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Asymptotic Estimates for the Ruin Probability of a Multidimensional Delay-Claim Risk Model with Dependent Claims

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Abstract. This paper studies a multidimensional delay-claim risk model in which an insurance company operates d $(d \geq 2)$ lines of business exposed to a common renewal counting process. Each catastrophic event simultaneously produces main and delayed claims across all business lines, where the delayed claims are settled after random delay periods. The surplus process incorporates a geometric Lévy price process to describe investment returns. Assuming that the main and delayed claims follow subexponential distributions and satisfy a conditional linear dependence structure, we derive asymptotic estimates for the finite-time ruin probability. The obtained results extend existing findings on delay-claim models to the multidimensional framework and contribute to a deeper understanding of ruin behavior under dependence and heavy-tailed risks.

AMS subject classifications: 62P05, 62E20, 91G05

Key words: Asymptotics, multidimensional risk model, dependence structure, delayed claim, subexponential class.

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1 Introduction

Nowadays, insurance companies typically manage multiple lines of business concurrently, such as health, motor, and homeowner's insurance. Since a single

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catastrophic event can give rise to claims across several lines, risk models have been developed to capture such interdependencies. In particular, models incorporating both main and delayed claims have been extensively investigated. Suppose an insurer operates d $(d\geq 2)$ lines of business simultaneously. A catastrophic event may generate a main claim in each line, which is settled immediately, while the associated delayed claim is processed after a random period. For instance, a car accident may result in property damage in multiple lines, leading to immediate claims, whereas compensation for personal injuries may be deferred, introducing dependencies between the main and delayed claims.

In this paper, we consider a multidimensional delay-claim risk model where the surplus process can be described as follows:

$$\begin{aligned}U(t)=\begin{pmatrix}U_1(t)\\U_2(t)\\\vdots\\U_d(t)\end{pmatrix}&=\begin{pmatrix}x_1e^{L(t)}\\x_2e^{L(t)}\\\vdots\\x_de^{L(t)}\end{pmatrix}+\begin{pmatrix}\int_0^t c_1(s)e^{L(t-s)}ds\\\int_0^t c_2(s)e^{L(t-s)}ds\\\vdots\\\int_0^t c_d(s)e^{L(t-s)}ds\end{pmatrix}-\begin{pmatrix}\sum_{i=1}^{N(t)}X_{1,i}e^{L(t-\tau_i)}\\\sum_{i=1}^{N(t)}X_{2,i}e^{L(t-\tau_i)}\\\vdots\\\sum_{i=1}^{N(t)}X_{d,i}e^{L(t-\tau_i)}\end{pmatrix}\\&\quad-\begin{pmatrix}\sum_{i=1}^{N(t)}Y_{1,i}e^{L(t-\tau_i-D_{1,i})}I_{\{\tau_i+D_{1,i}\leq t\}}\\\sum_{i=1}^{N(t)}Y_{2,i}e^{L(t-\tau_i-D_{2,i})}I_{\{\tau_i+D_{2,i}\leq t\}}\\\vdots\\\sum_{i=1}^{N(t)}Y_{d,i}e^{L(t-\tau_i-D_{d,i})}I_{\{\tau_i+D_{d,i}\leq t\}}\end{pmatrix},\quad t\geq 0,\end{aligned}\tag{1.1}$$

where $(x_1,\ldots,x_d)^\top$ denotes the vector of the initial reserve and $c_k(t)\geq 0$ is the density function of premium income at time t for $k=1,\ldots,d$. Assume $\{(X_{1,i},\ldots,X_{d,i});i\in\mathbb{N}_+\}$ denotes the i-th main claims of the business lines occurring simultaneously at time τ_i. For $k=1,\ldots,d$ and $i\in\mathbb{N}_+$, each main claim $X_{k,i}$ is associated with a delayed claim $Y_{k,i}$ occurring at time $\tau_i+D_{k,i}$, where $D_{k,i}$ denotes an uncertain delay time. Let $\{\theta_i;i\in\mathbb{N}_+\}$ be a sequence of nonnegative, independent, and identically distributed (i.i.d.) random variables representing claim inter-arrival times. The arrival times of the main claims $\tau_i=\sum_{j=1}^i\theta_j$, $i\in\mathbb{N}_+$ constitute a counting process $\{N(t);t\geq 0\}$ which is a renewal process with a finite renewal function

$$\lambda(t)=\mathbb{E}N(t)=\sum_{i=1}^{\infty}\mathbb{P}(\tau_i\leq t).$$

Additionally, for each $k=1,\ldots,d$, $\{D_{k,i};i\in\mathbb{N}_+\}$ is a sequence of nonnegative (possibly degenerate at 0) i.i.d. random variables with a generic random variable D_k with distribution H_k, and the price process of the investment portfolio is described as a geometric Lévy process $\{e^{L(t)};t\geq 0\}$ with $\{L(t);t\geq 0\}$ being a non-negative Lévy process which starts from 0 and has independent and stationary increments.

For the multidimensional risk model (1.1) described above, we define the finite-time ruin probability for any $T>0$ as follows:

$$\Psi(x_1,\ldots,x_d;T)=\mathbb{P}\left(T_{\max}\leq T\,|\,(U_1(0),\ldots,U_d(0))^\top=(x_1,\ldots,x_d)^\top\right),$$

where

$$T_{\max}=\inf\{t>0:\max\{U_1(t),\ldots,U_d(t)\}<0\}.$$

Here, $T_{\max}$ represents the first moment that the surplus of all lines of business simultaneously falls below zero.

The asymptotic analysis of ruin probabilities in multidimensional risk models has become an active area of research in risk theory (see [7, 9, 12, 13, 16], among others). In recent years, Fu and Liu [5] examined a model characterized by a common non-stationary and non-renewal arrival process. Yang and Su [26] explored the asymptotic behavior of both finite-time and infinite-time ruin probabilities within a multidimensional framework based on multivariate regular variation. Subsequently, Cheng *et al.* [3] introduced a new dependence structure formulated through multivariate regular variation and derived asymptotic results for multidimensional ruin probabilities in both discrete and continuous-time settings. While the study of multidimensional ruin models brings theoretical results closer to insurance practice, research on multidimensional models incorporating dependent delayed claims remains relatively limited.

The delay-claim risk model was originally introduced by Waters and Papatriandafylou [20]. Subsequent studies, such as [21, 27], examined its discrete-time versions. For continuous-time extensions, relevant works include [22, 28, 29], among others. Since catastrophic events are better represented by heavy-tailed rather than light-tailed claim distributions, Li [11] investigated the delay-claim model under the assumption of heavy-tailed, pairwise quasi-asymptotically independent claims. More recently, Wang *et al.* [19] analyzed a bidimensional delayed renewal risk model in which the main claims of two business lines share a common renewal counting process and derived an asymptotic expression for the finite-time ruin probability. Building on this, Yang *et al.* [24] further considered dependent structures where the main and delayed claim pairs follow bivariate Farlie-Gumbel-Morgenstern distributions. Sun *et al.* [17] studied the asymptotic behaviors for some common ruin probabilities of a bidimensional delay-claim risk model, in which each claim follows a subexponential distribution and each claim-size inherits some dependence structure.

Building on the above studies, this paper investigates a multidimensional delay-claim risk model with a dependent claim structure. In this model, a common renewal process simultaneously triggers main claims across multiple business lines, while each line experiences its own delayed claims thereafter. The surplus evolves under a geometric Lévy investment process. Assuming that both main and delayed claims follow subexponential distributions and satisfy a conditional linear dependence structure, we derive asymptotic estimates for the finite-time ruin probability. The results extend existing delay-claim models to the multidimensional setting and highlight the effect of dependence and heavy-tailed risks on ruin behavior.

The remainder of this paper is structured as follows. Section 2 provides the necessary preliminaries, including basic definitions and assumptions. Section 3 establishes the main asymptotic results and corresponding proofs for the theorem.

2 Preliminaries and assumptions

2.1 Preliminaries

Throughout this paper, all limit relations hold as $x\to\infty$ unless stated otherwise. For two positive functions f and g, we write $f(x)\lesssim g(x)$ if $\limsup f(x)/g(x)\le 1$, $f(x)\sim g(x)$ if $\lim f(x)/g(x)=1$, $f(x)=o(g(x))$ if $\lim f(x)/g(x)=0$, and $f(x)\asymp g(x)$ if $0<\liminf f(x)/g(x)<\limsup f(x)/g(x)<\infty$.

