\big)(x_0 - x)^{-1/2}, \tag{4.26}$$

and for $(x,y) \in [x_1, x_0 - \varepsilon] \times [x_1 \tan\vartheta_0, f(x_1)]$,

$$\frac{\partial w_2}{\partial x}(x,y) = -\frac{\hat{Q}_{\rm up}(x)}{x_1 q_0 \rho(q_0^2)} + O\big(\tau_\varepsilon(\delta)\big), \tag{4.27}$$

where $0 < \varepsilon < \min\{1, x_0 - x_1\}$, ς_1 and ς_2 are given by (2.30) and (2.31), respectively, while $\tau(\cdot)$ and $\tau_\varepsilon(\cdot)$ are given in the definition of $\mathscr{C}$. Owing to (4.10), one gets from the Hölder inequality that

$$\begin{aligned} w^2(x,y) &= \left(\int_x^{x_0} \frac{\partial w}{\partial \tilde{x}}(\tilde{x},y) d\tilde{x} \right)^2 \\ &\leq \int_x^{x_0} \frac{\hat{Q}_{\rm up}(\tilde{x})}{\cos\vartheta_0} d\tilde{x} \int_x^{x_0} \frac{\cos\vartheta_0}{\hat{Q}_{\rm up}(\tilde{x})} \left(\frac{\partial w}{\partial \tilde{x}}(\tilde{x},y) \right)^2 d\tilde{x} \\ &\leq \hat{\varphi}_1 \int_{x_1}^{x_0} \frac{\cos\vartheta_0}{\hat{Q}_{\rm up}(\tilde{x})} \left(\frac{\partial w}{\partial \tilde{x}}(\tilde{x},y) \right)^2 d\tilde{x}, \ \ (x,y) \in [x_1, x_0] \times [x_1 \tan\vartheta_0, f(x_1)], \end{aligned} \tag{4.28}$$

$$\begin{aligned} w^2\big(x, f(x_1)\big) \leq & \frac{1}{f(x_1) - x_1 \tan\vartheta_0} \int_{x_1\tan\vartheta_0}^{f(x_1)} w^2(x,y) dy \\ & + 2 \int_{x_1\tan\vartheta_0}^{f(x_1)} \left| w(x,y) \frac{\partial w}{\partial y}(x,y) \right| dy, \quad x \in [x_1, x_0]. \end{aligned} \tag{4.29}$$

Multiplying (4.6) by $-w$ and then integrating over $(x_1,x_0)\times(x_1\tan\vartheta_0,f(x_1))$ by parts and using (4.7)-(4.10), one gets that

$$
\begin{aligned}
&\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{\Phi_{\text{up},1}(x_0)l_1(x)}{(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))h_1(y)}\left(\frac{\partial w}{\partial x}(x,y)\right)^2dxdy\\
&\quad+\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))h_1(y)}{\Phi_{\text{up},1}(x_0)l_1(x)}K'(w_1(x,y))\left(\frac{\partial w}{\partial y}(x,y)\right)^2dxdy\\
&\quad+\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{((\Phi_{\text{up},1}(x_0)-\Phi_{\text{up},1}(x))\Phi'_{\text{in},1}(y))^2h_1(y)l_1(x)}{\Phi_{\text{up},1}(x_0)(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))}\\
&\qquad\qquad\qquad\qquad\times K'(w_1(x,y))\left(\frac{\partial w}{\partial x}(x,y)\right)^2dxdy\\
=&\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\left(\frac{\Phi_{\text{up},2}(x_0)l_2(x)}{(\Phi_{\text{up},2}(x_0)-\Phi_{\text{in},2}(y))h_2(y)}-\frac{\Phi_{\text{up},1}(x_0)l_1(x)}{(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))h_1(y)}\right)\\
&\qquad\qquad\qquad\qquad\times\frac{\partial w_2}{\partial x}(x,y)\frac{\partial w}{\partial x}(x,y)dxdy\\
&\quad-\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\left(E_1(x,y)\frac{\partial w}{\partial x}(x,y)+E_2(x,y)\frac{\partial w}{\partial y}(x,y)\right)dxdy. \qquad (4.30)
\end{aligned}
$$

