

Continuous Subsonic-Sonic Flows in Curved Convergent Nozzles with a Fixed Point

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Abstract. This paper concerns continuous subsonic-sonic potential flows in a two-dimensional, convergent nozzle, which is governed by a free boundary problem of a quasilinear degenerate elliptic equation. It is shown that for a given nozzle perturbed from a straight one, a given point on its wall where the curvature is zero, a given inlet which is a perturbation of an arc centered at the vertex, and a given incoming flow angle perturbed from the angle of the inner normal of the inlet, there exists uniquely a continuous subsonic-sonic flow whose velocity vector is along the normal direction at the sonic curve, which satisfies the slip conditions on the nozzle walls and whose sonic curve intersects the upper wall at the given point. Furthermore, the sonic curve of this flow is a free boundary, where the flow is singular in the sense that the speed is only $C^{1/2}$ Hölder continuous and the acceleration blows up. The perturbation problem is solved in the potential plane, where the flow is governed by a free boundary problem of a degenerate elliptic equation with three free boundaries and two nonlocal boundary conditions, and the equation is degenerate at one free boundary.

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1 Introduction

This paper concerns subsonic-sonic flows in curved convergent nozzles. Such problems naturally arise in physical experiments and engineering designs, and

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there are many experiments and numerical simulations and rigorous theories involved in this field [2,6]. A two-dimensional steady isentropic inviscid compressible flow is governed by the following Euler system:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \quad (1.1)$$

$$\frac{\partial}{\partial x}(P + \rho u^2) + \frac{\partial}{\partial y}(\rho uv) = 0, \quad (1.2)$$

$$\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(P + \rho v^2) = 0, \quad (1.3)$$

where (u, v) , P and ρ represent the velocity, pressure and density of the flow, respectively, and $P = P(\rho)$ is a smooth function. In particular, for a polytropic gas with adiabatic exponent $\gamma > 1$,

$$P(\rho) = \frac{1}{\gamma} \rho^\gamma \quad (1.4)$$

is the normalized pressure. Assume further that the flow is irrotational, i.e.

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \quad (1.5)$$

Then, the density ρ is expressed in terms of the speed q according to the Bernoulli law [2]

$$\rho(q^2) = \left(1 - \frac{\gamma-1}{2} q^2\right)^{1/(\gamma-1)}, \quad q = \sqrt{u^2 + v^2}, \quad 0 < q < \sqrt{2/(\gamma-1)}. \quad (1.6)$$

The sound speed c is defined as $c^2 = P'(\rho)$. At the sonic state, the sound speed is

$$c_* = \left(\frac{2}{\gamma+1}\right)^{1/2},$$

which is called the critical speed in the sense that the flow is subsonic when $q < c_*$, sonic when $q = c_*$, and supersonic when $q > c_*$. The system (1.1)-(1.6) can be transformed into the full potential equation

$$\operatorname{div}(\rho(|\nabla\varphi|^2)\nabla\varphi) = 0, \quad (1.7)$$

where φ is the velocity potential with $\nabla\varphi = (u, v)$, and ρ is defined by (1.6). It is noted that (1.7) is elliptic in the subsonic region, degenerate at the sonic state, while hyperbolic in the supersonic region.