A distribution F on $[0,\infty)$ is said to belong to the subexponential distribution class, denoted by $F\in\mathcal{S}$, if $\overline{F}(x)=1-F(x)>0$ for all $x\ge 0$ and the relation

$$\lim\frac{\overline{F^{n*}}(x)}{\overline{F}(x)}=n$$

holds for all (or, equivalently, for some) $n\ge 2$, where F^{n*} is the n-fold convolution of F with itself. Furthermore, [4, Lemma 1.3.5] tells that if $F\in\mathcal{S}$, then $F\in\mathcal{L}$, which stands for the class of long-tailed distributions defined as $\overline{F}(x)>0$ for all $x\ge 0$ and the relation

$$\lim\frac{\overline{F}(x+z)}{\overline{F}(x)}=1,\quad z\in(-\infty,\infty)$$

holds.

In this paper, we also require an additional class of distributions, denoted by $\mathcal{A}$, which was introduced in [10]. A distribution V supported on $(-\infty,\infty)$ is said to belong to the class $\mathcal{A}$ if it is subexponential and its right tail satisfies

$$\overline{V}^*(y)<1, \quad \text{for some} \quad y>1, \tag{2.1}$$

where $\overline{V}^*(y)=\limsup\overline{V}(xy)/\overline{V}(x)$. As pointed out in [10], the class $\mathcal{A}$ includes almost all well-known subexponential distributions. Further properties and discussions of the class $\mathcal{A}$ can be found in [10,18].

2.2 Assumptions

In practice, various dependence structures are employed to model relationships among claim sizes or claim inter-arrival times. For recent studies of ruin estimation under some dependence structure, we refer to pairwise (strong) quasi-asymptotically independence as discussed in [14], bivariate Farlie-Gumbel-Morgenstern distribution as presented in [23], and strongly asymptotic independence as explored in [2], among many others. Among these, Ko and Tang [8] introduced the following structure.

Assumption 2.1 ([8, Assumption 3.1])**.** Let $X_1,X_2,\dots,X_n$ be n random variables. There exist some constant $x_0=x_0(n)>0$ and a dominating coefficient $M_n>0$ such that for any $2\le m\le n$,

$$\frac{\mathbb{P}\left(\sum_{i=1}^{m-1}X_i>x-t|X_m=t\right)}{\mathbb{P}\left(\sum_{i=1}^{m-1}X_i>x-t\right)}\le M_n, \quad x\ge x_0$$

holds uniformly for all $t\in[x_0,x]$.

Assumption 2.1 is introduced to characterize the dependence structure of the claim sizes, forming the basis for our model. We next state some fundamental assumptions for the risk model (1.1).

Assumption 2.2. Assume that $\{(X_{k,i},Y_{k,i});i\in\mathbb{N}_+\}$ is a sequence of i.i.d. copy of nonnegative random pair (X_k,Y_k) with marginal distributions (F_k,G_k), respectively, for $k=1,\dots,d$. Assume that $\{D_{k,i};i\in\mathbb{N}_+\}$, $\{(X_{k,i},Y_{k,i});i\in\mathbb{N}_+\}$, $\{L(t);t\ge0\}$ and $\{N(t);t\ge0\}$ are mutually independent for $k=1,\dots,d$. Additionally, for $k=1,\dots,d$, assume that the premium density function $c_k(t)$ is bounded, i.e., $0\le c_k(t)\le c_0$ for some $c_0>0$ and all $t\ge0$.

3 Main results

In this section, we introduce our core findings, including the asymptotic estimation of ruin probability for the multidimensional model. Before presenting the estimation of ruin probability, we denote the following terms for simplicity. Define the number of permutations as

$$\begin{aligned}&S_d=\big\{(h_1,\dots,h_d)\mid h_l\in\{1,\dots,d\}\ \text{ for }\ l=1,\dots,d\ \text{ and }\ h_l\neq h_j\ \text{ for }\ l\neq j\big\},\\&S_{d-k}=\big\{(h_1,\dots,h_{d-k})\mid h_l\in\{1,\dots,d\}\ \text{ for }\ l=1,\dots,d-k\ \text{ and }\ h_l\neq h_j\ \text{ for }\ l\neq j\big\},\end{aligned}$$

and denote $Q(k)=\{1,\dots,d\}/\{h_1,\dots,h_{d-k}\}$ for $k=2,\dots,d-1$. Let W_i is either X_i or Y_i for $1\le i\le d$, and write $\Gamma=\{(W_1,\dots,W_d)|W_i\in\{X_i,Y_i\}$ for $1\le i\le d\}$. We can observe that since W_i is either X_i or Y_i, the cardinality of the set $|\Gamma|$ is 2^d. For example, $(X_1,\dots,X_{d-1},Y_d)$ is an element in Γ, with $W_i=X_i$ for $i=1,\dots,d-1$ and $W_d=Y_d$.

Then Theorem 3.1 presents the asymptotic estimate of ruin probabilities when the main claims and the corresponding delayed claims are dependent through the dependence structure given in Assumption 2.1.

Theorem 3.1. *Consider the multidimensional risk model* (1.1) *satisfies Assumption* 2.2. *Assume that for any fixed $k=1,\dots,d$ and $i\in\mathbb{N}_+$, $(X_{k,i},Y_{k,i})$ satisfies Assumption* 2.1 *as $m=2$, that is, for some large $x_{k,i}>0$ and a constant $M_{k,i}>0$,*

$$\frac{\mathbb{P}(X_{k,i}>x-t|Y_{k,i}=t)}{\mathbb{P}(X_{k,i}>x-t)}\le M_{k,i},\quad x\ge x_{k,i}$$

holds uniformly for all $t\in[x_{k,i},x]$. Let $T>0$ be any fixed time such that $\mathbb{P}(\tau_1\le T)>0$. If $F_i,G_i\in\mathcal{A}$ and $\overline{F}_i(x)\asymp\overline{G}_j(x)$ for $i,j=1,\dots,d$, then as $\min\{x_1,\dots,x_d\}\to\infty$,

$$\Psi(x_1,\dots,x_d;T)\sim\sum_{(W_1,\dots,W_d)\in\Gamma}\chi\big((W_1,\dots,W_d),x_1,\dots,x_d;T\big),$$

where

$$\begin{aligned}&\chi\big((W_1,\dots,W_d),x_1,\dots,x_d;T\big)\\=&\sum_{(h_1,\dots,h_d)\in S_d}\int_{R_d}\prod_{p=1}^{d}\mathbb{P}\Big(W_{h_p}>x_{h_p}e^{L(\sum_{j=1}^{p}u_j+v_{h_p}I_{\{W_{h_p}=Y_{h_p}\}})}\Big)\prod_{j=1}^{d}\lambda(du_j)H_j(dv_j)\\&+\sum_{k=2}^{d-1}\sum_{\substack{(h_1,\dots,h_{d-k})\\\in S_{d-k}}}\Bigg[\int_{R_{1,k}(h_1,\dots,h_{d-k})}\prod_{p=1}^{d-k}\mathbb{P}\Big(W_{h_p}>x_{h_p}e^{L(\sum_{j=1}^{p+1}u_j+v_{h_p}I_{\{W_{h_p}=Y_{h_p}\}})}\Big)\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{m\in Q(k)} \mathbb{P}\left(W_m > x_m e^{L\left(u_1+v_m I_{\{W_m=Y_m\}}\right)}\right) \\
&\times \prod_{j=1}^{d-k+1} \lambda(du_j) \prod_{j=1}^{d} H_j(dv_j) \\
&+ \int_{R_{2,k}(h_1,\dots,h_{d-k})} \prod_{p=2}^{d-k} \mathbb{P}\left(W_{h_p} > x_{h_p} e^{L(\sum_{j=1}^{p+1} u_j + v_{h_p} I_{\{W_{h_p}=Y_{h_p}\}})}\right) \\
&\times \mathbb{P}\left(W_{h_1} > x_{h_1} e^{L(u_1+v_{h_1} I_{\{W_{h_1}=Y_{h_1}\}})}\right) \\
&\times \prod_{m\in Q(k)} \mathbb{P}\left(W_m > x_m e^{L(u_1+u_2+v_m I_{\{W_m=Y_m\}})}\right) \prod_{j=1}^{d-k+1} \lambda(du_j) \prod_{j=1}^{d} H_j(dv_j) \\
&+\cdots+ \int_{R_{d-k+1,k}(h_1,\dots,h_{d-k})} \prod_{m\in Q(k)} \mathbb{P}\left(W_m > x_m e^{L(\sum_{j=1}^{d-k+1} u_j + v_m I_{\{W_m=Y_m\}})}\right) \\
&\times \prod_{p=1}^{d-k} \mathbb{P}\left(W_{h_p} > x_{h_p} e^{L(\sum_{j=1}^{p} u_j + v_{h_p} I_{\{W_{h_p}=Y_{h_p}\}})}\right) \prod_{j=1}^{d-k+1} \lambda(du_j) \prod_{j=1}^{d} H_j(dv_j) \Bigg] \\
&+ \int_{R_1} \prod_{m=1}^{d} \mathbb{P}\left(W_m > x_m e^{L(u+v_m I_{\{W_m=Y_m\}})}\right) \lambda(du) \prod_{j=1}^{d} H_j(dv_j),
\end{aligned}
$$