Each term in (4.30) will be estimated by using (4.11)-(4.29). First, it follows from (4.15), (4.17), (4.20) and (4.25) that

$$
\begin{aligned}
&\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{\Phi_{\text{up},1}(x_0)l_1(x)}{(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))h_1(y)}\left(\frac{\partial w}{\partial x}(x,y)\right)^2dxdy\\
&\quad+\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))h_1(y)}{\Phi_{\text{up},1}(x_0)l_1(x)}K'(w_1(x,y))\left(\frac{\partial w}{\partial y}(x,y)\right)^2dxdy\\
&\quad+\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{((\Phi_{\text{up},1}(x_0)-\Phi_{\text{up},1}(x))\Phi'_{\text{in},1}(y))^2h_1(y)l_1(x)}{\Phi_{\text{up},1}(x_0)(\Phi_{\text{up},1}(x_0)-\Phi_{\text{in},1}(y))}\\
&\qquad\qquad\qquad\qquad\times K'(w_1(x,y))\left(\frac{\partial w}{\partial x}(x,y)\right)^2dxdy\\
\geq&\left(q_0\rho(q_0^2)\cos\vartheta_0+O(\tau(\delta))\right)L_1+\left(\frac{\varsigma_2}{q_0\rho(q_0^2)\cos\vartheta_0}+O(\tau(\delta))\right)L_2, \qquad (4.31)
\end{aligned}
$$

where

$$
L_1=\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{\cos\Theta_{\text{in}}(y)-\sin\Theta_{\text{in}}(y)g'(y)}{\hat{Q}_{\text{up}}(x)}\left(\frac{\partial w}{\partial x}(x,y)\right)^2dxdy, \qquad (4.32)
$$

$$L_2=\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{(x_0-x)^{-1/2}\hat{Q}_{\rm up}(x)}{\cos\Theta_{\rm in}(y)-\sin\Theta_{\rm in}(y)g'(y)}\left(\frac{\partial w}{\partial y}(x,y)\right)^2dxdy. \tag{4.33}$$

Second, consider the first term on the right side of (4.30). Owing to (4.11)-(4.29) and the Hölder inequality, through a series of precise estimations similar to those in [16,20,23], it is obtained that

$$\begin{aligned}
&\left|\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\left(\frac{\Phi_{{\rm up},2}(x_0)l_2(x)}{(\Phi_{{\rm up},2}(x_0)-\Phi_{{\rm in},2}(y))h_2(y)}-\frac{\Phi_{{\rm up},1}(x_0)l_1(x)}{(\Phi_{{\rm up},1}(x_0)-\Phi_{{\rm in},1}(y))h_1(y)}\right)\right.\\
&\qquad\left.\times\frac{\partial w_2}{\partial x}(x,y)\frac{\partial w}{\partial x}(x,y)dxdy\right|\\
=&\left|\int_{x_1}^{x_0-\varepsilon}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{(\Phi_{{\rm up},1}(x_0)\Phi_{{\rm in},2}(y))-\Phi_{{\rm up},2}(x_0)\Phi_{{\rm in},1}(y))}{(\Phi_{{\rm up},1}(x_0)-\Phi_{{\rm in},1}(y))(\Phi_{{\rm up},2}(x_0)-\Phi_{{\rm in},2}(y))}\right.\\
&\qquad\times\frac{l_2(x)}{h_2(y)}\frac{\partial w_2}{\partial x}(x,y)\frac{\partial w}{\partial x}(x,y)dxdy\\
&+\int_{x_0-\varepsilon}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{(\Phi_{{\rm up},1}(x_0)\Phi_{{\rm in},2}(y))-\Phi_{{\rm up},2}(x_0)\Phi_{{\rm in},1}(y))}{(\Phi_{{\rm up},1}(x_0)-\Phi_{{\rm in},1}(y))(\Phi_{{\rm up},2}(x_0)-\Phi_{{\rm in},2}(y))}\\
&\qquad\times\frac{l_2(x)}{h_2(y)}\frac{\partial w_2}{\partial x}(x,y)\frac{\partial w}{\partial x}(x,y)dxdy\\
&+\int_{x_1}^{x_0-\varepsilon}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{\Phi_{{\rm up},1}(x_0)}{\Phi_{{\rm up},1}(x_0)-\Phi_{{\rm in},1}(y)}\left(\frac{l_2(x)}{h_2(y)}-\frac{l_1(x)}{h_1(y)}\right)\\
&\qquad\times\frac{\partial w_2}{\partial x}(x,y)\frac{\partial w}{\partial x}(x,y)dxdy\\
&+\int_{x_0-\varepsilon}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{\Phi_{{\rm up},1}(x_0)}{\Phi_{{\rm up},1}(x_0)-\Phi_{{\rm in},1}(y)}\left(\frac{l_2(x)}{h_2(y)}-\frac{l_1(x)}{h_1(y)}\right)\\
&\qquad\left.\times\frac{\partial w_2}{\partial x}(x,y)\frac{\partial w}{\partial x}(x,y)dxdy\right|\\
\le&\left(\frac{\rho(q_0^2)\hat{\varphi}_1\cos^2\vartheta_0}{-x_1}+\nu_1+\frac{(x_0-x_1)\vartheta_0\cos^2\vartheta_0}{4\nu_1x_1^2\varsigma_1^2q_0\tan\vartheta_0}+\frac{2(x_0-x_1)^{1/2}\vartheta_0^2\cos^2\vartheta_0}{\nu_1\nu_2\varsigma_1^4q_0^4}\right)L_1\\
&\qquad+\big(O(\varepsilon)+O\big(\tau(\delta)\big)\big)L_1+\left(\frac{\nu_2\cos\vartheta_0}{4\nu_1x_1^2}+O\big(\tau(\delta)\big)\right)L_2,
\end{aligned}\tag{4.34}$$

