

The Exponential Decaying Noncommutative Differential Forms and Convolution Algebras

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Abstract. We construct an algebra of rapidly decaying C_0 functions on an étale (Lie) groupoid, which extends the standard algebra of compactly supported noncommutative differential forms. In particular, using the theory of bisections, we prove that this algebra is closed under convolution. This construction clarifies the superconnection proof of Gorokhovsky and Lott.

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1 Introduction

Gorokhovsky and Lott [4] gave a superconnection heat kernel proof, in the style of Bismut, of Connes' index theorem. They considered a smooth étale groupoid G acting on a G -proper manifold and a G -Dirac type operator D . Given a closed graded trace η on $\Omega_\omega^\bullet(G)$, they proved that

$$\langle \text{ch}(\text{Ind}(D)), \eta \rangle = \int_M \hat{A}(T\mathcal{F}) \text{ch}(V) v^* \Phi_\eta \in \mathbb{C}. \quad (1.1)$$

As pointed out in [8], Connes' index theorem for G -proper manifolds unified most of the existing index theorems at that time under a single statement.

Before delving further into the work of Gorokhovsky and Lott, we first recall some essential background on index theory. As a starting point, we recall the family index theorem, which considers a family of elliptic pseudodifferential operators that depend continuously on a parameter from some compact space B .

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The index of such a family of operators is an element in the K -theory of B . A special case occurs when the kernel and the cokernel are vector bundles. In such a case, the index is the difference of the classes of these bundles [1].

One can reformulate this situation by considering the C^* -algebra $C(B)$ instead of the topology of B . For instance, a vector bundle over B is just a finitely generated projective $C(B)$ -module. Any family of elliptic operators is invertible modulo fiberwise smoothing operators. Hence, its index lies in the C^* -algebraic K -theory of the algebra of fiberwise smoothing operators, which is isomorphic to $K_0(B)$. For general C^* -algebras, the Mishchenko-Fomenko index theorem [11] for elliptic operators on Hilbert C^* -modules formalized and generalized this point of view. The index of such an elliptic operator is an element in the K -theory of the C^* -algebra.

The index problem is simplified by considering the Chern character of $\text{Ind}(D)$, which lies in de Rham cohomology $H^*(B)$. Bismut's elementary proof of the local family index theorem (at the level of Chern characters) used the superconnection formalism. One considers a family of generalized Laplacians with differential form coefficients, rather than the family of Dirac Laplacians $(D_z)_{z \in B}^2$. When combined, the fiber supertraces of the heat kernels corresponding to this new family of generalized Laplacians produce a closed differential form on B , known as the superconnection Chern form. It can be proven that $\text{ch}(\text{Ind}(D))$ equals the cohomology class of the superconnection Chern form (cf. [2, Theorem 9.33]). For more details, see [2, 7, 15].

Replacing $C(B)$ by the (noncommutative) convolution algebra of an étale groupoid in the superconnection formalism described above, Gorokhovsky and Lott extended the local family index theorem. According to Connes' index theorem, the index of D belongs to the K -theory group $K_0(C_r^*(G))$. Since $C_r^*(G)$ lacks a naturally dense smooth subalgebra stable under the holomorphic functional calculus, Gorokhovsky and Lott overcame this problem by defining $\text{Ind}(D)$ as the K -theory group element represented by the difference between the index projection and a standard projection. In their setting, $\text{Ind}(D)$ (refined index class) is an element in the K -theory of a certain algebra of smoothing $C^*(G)$ -operators. They then developed homological computations and defined a graded differential algebra (GDA) that can be considered a space of noncommutative forms in the general étale groupoid case. Its cohomology generalizes the de Rham cohomology. The key ingredient of the superconnection proof is the heat kernel, which is defined by [4, (6.42)]

$$\begin{aligned} e^{-D(u)^2} &= \exp(-D(u)^2) \\ &= \exp(-\Delta_u) + \int_0^1 \exp(-\sigma_1 \Delta_u) \star \mathfrak{F} \star \exp(-(1-\sigma_1) \Delta_u) d\sigma_1 \end{aligned}$$