and the regions of integration are defined as

$$
\begin{aligned}
R_d &= \left\{ (u_1,\dots,u_d,v_1,\dots,v_d) \in (0,+\infty)^{2d} \,\Big|\, \sum_{j=1}^{p} u_j + v_{h_p} I_{\{W_{h_p}=Y_{h_p}\}} \le T \ \textit{for} \ \ p=1,\dots,d \right\}, \\
R_{1,k}(h_1,\dots,h_{d-k}) &= \Bigg\{ (u_1,\dots,u_{d-k+1},v_1,\dots,v_d) \in (0,+\infty)^{2d-k+1} \,\Big| \\
&\qquad \sum_{j=1}^{p+1} u_j + v_{h_p} I_{\{W_{h_p}=Y_{h_p}\}} \le T \ \textit{for} \ \ p=1,\dots,d-k \ \ \textit{and} \\
&\qquad u_1 + v_m I_{\{W_m=Y_m\}} \le T \ \textit{for} \ \ m\in Q(k) \Bigg\}, \\
R_{2,k}(h_1,\dots,h_{d-k}) &= \Bigg\{ (u_1,\dots,u_{d-k+1},v_1,\dots,v_d) \in (0,+\infty)^{2d-k+1} \,\Big| \\
&\qquad u_1 + v_{h_1} I_{\{W_{h_1}=Y_{h_1}\}} \le T,
\end{aligned}
$$

$$\sum_{j=1}^{p+1} u_j + v_{h_p} I_{\{W_{h_p}=Y_{h_p}\}} \le T \ \textit{for} \ \ p=2,\dots,d-k \ \textit{and}$$

$$u_1+u_2+v_m I_{\{W_m=Y_m\}} \le T \ \textit{for} \ \ m\in Q(k) \Big\},$$

$\cdots$

$$R_{d-k+1,k}(h_1,\dots,h_{d-k}) = \Big\{ (u_1,\dots,u_{d-k+1},v_1,\dots,v_d)\in(0,+\infty)^{2d-k+1} \ \Big|$$

$$\sum_{j=1}^{p} u_j + v_{h_p} I_{\{W_{h_p}=Y_{h_p}\}} \le T \ \textit{for} \ \ p=1,\dots,d-k \ \textit{and}$$

$$\sum_{j=1}^{d-k+1} u_j + v_m I_{\{W_m=Y_m\}} \le T \ \textit{for} \ \ m\in Q(k) \Big\},$$

$$R_1 = \Big\{ (u,v_1,\dots,v_d)\in(0,+\infty)^{d+1} \,\Big|\, u+v_m I_{\{W_m=Y_m\}} \le T \ \textit{for} \ \ m=1,\dots,d \Big\}.$$

3.1 Lemmas and proofs

We present some lemmas that study tail behaviors and characteristics of random variables with dependent structures. The first lemma establishes a Kesten's bound for random variables satisfying Assumption 2.1. We refer to [25, Lemma 2.1] for the proof details.

Lemma 3.1 ([25, Lemma 2.1])**.** *Let $\{X_k,k\in\mathbb{N}\}$ be a sequence of identically distributed real-valued random variables with a common distribution $F\in\mathcal{S}$ and satisfying Assumption* 2.1. *Then, for any $\varepsilon>0$, there exists a positive constant K_ε such that*

$$\mathbb{P}\left(\sum_{k=1}^{n} X_k > x\right) \le K_\varepsilon (1+\varepsilon)^n \overline{F}(x)$$

holds for all $n\in\mathbb{N}$ and all $x\ge 0$.

The following lemma plays a crucial role in the proof of Theorem 3.1.

Lemma 3.2. *Under the conditions of Theorem* 3.1*, for $T>0$,*

$$\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\sum_{i=1}^{N(T)}\left(X_{k,i}e^{-L(\tau_i)} + Y_{k,i}e^{-L\left(\tau_i+D_{k,i}\right)} I_{\left\{\tau_i+D_{k,i}\le T\right\}}\right) > x_k\right\}\right)$$
$$\sim \chi\big((W_1,\dots,W_d),x_1,\dots,x_d;T\big), \tag{3.1}$$

where the symbols are consistent with those in Theorem 3.1.

Proof. We choose an integer m large enough such that the following holds:

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\sum_{i=1}^{N(T)}\left(X_{k,i}e^{-L(\tau_i)}+Y_{k,i}e^{-L\left(\tau_i+D_{k,i}\right)}I_{\left\{\tau_i+D_{k,i}\le T\right\}}\right)>x_k\right\}\right)\\
&=\left(\sum_{n=m+1}^{\infty}+\sum_{n=1}^{m}\right)\\
&\quad\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\sum_{i=1}^{n}\left(X_{k,i}e^{-L(\tau_i)}+Y_{k,i}e^{-L\left(\tau_i+D_{k,i}\right)}I_{\left\{\tau_i+D_{k,i}\le T\right\}}\right)>x_k\right\},N(T)=n\right)\\
&:=I_1(x_1,\ldots,x_d;T)+I_2(x_1,\ldots,x_d;T).
\end{aligned}$$

By conditioning on $\tau_1, L(s)$, we have

$$\begin{aligned}
&I_1(x_1,\ldots,x_d;T)\\
&\le\sum_{n=m+1}^{\infty}\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\sum_{i=1}^{n}\left(X_{k,i}e^{-L(\tau_1)}+Y_{k,i}e^{-L(\tau_1)}\right)>x_k\right\},N(T)=n\right)\\
&=\sum_{n=m+1}^{\infty}\int_{0-}^{T}\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\sum_{i=1}^{n}\left(X_{k,i}e^{-L(s)}+Y_{k,i}e^{-L(s)}\right)>x_k\right\}\right)\\
&\qquad\times\mathbb{P}\big(N(T-s)=n-1\big)\mathbb{P}(\tau_1\in ds)\\
&=\sum_{n=m+1}^{\infty}\int_{0-}^{T}\int_{0}^{\infty}\prod_{k=1}^{d}\mathbb{P}\left(\sum_{i=1}^{n}\left(X_{k,i}e^{-r}+Y_{k,i}e^{-r}\right)>x_k\right)\\
&\qquad\times\mathbb{P}\big(N(T-s)=n-1\big)\mathbb{P}(L(s)\in dr)\mathbb{P}(\tau_1\in ds),
\end{aligned}$$

where the last step is due to the independence of $\{X_{k,i}+Y_{k,i};1\le i\le n\}$ for $1\le k\le d$. Let $Z_{k,i}=X_{k,i}+Y_{k,i}$ and $Z_k=X_k+Y_k$ with distributions F_k^*, then $F_k^*\in\mathcal{S}$ due to [15, Lemma 1]. Applying Lemma 3.1 yields