where $\varepsilon\in(0,\min\{1,x_0-x_1\})$, ν_1, $\nu_2>0$ will be determined. Then, for the second term on the right-hand side of (4.30), one can get that

$$
\begin{aligned}
&\left|\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\left(E_1(x,y)\frac{\partial w}{\partial x}(x,y)+E_2(x,y)\frac{\partial w}{\partial y}(x,y)\right)dxdy\right| \\
&\leq O\big(\tau(\delta)\big)(L_1+L_2).
\end{aligned}
\tag{4.35}
$$

Substitute (4.31), (4.34), (4.35) into (4.30) to get

$$
\begin{aligned}
&\Big(q_0\rho(q_0^2)\cos\vartheta_0+O\big(\tau(\delta)\big)\Big)L_1+\left(\frac{\varsigma_2}{q_0\rho(q_0^2)\cos\vartheta_0}+O\big(\tau(\delta)\big)\right)L_2 \\
\leq&\left(\frac{\rho(q_0^2)\hat{\varphi}_1\cos^2\vartheta_0}{-x_1}+\nu_1+\frac{(x_0-x_1)\vartheta_0\cos^2\vartheta_0}{4\nu_1x_1^2\varsigma_1^2q_0\tan\vartheta_0}+\frac{2(x_0-x_1)^{1/2}\vartheta_0^2\cos^2\vartheta_0}{\nu_1\nu_2\varsigma_1^4q_0^4}\right)L_1 \\
&+\big(O(\varepsilon)+O\big(\tau(\delta)\big)\big)L_1+\left(\frac{\nu_2\cos\vartheta_0}{4\nu_1x_1^2}+O\big(\tau(\delta)\big)\right)L_2.
\end{aligned}
$$

Choose ν_1 and ν_2 to satisfy (2.28) and (2.29). Thus, there exist $\varepsilon\in(0,\min\{1,x_0-x_1\})$ and $\delta_2>0$ depending only on γ, ϑ_0, x_0, x_1, $\underline{M}$, $\overline{M}$, $\tau(\cdot)$ and $\tau_\varepsilon(\cdot)$ such that for any $0<\delta\leq\delta_2$,

$$
\begin{aligned}
&\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{\cos\Theta_{\rm in}(y)-\sin\Theta_{\rm in}(y)g'(y)}{\hat{Q}_{\rm up}(x)}\left(\frac{\partial w}{\partial x}(x,y)\right)^2dxdy \\
&+\int_{x_1}^{x_0}\int_{x_1\tan\vartheta_0}^{f(x_1)}\frac{(x_0-x)^{-1/2}\hat{Q}_{\rm up}(x)}{\cos\Theta_{\rm in}(y)-\sin\Theta_{\rm in}(y)g'(y)}\left(\frac{\partial w}{\partial y}(x,y)\right)^2dxdy\leq 0.
\end{aligned}
$$

This, together with (4.10), shows that

$$
w(x,y)=0,\quad (x,y)\in[x_1,x_0]\times[x_1\tan\vartheta_0,f(x_1)].
$$

Therefore, $(q_1,\varphi_{1,1},m_{{\rm in},1})=(q_2,\varphi_{1,2},m_{{\rm in},2})$. □

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