Commun. Math. Res.
doi: 10.4208/cmr.2026-0019

Vol. **x**, No. x, pp. 1-29
x 2026

A Symmetric Mountain Pass Lemma and Its Application to an Equation for a Class of Quasilinear Elliptic Operators in a Variable Exponent Sobolev Space

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Received 30 January 2026; Accepted 18 March 2026

Abstract. The purpose of this paper is to solve equation for a class of quasilinear elliptic operators containing the $p(\cdot)$-Laplacian and the mean curvature operator with mixed boundary conditions. More precisely, we are concerned with the problem that has the Dirichlet condition in one part of the boundary and the Steklov condition in another. Using a symmetric mountain pass lemma and its corollary, we show the existence of infinitely many weak solutions of the equation and the boundedness of the sequence of solutions, or convergence to zero of the sequence of solutions according to the hypotheses about the data functions.

AMS subject classifications: 35J66, 35D30, 35J62, 35J57

Key words: Mountain pass lemma, $p(\cdot)$-Laplacian type equation, mean curvature operator, mixed boundary value problem.

1 Introduction

In this paper, we consider the following problem under the mixed boundary value condition:

$$\begin{cases} -\operatorname{div}\left[\boldsymbol{a}(x,\nabla u(x))\right]=f\left(x,u(x)\right) & \text{in } \Omega, \\ u(x)=0 & \text{on } \Gamma_1, \\ \boldsymbol{n}(x)\cdot\boldsymbol{a}\left(x,\nabla u(x)\right)=g\left(x,u(x)\right) & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

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Here Ω is a bounded domain of $\mathbb{R}^N$ $(N\geq 2)$ with a Lipschitz-continuous ($C^{0,1}$ for short) boundary Γ satisfying that Γ_1 and Γ_2 are disjoint open subsets of Γ such that

$$\overline{\Gamma_1}\cup\overline{\Gamma_2}=\Gamma, \quad \Gamma_1\neq\emptyset, \tag{1.2}$$

and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. The function $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$. Here we say that $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$, if for a.e. $x\in\Omega$, the map $\mathbb{R}^N\ni\boldsymbol{\xi}\mapsto\boldsymbol{a}(x,\boldsymbol{\xi})$ is continuous and for every $\boldsymbol{\xi}\in\mathbb{R}^N$, the map $\Omega\ni x\mapsto\boldsymbol{a}(x,\boldsymbol{\xi})$ is measurable on Ω. The operator $u\mapsto\operatorname{div}[\boldsymbol{a}(x,\nabla u(x))]$ is more general than the $p(\cdot)$-Laplacian

$$\Delta_{p(x)}u(x)=\operatorname{div}[|\nabla u(x)|^{p(x)-2}\nabla u(x)]$$

and the mean curvature operator $\operatorname{div}[(1+|\nabla u(x)|^2)^{(p(x)-2)/2}\nabla u(x)]$. This generality brings about difficulties and requires some conditions.

We impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $g:\Gamma_2\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying some structure conditions.

The study of differential equations with $p(\cdot)$-growth conditions has recently become a very interesting topic. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics [33], in electrorheological fluids [13,23,27,29]. However, since we find a few papers associated with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1) (for example, [2–4, 8, 10]), we believe this paper has merit.

As Fan [17] takes the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as the base space, it is necessary to add the term $|u(x)|^{p(x)-2}u(x)$ to the left-hand side of the first equation of (1.1). However, since we consider the Dirichlet boundary condition on $\Gamma_1\neq\emptyset$ in this paper, we can delete such a term according to a Poincaré-type inequality due to Ciarlet and Dinca [11]. In this paper, we extend these results to a class of quasilinear elliptic operators containing the $p(\cdot)$-Laplacian and the mean curvature operator. In the previous paper [6], we treated the case $p(x)\geq 2$ on $\overline{\Omega}$, however, in the present paper, we consider the case with not only the case $p(x)\geq 2$ on $\overline{\Omega}$, but also the case $p(x)>1$ on $\overline{\Omega}$. The case $p(x)>1$ on $\overline{\Omega}$ is more difficult. To overcome this, we apply the Glowinski and Marroco technique [22] which is used in the case $p(x)=p=\text{const.}>1$. As a result, we can derive the uniform monotonicity of $\boldsymbol{a}(x,\nabla u(x))$ in the case where $p(x)>1$ on $\overline{\Omega}$. This is a new result. By this, we can show that the corresponding functional satisfies the Paley-Smale condition ((PS)-condition for short).