$$\begin{aligned}
&I_1(x_1,\ldots,x_d;T)\\
&\le\sum_{n=m+1}^{\infty}\int_{0-}^{T}\int_{0}^{\infty}\prod_{k=1}^{d}\mathbb{P}\left(\sum_{i=1}^{n}Z_{k,i}e^{-r}>x_k\right)\mathbb{P}\big(N(T-s)=n-1\big)\\
&\qquad\times\mathbb{P}\big(L(s)\in dr\big)\mathbb{P}(\tau_1\in ds)\\
&\le\sum_{n=m+1}^{\infty}\int_{0-}^{T}\int_{0}^{\infty}\prod_{k=1}^{d}K_k(1+\varepsilon_k)^n\mathbb{P}(Z_ke^{-r}>x_k)\mathbb{P}\big(N(T-s)=n-1\big)\\
&\qquad\times\mathbb{P}\big(L(s)\in dr\big)\mathbb{P}(\tau_1\in ds)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=m+1}^{\infty} K(1+\varepsilon)^{d(n-1)}(1+\varepsilon)^{d} \int_{0-}^{T}\int_{0}^{\infty} \prod_{k=1}^{d} \mathbb{P}(Z_k e^{-r} > x_k)\mathbb{P}\big(N(T-s)=n-1\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{P}\big(L(s)\in dr\big)\mathbb{P}(\tau_1\in ds) \\
&\leq K(1+\varepsilon)^{d} \int_{0-}^{T}\int_{0}^{\infty} \mathbb{E}\Big((1+\varepsilon)^{dN(T-s)} I_{\{N(T-s)\geq m\}}\Big) \prod_{k=1}^{d} \overline{F}_k^{*}(x_k e^{r}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbb{P}\big(L(s)\in dr\big)\mathbb{P}(\tau_1\in ds) \\
&\leq K(1+\varepsilon)^{d} \mathbb{E}\Big((1+\varepsilon)^{dN(T)} I_{\{N(T)\geq m\}}\Big) \int_{0-}^{T} \prod_{k=1}^{d} \mathbb{P}\big(Z_k > x_k e^{L(s)}\big)\lambda(ds),
\end{aligned}
$$

where $K=\prod_{k=1}^{d} K_k$ and $\varepsilon=\max\{\varepsilon_1,\ldots,\varepsilon_d\}$. By conditioning on $L(s)$, [15, Lemma 1] and $\overline{F}_k \asymp \overline{G}_k$ for $k=1,\ldots,d$, we have

$$
\prod_{k=1}^{d} \mathbb{P}\Big(Z_k > x_k e^{L(s)}\Big) \lesssim M \prod_{k=1}^{d} \mathbb{P}\Big(X_k > x_k e^{L(s)}\Big)
$$

for some $M\geq 2^d$. By [6, Lemma 3.2], we can choose some ε small enough such that $\mathbb{E}((1+\varepsilon)^{dN(T)})<\infty$ and $\mathbb{E}((1+\varepsilon)^{dN(T)} I_{\{N(T)\geq m\}})\to 0$ as $m\to\infty$. Thus,

$$
\begin{aligned}
I_1 &\lesssim KM(1+\varepsilon)^{d} \mathbb{E}\Big((1+\varepsilon)^{dN(T)} I_{\{N(T)\geq m\}}\Big) \int_{0-}^{T} \prod_{k=1}^{d} \mathbb{P}\Big(X_k > x_k e^{L(s)}\Big)\lambda(ds) \\
&= o\left(\int_{0-}^{T} \prod_{k=1}^{d} \mathbb{P}\Big(X_k > x_k e^{L(s)}\Big)\lambda(ds)\right) \\
&= o\big(\chi((X_1,\ldots,X_d),x_1,\ldots,x_d;T)\big).
\end{aligned} \tag{3.2}
$$

We set

$$
\begin{aligned}
\Omega_n = \Big\{ &\big(s_1,s_2,\ldots,s_{n+1},\ w_{1,1},w_{1,2},\ldots,w_{1,n},\ldots,w_{d,1},w_{d,2},\ldots,w_{d,n} \\
&\quad r_1,\ldots,r_n,r_{1,1},\ldots,r_{1,n},\ldots,r_{d,1},\ldots,r_{d,n}\big): \\
&\quad 0\leq s_1\leq\cdots\leq s_n\leq T< s_{n+1},\ w_{k,i},r_i,r_{k,i}\geq 0,\ \text{ for } i=1,\ldots,n,\ k=1,\ldots,d \Big\}
\end{aligned}
$$

and condition on τ_i for $1\leq i\leq n+1$, $D_{k,i},L(s_i),L(s_i+w_{k,i})$ for $1\leq i\leq n, 1\leq k\leq d$. Let $\xi_{k,i}$ is either $X_{k,i}$ or $Y_{k,i}$ for $1\leq k\leq d$, and write

$$
\Gamma' = \big\{(\xi_{1,i},\ldots,\xi_{d,i}) \,|\, \xi_{k,i}\in\{X_{k,i},Y_{k,i}\} \ \text{ for } \ 1\leq k\leq d\big\}.
$$

Then, we have

$$
\begin{aligned}
&I_2(x_1,\ldots,x_d;T)\\
=&\sum_{n=1}^{m}\int_{\Omega_n}\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\sum_{i=1}^{n}\left(X_{k,i}e^{-r_i}+Y_{k,i}e^{-r_{k,i}}I_{\{s_i+w_{k,i}\le T\}}\right)>x_k\right\}\right)\\
&\quad\times\mathbb{P}\big(\tau_1\in ds_1,\ldots,\tau_{n+1}\in ds_{n+1},D_{1,1}\in dw_{1,1},\ldots,D_{d,n}\in dw_{d,n},\\
&\qquad L(s_1)\in dr_1,\ldots,L(s_n)\in dr_n,L(s_1+w_{1,1})\in dr_{1,1},\ldots,L(s_n+w_{d,n})\in dr_{d,n}\big).
\end{aligned}
$$

Note that the integrand in the above equation can be rewritten as

$$
\begin{aligned}
&\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\sum_{i=1}^{n}\left(X_{k,i}e^{-r_i}+Y_{k,i}e^{-r_{k,i}}I_{\{s_i+w_{k,i}\le T\}}\right)>x_k\right\}\right)\\
=&\prod_{k=1}^{d}\mathbb{P}\left(\sum_{i=1}^{n}\left(X_{k,i}e^{-r_i}+Y_{k,i}e^{-r_{k,i}}I_{\{s_i+w_{k,i}\le T\}}\right)>x_k\right)\\
\sim&\prod_{k=1}^{d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\left(\mathbb{P}\left(X_{k,i_k}e^{-r_{i_k}}>x_k\right)+\mathbb{P}\left(Y_{k,i_k}e^{-r_{k,i_k}}I_{\{s_{i_k}+w_{k,i_k}\le T\}}>x_k\right)\right),
\end{aligned}
$$

where the last step is due to [1, Lemma 4.2]. Expanding the product $\prod_{k=1}^{d}(\cdot)$ gives rise to 2^d summation terms, which can be described through the following combinatorial representation. Thus,

$$
\begin{aligned}
&I_2(x_1,\ldots,x_d;T)\sim\sum_{n=1}^{m}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\\
&\int_{\Omega_n}\sum_{\substack{(\xi_{1,i_1},\ldots,\xi_{d,i_d})\\ \in\Gamma'}}\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\xi_{k,i_k}\left(e^{-r_{i_k}}I_{\{\xi_{k,i_k}=X_{k,i_k}\}}+e^{-r_{k,i_k}}I_{\{\xi_{k,i_k}=Y_{k,i_k},s_{i_k}+w_{k,i_k}\le T\}}\right)>x_k\right\}\right)\\
&\quad\times\mathbb{P}\big(\tau_1\in ds_1,\ldots,\tau_{n+1}\in ds_{n+1},D_{1,1}\in dw_{1,1},\ldots,D_{d,n}\in dw_{d,n},\\
&\qquad L(s_1)\in dr_1,\ldots,L(s_n)\in dr_n,L(s_1+w_{1,1})\in dr_{1,1},\ldots,L(s_n+w_{d,n})\in dr_{d,n}\big)\\
=&\sum_{n=1}^{m}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{(\xi_{1,i_1},\ldots,\xi_{d,i_d})\\ \in\Gamma'}}\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{\xi_{k,i_k}\left(e^{-L(\tau_{i_k})}I_{\{\xi_{k,i_k}=X_{k,i_k}\}}\right.\right.\right.\\
&\qquad\left.\left.\left.+e^{-L(\tau_{i_k}+D_{k,i_k})}I_{\{\xi_{k,i_k}=Y_{k,i_k},\tau_{i_k}+w_{k,i_k}\le T\}}\right)>x_k\right\},N(T)=n\right)\\
=:&\sum_{(\xi_{1*},\ldots,\xi_{d*})\in\Gamma^*}I_2(\xi_{1*},\ldots,\xi_{d*}),
\end{aligned}\tag{3.3}
$$

where

$$\Gamma^*=\left\{(\xi_{1*},\ldots,\xi_{d*})\,|\,\xi_{k*}\in\{X_{k*},Y_{k*}\}\ \text{ for }\ 1\leq k\leq d\right\}.$$

Each element in Γ^* is coincidence with the choice of $(\xi_{1,i_1},\ldots,\xi_{d,i_d})$. Thus, for case $(\xi_{1*},\ldots,\xi_{d*})=(X_{1*},\ldots,X_{d*})$, we have

$$\begin{aligned}
&\sum_{(\xi_{1*},\ldots,\xi_{d*})\in\Gamma^*} I_2(X_{1*},\ldots,X_{d*})\\
&=\left(\sum_{n=1}^{\infty}-\sum_{n=m+1}^{\infty}\right)\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{X_{k,i_k}e^{-L(\tau_{i_k})}>x_k\right\},N(T)=n\right)\\
&=:I_{21}(X_{1*},\ldots,X_{d*})-I_{22}(X_{1*},\ldots,X_{d*}).
\end{aligned}\tag{3.4}$$

Whenever it causes no ambiguity, we omit explicit arguments in the expressions of the form $I_{21}(X_{1*},\ldots,X_{d*})$, and this convention will be used hereafter. We also follow the standard convention that a summation over an empty index set is defined as zero. Since the proof for the d-dimensional case is technically intricate and less transparent, we present the detailed argument for the three-dimensional case to improve clarity. The same reasoning applies to the general d-dimensional setting, ensuring the validity of the overall result.

Interchanging the order of the sums in $I_{21}(X_{1*},X_{2*},X_{3*})$ leads to

$$\begin{aligned}
I_{21}=&\sum_{i_1=1}^{\infty}\sum_{n=i_1}^{\infty}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\mathbb{P}\Big(X_{1,i_1}e^{-L(\tau_{i_1})}>x_1,X_{2,i_2}e^{-L(\tau_{i_2})}>x_2,\\
&\qquad X_{3,i_3}e^{-L(\tau_{i_3})}>x_3,N(T)=n\Big)\\
=&\sum_{i_1=1}^{\infty}\sum_{n=i_1}^{\infty}\left(\sum_{i_2=1}^{i_1-1}\left(\sum_{i_3=1}^{i_2-1}+\sum_{i_3=i_2}+\sum_{i_3=i_2+1}^{i_1-1}+\sum_{i_3=i_1}+\sum_{i_3=i_1+1}^{n}\right)\right.\\
&\quad+\sum_{i_2=i_1}\left(\sum_{i_3=1}^{i_1-1}+\sum_{i_3=i_2}+\sum_{i_3=i_2+1}^{n}\right)\\
&\quad\left.+\sum_{i_2=i_1+1}^{n}\left(\sum_{i_3=1}^{i_1-1}+\sum_{i_3=i_1}+\sum_{i_3=i_1+1}^{i_2-1}+\sum_{i_3=i_2}+\sum_{i_3=i_2+1}^{n}\right)\right)\\
&\mathbb{P}\Big(X_{1,i_1}e^{-L(\tau_{i_1})}>x_1,X_{2,i_2}e^{-L(\tau_{i_2})}>x_2,X_{3,i_3}e^{-L(\tau_{i_3})}>x_3,N(T)=n\Big)\\
=&\sum_{i_3=1}^{\infty}\sum_{i_2=i_3+1}^{\infty}\sum_{i_1=i_2+1}^{\infty}\mathbb{P}\Big(X_{1,i_1}e^{-L(\tau_{i_1})}>x_1,X_{2,i_2}e^{-L(\tau_{i_2})}>x_2,
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad X_{3,i_3}e^{-L(\tau_{i_3})}>x_3,\tau_{i_1}\le T\Big)\\
&+\cdots+\sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}\sum_{i_3=i_2+1}^{\infty}\mathbb{P}\Big(X_{1,i_1}e^{-L(\tau_{i_1})}>x_1,X_{2,i_2}e^{-L(\tau_{i_2})}>x_2,\\
&\qquad\qquad X_{3,i_3}e^{-L(\tau_{i_3})}>x_3,\tau_{i_3}\le T\Big)\\
=:&\, I_{21(1)}+\cdots+I_{21(13)}.
\end{aligned}
\tag{3.5}
$$

For $I_{21(1)}$ (case $i_1>i_2>i_3$), we have

$$
\begin{aligned}
I_{21(1)}=\sum_{i_3=1}^{\infty}\sum_{i_2=i_3+1}^{\infty}\sum_{i_1=i_2+1}^{\infty}\mathbb{P}\Big(&X_{1,i_1}e^{-L(\tau_{i_3}+(\tau_{i_2}-\tau_{i_3})+(\tau_{i_1}-\tau_{i_2}))}>x_1,\\
&X_{2,i_2}e^{-L(\tau_{i_3}+(\tau_{i_2}-\tau_{i_3}))}>x_2,\\
&X_{3,i_3}e^{-L(\tau_{i_3})}>x_3,\tau_{i_3}+(\tau_{i_2}-\tau_{i_3})+(\tau_{i_1}-\tau_{i_2})\le T\Big).
\end{aligned}
$$

Recall that both $\{L(t);t\ge 0\}$ and $\{\tau_i;i\ge 0\}$ possess the property of independent and stationary increments. Hence, conditioning on τ_{i_3}, $\tau_{i_2}-\tau_{i_3},\tau_{i_1}-\tau_{i_2}$ and $L(\cdot)$ yields

$$
\begin{aligned}
I_{21(1)}=&\sum_{i_3=1}^{\infty}\sum_{i_2=i_3+1}^{\infty}\sum_{i_1=i_2+1}^{\infty}\\
&\int_{0-}^{T}\int_{0-}^{T-u_3}\int_{0-}^{T-u_3-u_2}\mathbb{P}\Big(X_1>x_1e^{L(u_1+u_2+u_3)}\Big)\mathbb{P}\Big(X_2>x_2e^{L(u_2+u_3)}\Big)\\
&\qquad\times\mathbb{P}\Big(X_3>x_3e^{L(u_3)}\Big)\mathbb{P}(\tau_{i_1-i_2}\in du_1)\mathbb{P}(\tau_{i_2-i_3}\in du_2)\mathbb{P}(\tau_{i_3}\in du_3)\\
=&\iiint\limits_{\substack{u_1,u_2,u_3>0,\\ u_1+u_2+u_3\le T}}\mathbb{P}\Big(X_1>x_1e^{L(u_1+u_2+u_3)}\Big)\mathbb{P}\Big(X_2>x_2e^{L(u_2+u_3)}\Big)\mathbb{P}\Big(X_3>x_3e^{L(u_3)}\Big)\\
&\qquad\times\lambda(du_1)\lambda(du_2)\lambda(du_3).
\end{aligned}
\tag{3.6}
$$

Similarly, we can derive $I_{21(3)},I_{21(5)},I_{21(9)},I_{21(11)},I_{21(13)}$. For $I_{21(2)}$ (case $i_1>i_2=i_3$),

$$
\begin{aligned}
I_{21(2)}=&\sum_{i_2=1}^{\infty}\sum_{i_1=i_2+1}^{\infty}\mathbb{P}\Big(X_{1,i_1}e^{-L(\tau_{i_2}+(\tau_{i_1}-\tau_{i_2}))}>x_1,\,X_{2,i_2}e^{-L(\tau_{i_2})}>x_2,\\
&\qquad X_{3,i_3}e^{-L(\tau_{i_2})}>x_3,\tau_{i_2}+(\tau_{i_1}-\tau_{i_2})\le T\Big)\\
=&\sum_{i_2=1}^{\infty}\sum_{i_1=i_2+1}^{\infty}\int_{0-}^{T}\int_{0-}^{T-u_2}\mathbb{P}\Big(X_1>x_1e^{L(u_1+u_2)}\Big)\mathbb{P}\Big(X_2>x_2e^{L(u_2)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad \times \mathbb{P}\left(X_3 > x_3 e^{L(u_2)}\right) \mathbb{P}(\tau_{i_1-i_2} \in du_1) \mathbb{P}(\tau_{i_2} \in du_2) \\
&= \iint_{\substack{u_1,u_2>0,\\ u_1+u_2\le T}} \mathbb{P}\left(X_1 > x_1 e^{L(u_1+u_2)}\right) \mathbb{P}\left(X_2 > x_2 e^{L(u_2)}\right) \\
&\qquad\qquad \times \mathbb{P}\left(X_3 > x_3 e^{L(u_2)}\right) \lambda(du_1)\lambda(du_2). \qquad (3.7)
\end{aligned}
$$

Similarly, we can derive $I_{21(4)}, I_{21(6)}, I_{21(8)}, I_{21(10)}, I_{21(12)}$. For $I_{21(7)}$ (case $i_1=i_2=i_3$), it holds that

$$
\begin{aligned}
I_{21(7)} &= \sum_{i_1=1}^{\infty} \mathbb{P}\left(X_{1,i_1} e^{-L(\tau_{i_1})} > x_1, X_{2,i_1} e^{-L(\tau_{i_1})} > x_2, X_{3,i_1} e^{-L(\tau_{i_1})} > x_3, \tau_{i_1} \le T\right) \\
&= \sum_{i_1=1}^{\infty} \int_{0-}^{T} \mathbb{P}\left(X_1 > x_1 e^{L(u)}\right) \mathbb{P}\left(X_2 > x_2 e^{L(u)}\right) \mathbb{P}\left(X_3 > x_3 e^{L(u)}\right) \mathbb{P}(\tau_{i_1} \in du) \\
&= \int_{0-}^{T} \mathbb{P}\left(X_1 > x_1 e^{L(u)}\right) \mathbb{P}\left(X_2 > x_2 e^{L(u)}\right) \mathbb{P}\left(X_3 > x_3 e^{L(u)}\right) \lambda(du). \qquad (3.8)
\end{aligned}
$$

Combining (3.5)-(3.8) leads to the result for I_{21} in 3-dimensional case

$$
\begin{aligned}
I_{21} &= \sum_{\substack{(h_1,h_2,h_3)\\ \in S_3}} \int_{R_3} \prod_{p=1}^{3} \mathbb{P}\left(X_{h_p} > x_{h_p} e^{L(\sum_{j=1}^{p} u_j)}\right) \prod_{j=1}^{3} \lambda(du_j) \\
&\quad + \sum_{h_1 \in S_1} \Bigg[\int_{R_{1,2(h_1)}} \mathbb{P}\left(X_{h_1} > x_{h_1} e^{L(u_1+u_2)}\right) \prod_{m\in h_2,h_3} \mathbb{P}\left(X_m > x_m e^{L(u_1)}\right) \lambda(du_1)\lambda(u_2) \\
&\qquad\qquad + \int_{R_{2,2(h_1)}} \prod_{m\in h_2,h_3} \mathbb{P}\left(X_m > x_m e^{L(u_1+u_2)}\right) \mathbb{P}\left(X_{h_1} > x_{h_1} e^{L(u_1)}\right) \lambda(u_1)\lambda(u_2) \Bigg] \\
&\quad + \int_{R_1} \prod_{m=1}^{3} \mathbb{P}\left(X_m > x_m e^{L(u)}\right) \lambda(du) \\
&= \chi\big((X_1,X_2,X_3), x_1,x_2,x_3;T\big),
\end{aligned}
$$

where S_3, S_1, R_3, $R_{1,2(h_1)}$, $R_{2,2(h_1)}$, R_1 are defined as in Theorem 3.1 with $d=3$, $k=2$, $(W_1,W_2,W_3)=(X_1,X_2,X_3)$, which can be further extended to the d-dimensional case. Thus, we have

$$
I_{21} = \chi\big((X_1,\dots,X_d), x_1,\dots,x_d;T\big). \qquad (3.9)
$$

For I_{22}, it holds that

$$\begin{aligned}
I_{22} &= \sum_{n=m+1}^{\infty}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{X_{k,i_k}e^{-L(\tau_{i_k})}>x_k\right\},N(T)=n\right)\\
&\le \sum_{n=m+1}^{\infty}\sum_{1\le i_1,\dots,i_d\le n}\mathbb{P}\left(\bigcap_{k=1}^{d}\left\{X_{k,i_k}e^{-L(\tau_1)}>x_k\right\},\tau_n\le T<\tau_{n+1}\right)\\
&\le \sum_{n=m+1}^{\infty}n^d\int_{0-}^{T}\prod_{k=1}^{d}\mathbb{P}\left(X_k>x_ke^{L(s)}\right)\mathbb{P}\big(N(T-s)=n-1\big)\mathbb{P}(\tau_1\in ds)\\
&\le \mathbb{E}\left(\big(N(T)+1\big)^dI_{\{N(T)\ge m\}}\right)\int_{0-}^{T}\prod_{k=1}^{d}\mathbb{P}\left(X_k>x_ke^{L(s)}\right)\mathbb{P}(\tau_1\in ds)\\
&= o\left(\int_{0-}^{T}\prod_{k=1}^{d}\mathbb{P}\left(X_k>x_ke^{L(s)}\right)\lambda(ds)\right),
\end{aligned}\tag{3.10}$$

where in the last step we use $\mathbb{E}((N(T)+1)^d)<\infty$ and $\mathbb{E}((N(T)+1)^dI_{\{N(T)\ge m\}})\to 0$ as $m\to\infty$. Hence, plugging (3.9) and (3.10) into (3.4) yields

$$I_2(X_{1*},\dots,X_{d*})\sim\chi\big((X_1,\dots,X_d),x_1,\dots,x_d;T\big).\tag{3.11}$$

For $I_2(Y_{1*},X_{2*},\dots,X_{d*})$, we have

$$\begin{aligned}
&I_2(Y_{1*},X_{2*},\dots,X_{d*})\\
&=\left(\sum_{n=1}^{\infty}-\sum_{n=m+1}^{\infty}\right)\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\mathbb{P}\Bigg(Y_{1,i_1}e^{-L(\tau_{i_1}+D_{1,i_1})}I_{\{\tau_{i_1}+D_{1,i_1}\le T\}}>x_1,\\
&\qquad\qquad\bigcap_{k=2}^{d}\left\{X_{k,i_k}e^{-L(\tau_{i_k})}>x_k\right\},N(T)=n\Bigg)\\
&=: I_{21}^{Y}(Y_{1*},X_{2*},\dots,X_{d*})-I_{22}^{Y}(Y_{1*},X_{2*},\dots,X_{d*}).
\end{aligned}\tag{3.12}$$

The 3-dimensional case for $I_{21}^{Y}(Y_{1*},X_{2*},\dots,X_{d*})$ is considered analogously to $I_{21}(X_{1*},\dots,X_{d*})$. Changing the order of summation in I_{21}^{Y} yields

$$\begin{aligned}
I_{21}^{Y}=\sum_{i_1=1}^{\infty}\sum_{n=i_1}^{\infty}\Bigg(&\sum_{i_2=1}^{i_1-1}\left(\sum_{i_3=1}^{i_2-1}+\sum_{i_3=i_2}+\sum_{i_3=i_2+1}^{i_1-1}+\sum_{i_3=i_1}+\sum_{i_3=i_1+1}^{n}\right)\\
&+\sum_{i_2=i_1}\left(\sum_{i_3=1}^{i_1-1}+\sum_{i_3=i_2}+\sum_{i_3=i_2+1}^{n}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i_2=i_1+1}^{n}\left(\sum_{i_3=1}^{i_1-1}+\sum_{i_3=i_1}+\sum_{i_3=i_1+1}^{i_2-1}+\sum_{i_3=i_2}+\sum_{i_3=i_2+1}^{n}\right)\Bigg)\\
&\times\mathbb{P}\Big(Y_{1,i_1}e^{-L(\tau_{i_1}+D_{1,i_1})}I_{\{\tau_{i_1}+D_{1,i_1}\le T\}}>x_1,X_{2,i_2}e^{-L(\tau_{i_2})}>x_2,\\
&\qquad X_{3,i_3}e^{-L(\tau_{i_3})}>x_3,N(T)=n\Big)\\
&=:I^Y_{21(1)}+\cdots+I^Y_{21(13)}.
\end{aligned}\tag{3.13}
$$

For $I^Y_{21(1)}$, we have

$$
\begin{aligned}
I^Y_{21(1)}=\sum_{i_3=1}^{\infty}\sum_{i_2=i_3+1}^{\infty}\sum_{i_1=i_2+1}^{\infty}\mathbb{P}\Big(&Y_{1,i_1}e^{-L(\tau_{i_3}+(\tau_{i_2}-\tau_{i_3})+(\tau_{i_1}-\tau_{i_2})+D_{1,i_1})}I_{\{\tau_i+D_{1,i_1}\le T\}}>x_1,\\
&X_{2,i_2}e^{-L(\tau_{i_3}+(\tau_{i_2}-\tau_{i_3}))}>x_2,\\
&X_{3,i_3}e^{L(\tau_{i_3})}>x_3,\tau_{i_3}+(\tau_{i_2}-\tau_{i_3})+(\tau_{i_1}-\tau_{i_2})\le T\Big).
\end{aligned}
$$

Following similar derivations as in (3.6)-(3.8), we can derive $I^Y_{21(1)}$, $I^Y_{21(3)}$, $I^Y_{21(5)}$, $I^Y_{21(9)}$, $I^Y_{21(11)}$, $I^Y_{21(13)}$, for instance,

$$
\begin{aligned}
I^Y_{21(1)}=\iiiint\limits_{\substack{u_1,u_2,u_3,w>0,\\u_1+u_2+u_3+w\le T}}&\mathbb{P}\Big(Y_1>x_1e^{L(u_1+u_2+u_3+w)}\Big)\mathbb{P}\Big(X_2>x_2e^{L(u_2+u_3)}\Big)\\
&\times\mathbb{P}\Big(X_3>x_3e^{L(u_3)}\Big)H_1(dw)\lambda(du_1)\lambda(du_2)\lambda(du_3).
\end{aligned}\tag{3.14}
$$

Similarly for $I^Y_{21(2)}, I^Y_{21(4)}, I^Y_{21(6)}, I^Y_{21(8)}, I^Y_{21(10)}, I^Y_{21(12)}$, for example,

$$
\begin{aligned}
I^Y_{21(2)}=\iiint\limits_{\substack{u_1,u_2,w>0,\\u_1+u_2+w\le T}}&\mathbb{P}\Big(Y_1>x_1e^{L(u_1+u_2+w)}\Big)\mathbb{P}\Big(X_2>x_2e^{L(u_2)}\Big)\\
&\times\mathbb{P}\Big(X_3>x_3e^{L(u_2)}\Big)H_1(dw)\lambda(du_1)\lambda(du_2),
\end{aligned}\tag{3.15}
$$

$$
\begin{aligned}
I^Y_{21(7)}=\iint\limits_{\substack{u,w>0,\\u+w\le T}}&\mathbb{P}\Big(Y_1>x_1e^{L(u+w)}\Big)\mathbb{P}\Big(X_2>x_2e^{L(u)}\Big)\\
&\times\mathbb{P}\Big(X_3>x_3e^{L(u)}\Big)H_1(dw)\lambda(du).
\end{aligned}\tag{3.16}
$$

Combining (3.13)-(3.16) leads to

$$
I^Y_{21}=\sum_{\substack{(h_1,h_2,h_3)\\\in S_3}}\int_{R_3}\prod_{p=1}^{3}\mathbb{P}\Big(W_{h_p}>x_{h_p}e^{L(\sum_{j=1}^{p}u_j+vI_{\{h_p=1\}})}\Big)\prod_{j=1}^{3}\lambda(du_j)H_1(dv)
$$

$$
\begin{aligned}
&+\sum_{h_1\in S_1}\Bigg[\int_{R_{1,2(h_1)}}\mathbb{P}\left(W_{h_1}>x_{h_1}e^{L(u_1+u_2+vI_{\{h_1=1\}})}\right)\\
&\qquad\times\prod_{m\in h_2,h_3}\mathbb{P}\left(W_m>x_me^{L(u_1+vI_{\{m=1\}})}\right)\lambda(du_1)\lambda(u_2)H_1(dv)\\
&\qquad+\int_{R_{2,2(h_1)}}\prod_{m\in h_2,h_3}\mathbb{P}\left(W_m>x_me^{L(u_1+u_2+vI_{\{m=1\}})}\right)\\
&\qquad\times\mathbb{P}\left(W_{h_1}>x_{h_1}e^{L(u_1+vI_{\{h_1=1\}})}\right)\lambda(du_1)\lambda(u_2)H_1(dv)\Bigg]\\
&+\int_{R_1}\prod_{m=1}^{3}\mathbb{P}\left(W_m>x_me^{L(u+vI_{\{m=1\}})}\right)\lambda(du)H_1(dv)\\
=&\chi\big((Y_1,X_2,X_3),x_1,x_2,x_3;T\big),
\end{aligned}
$$

where $(W_1,W_2,W_3)=(Y_1,X_2,X_3)$. Similar to I_{21}, we can extend the result to d-dimensional case

$$
I_{21}^{Y}=\chi\big((Y_1,X_2,\ldots,X_d),x_1,\ldots,x_d;T\big). \tag{3.17}
$$

For I_{22}^{Y}, it holds that

$$
\begin{aligned}
I_{22}^{Y}=&\sum_{n=m+1}^{\infty}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\mathbb{P}\Bigg(Y_{1,i_1}e^{-L(\tau_{i_1}+D_{1,i_1})}I_{\{\tau_{i_1}+D_{1,i_1}\le T\}}>x_1,\\
&\qquad\bigcap_{k=2}^{d}\left\{X_{k,i_k}e^{-L(\tau_{i_k})}>x_k\right\},N(T)=n\Bigg)\\
\le&\sum_{n=m+1}^{\infty}\sum_{1\le i_1,\ldots,i_d\le n}\mathbb{P}\Bigg(Y_{1,i_1}e^{-L(\tau_1+D_1)}I_{\{\tau_1+D_1\le T\}}>x_1,\\
&\qquad\bigcap_{k=2}^{d}\left\{X_{k,i_k}e^{-L(\tau_1)}>x_k\right\},\tau_n\le T\le\tau_{n+1}\Bigg)\\
\le&\sum_{n=m+1}^{\infty}n^d\iint\limits_{\substack{u,w>0,\\u+w\le T}}\mathbb{P}\left(Y_1>x_1e^{L(u+w)}\right)\prod_{k=2}^{d}\mathbb{P}\left(X_k>x_ke^{L(u)}\right)\mathbb{P}\big(N(T-s)=n-1\big)\\
&\qquad\times\mathbb{P}(D_1\in dw)\mathbb{P}(\tau_1\in du)\\
\le&\mathbb{E}\left(\big(N(T)+1\big)^dI_{\{N(T)\ge m\}}\right)\iint\limits_{\substack{u,w>0,\\u+w\le T}}\mathbb{P}\left(Y_1>x_1e^{L(u+w)}\right)\prod_{k=2}^{d}\mathbb{P}\left(X_k>x_ke^{L(u)}\right)
\end{aligned}
$$

$$\times \mathbb{P}(D_1\in dw)\mathbb{P}(\tau_1\in du)$$
$$=o\big(\chi((Y_1,X_2,\ldots,X_d),x_1,\ldots,x_d;T)\big), \tag{3.18}$$

where D_1 is i.i.d. as $\{D_{1,i};i\in\mathbb{N}_+\}$ and is independent of other sources of randomness. In the last step we use [6, Lemma 3.2] and the fact that $\mathbb{E}((N(T)+1)^d)<\infty$ and $\mathbb{E}((N(T)+1)^d I_{\{N(T)\geq m\}})\to 0$ as $m\to\infty$. Hence, plugging (3.17) and (3.18) into (3.12) yields

$$I_{22}^Y\sim\chi\big((Y_1,X_2,\ldots,X_d),x_1,\ldots,x_d;T\big). \tag{3.19}$$

Similarly, we can derive the following asymptotic result:

$$I_2(\xi_{1*},\ldots,\xi_{d*})\sim\chi\big((W_1,\ldots,W_d),x_1,\ldots,x_d;T\big). \tag{3.20}$$

Plugging (3.11), (3.19), (3.20) into (3.3) and combining with (3.2), the proof is completed. □

3.2 Proof of Theorem 3.1

By Lemma 3.2, we have

$$\begin{aligned}\Psi(x_1,\ldots,x_d;T)&\leq\mathbb{P}\left(\bigcap_{k=1}^d\left\{\sum_{i=1}^{N(T)}\left(X_{k,i}e^{-L(\tau_i)}+Y_{k,i}e^{-L(\tau_i+D_{k,i})}I_{\{\tau_i+D_{k,i}\leq T\}}\right)>x_k\right\}\right)\\&\sim\sum_{(W_1,\ldots,W_d)\in\Gamma}\chi\big((W_1,\ldots,W_d),x_1,\ldots,x_d;T\big).\end{aligned}$$

Applying Lemma 3.2, we have

$$\begin{aligned}&\Psi(x_1,\ldots,x_d;T)\\&\geq\mathbb{P}\left(\bigcap_{k=1}^d\left\{\sum_{i=1}^{N(T)}\left(X_{k,i}e^{-L(\tau_i)}+Y_{k,i}e^{-L(\tau_i+D_{k,i})}I_{\{\tau_i+D_{k,i}\leq T\}}\right)>x_k+c_0T\right\}\right)\\&\sim\sum_{(W_1,\ldots,W_d)\in\Gamma}\chi\big((W_1,\ldots,W_d),x_1+c_0T,\ldots,x_d+c_0T;T\big).\end{aligned}$$

Observing the form of $\sum_{(W_1,\ldots,W_d)\in\Gamma}\chi((W_1,\ldots,W_d),x_1,\ldots,x_d;T)$ and recalling that $F_k\in\mathcal{S}\subset\mathcal{L},G_k\in\mathcal{S}\subset\mathcal{L}$ for $k=1,\ldots,d$ and (3.1), we have, for $c_0T>0$,

$$\begin{aligned}&\sum_{(W_1,\ldots,W_d)\in\Gamma}\chi\big((W_1,\ldots,W_d),x_1+c_0T,\ldots,x_d+c_0T;T\big)\\&\sim\sum_{(W_1,\ldots,W_d)\in\Gamma}\chi\big((W_1,\ldots,W_d),x_1,\ldots,x_d;T\big).\end{aligned}$$

Thus, we have

$$\Psi(x_1,\ldots,x_d;T) \gtrsim \sum_{(W_1,\ldots,W_d)\in\Gamma} \chi\big((W_1,\ldots,W_d),x_1,\ldots,x_d;T\big).$$

This completes the proof of Theorem 3.1.

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References

[1] Z. Chen, M. Li, and D. Cheng, *Asymptotics for sum-ruin probabilities of a bidimensional risk model with heavy-tailed claims and stochastic returns*, Stochastics 96(2) (2024), 947–967.

[2] D. Cheng, Y. Yang, and X. Wang, *Asymptotic finite-time ruin probabilities in a dependent bidimensional renewal risk model with subexponential claims*, Jap. J. Ind. Appl. Math. 37 (2020), 657–675.

[3] M. Cheng, D. G. Konstantinides, and D. Wang, *Multivariate regularly varying insurance and financial risks in multidimensional risk models*, J. Appl. Probab. 61(4) (2024), 1319–1342.

[4] P. Embrechts, C. Klüppelberg, and T. Mikosch, *Modelling Extremal Events*, Springer-Verlag, 1997.

[5] K.-A. Fu and Y. Liu, *Ruin probabilities for a multidimensional risk model with non-stationary arrivals and subexponential claims*, Probab. Eng. Inf. Sci. 36(3) (2022), 799–811.

[6] X. Hao and Q. Tang, *A uniform asymptotic estimate for discounted aggregate claims with subexponential tails*, Insur. Math. Econ. 43(1) (2008), 116–120.

[7] W. Huang, C. Weng, and Y. Zhang, *Multivariate risk models under heavy-tailed risks*, Appl. Stoch. Models Bus. Ind. 30 (2014), 341–360.

[8] B. Ko and Q. Tang, *Sums of dependent nonnegative random variables with subexponential tails*, J. Appl. Probab. 45(1) (2008), 85–94.

[9] D. G. Konstantinides and J. Li, *Asymptotic ruin probabilities for a multidimensional renewal risk model with multivariate regularly varying claims*, Insur. Math. Econ. 69 (2016), 38–44.

[10] D. Konstantinides, Q. Tang, and G. Tsitsiashvili, *Estimates for the ruin probability in the classical risk model with constant interest force in the presence of heavy tails*, Insur. Math. Econ. 31(3) (2002), 447–460.

[11] J. Li, *On pairwise quasi-asymptotically independent random variables and their applications*, Stat. Probab. Lett. 83(9) (2013), 2081–2087.

[12] J. Li, *Uniform asymptotics for a multi-dimensional time-dependent risk model with multivariate regularly varying claims and stochastic return*, Insur. Math. Econ. 71 (2016), 195–204.

[13] X. Li, J. Wu, and J. Zhuang, *Asymptotic multivariate finite-time ruin probability with statistically dependent heavy-tailed claims*, Methodol. Comput. Appl. Probab. 17 (2015), 463–477.

[14] X. Liu, Q. Gao, and Z. Dong, *Asymptotics for a diffusion-perturbed risk model with dependence structures, constant interest force, and a random number of delayed claims*, Stoch. Models 40(1) (2024), 97–122.

[15] D. Lu and M. Yuan, *Asymptotic finite-time ruin probabilities for a bidimensional delay-claim risk model with subexponential claims*, Methodol. Comput. Appl. Probab. 24(4) (2022), 2265–2286.

[16] G. Samorodnitsky and J. Sun, *Multivariate subexponential distributions and their applications*, Extremes 19 (2016), 171–196.

[17] Y. Sun, M. Yuan, and D. Lu, *Asymptotic estimates for ruin probabilities in a bidimensional delay-claim risk model with subexponential claims*, Commun. Statist. Theory Methods 54(11) (2024), 3149–3173.

[18] Q. Tang and Z. Yuan, *Randomly weighted sums of subexponential random variables with application to capital allocation*, Extremes 17(3) (2014), 467–493.

[19] S. Wang, Y. Yang, Y. Liu, and L. Yang, *Asymptotics for a bidimensional renewal risk model with subexponential main claims and delayed claims*, Methodol. Comput. Appl. Probab. 25(3) (2023), Paper No. 76.

[20] H. R. Waters and A. Papatriandafylou, *Ruin probabilities allowing for delay in claims settlement*, Insur. Math. Econ. 4(2) (1985), 113–122.

[21] Y. Xiao and J. Guo, *The compound binomial risk model with time-correlated claims*, Insur. Math. Econ. 41(1) (2007), 124–133.

[22] J. Xie and W. Zou, *Expected present value of total dividends in a delayed claims risk model under stochastic interest rates*, Insur. Math. Econ. 46(2) (2010), 415–422.

[23] H. Yang and J. Li, *Asymptotic finite-time ruin probability for a bidimensional renewal risk model with constant interest force and dependent subexponential claims*, Insur. Math. Econ. 58 (2014), 185–192.

[24] Y. Yang, B. Geng, and S. Wang, *On asymptotic ruin probability for a bidimensional renewal risk model with dependent and subexponential main claims and delayed claims*, Jap. J. Ind. Appl. Math. 41(2) (2024), 1189–1205.

[25] Y. Yang, J. Lin, C. Huang, and X. Ma, *The finite-time ruin probability in two non-standard renewal risk models with constant interest rate and dependent subexponential claims*, J. Korean Stat. Soc. 41(2) (2012), 213–224.

[26] Y. Yang and Q. Su, *Asymptotic behavior of ruin probabilities in a multidimensional risk*

model with investment and multivariate regularly varying claims, J. Math. Anal. Appl. 525(2) (2023), Paper No. 127319.

[27] K. C. Yuen and J. Y. Guo, *Ruin probabilities for time-correlated claims in the compound binomial model*, Insur. Math. Econ. 29(1) (2001), 47–57.

[28] K. C. Yuen, J. Guo, and K. W. Ng, *On ultimate ruin in a delayed-claims risk model*, J. Appl. Probab. 42(1) (2005), 163–174.

[29] W. Zou and J.-H. Xie, *On the probability of ruin in a continuous risk model with delayed claims*, J. Korean Math. Soc. 50(1) (2013), 111–125.