The paper is organized as follows. In Section 2, we recall some results on variable exponent Lebesgue-Sobolev spaces. In Section 3, we give a mountain pass lemma and its corollary in which we show the existence of infinitely many critical points. In Section 4, we present some assumptions and preparations to the main theorems. Finally, in Section 5, we give the main theorems (Theorems 5.1, 5.2) and their proofs.

2 Preliminaries

Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^N$ $(N\geq 2)$ with a $C^{0,1}$-boundary Γ and Ω is locally on the same side of Γ. Moreover, we assume that Γ satisfies (1.2).

In the present paper, we only consider real vector spaces. For any space B, we denote B^N by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a}=(a_1,\dots,a_N)$ and $\boldsymbol{b}=(b_1,\dots,b_N)$ in $\mathbb{R}^N$ by $\boldsymbol{a}\cdot\boldsymbol{b}=\sum_{i=1}^N a_ib_i$ and $|\boldsymbol{a}|=(\boldsymbol{a}\cdot\boldsymbol{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle\cdot,\cdot\rangle_{B^*,B}$.

We recall some well-known results on variable exponent Lebesgue and Sobolev spaces. See Fan and Zhang [19], Kováčik and Rácosník [25], Diening *et al.* [14] and references therein for more detail. Furthermore, we consider some new properties on variable exponent Lebesgue space. Define

$$C(\overline{\Omega})=\{p;p \text{ is a continuous function on } \overline{\Omega}\},$$

and for any $p\in C(\overline{\Omega})$, put

$$\begin{aligned} p^+&=p^+(\Omega)=\sup_{x\in\Omega}p(x)=\max_{x\in\overline{\Omega}}p(x),\\ p^-&=p^-(\Omega)=\inf_{x\in\Omega}p(x)=\min_{x\in\overline{\Omega}}p(x). \end{aligned}$$

For any $p\in C(\overline{\Omega})$ with $p^-\geq 1$ and for any measurable function u on Ω, a modular $\rho_{p(\cdot)}=\rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u)=\int_\Omega |u(x)|^{p(x)}dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega)=\{u;u:\Omega\to\mathbb{R} \ \text{ is a measurable function satisfying } \ \rho_{p(\cdot)}(u)<\infty\}$$

equipped with the (Luxemburg) norm

$$\|u\|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\lambda>0;\,\rho_{p(\cdot)}\left(\frac{u}{\lambda}\right)\le 1\right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define the Sobolev space

$$W^{1,p(\cdot)}(\Omega)=\{u\in L^{p(\cdot)}(\Omega);\,|\nabla u|\in L^{p(\cdot)}(\Omega)\}$$

endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)}=\|u\|_{L^{p(\cdot)}(\Omega)}+\||\nabla u|\|_{L^{p(\cdot)}(\Omega)}.$$

The following three propositions are well known (see [20,21,32]).

Proposition 2.1. *Let $p\in C(\overline{\Omega})$ with $p^-\ge 1$, and let $u,u_n\in L^{p(\cdot)}(\Omega)$ $(n=1,2,\ldots)$. Then we have the following properties:*

- *(i)* $\|u\|_{L^{p(\cdot)}(\Omega)}<1(=1,>1) \iff \rho_{p(\cdot)}(u)<1(=1,>1)$.
- *(ii)* $\|u\|_{L^{p(\cdot)}(\Omega)}>1 \implies \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}\le\rho_{p(\cdot)}(u)\le\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.
- *(iii)* $\|u\|_{L^{p(\cdot)}(\Omega)}<1 \implies \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\le\rho_{p(\cdot)}(u)\le\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.
- *(iv)* $\lim_{n\to\infty}\|u_n-u\|_{L^{p(\cdot)}(\Omega)}=0 \iff \lim_{n\to\infty}\rho_{p(\cdot)}(u_n-u)=0$.
- *(v)* $\|u_n\|_{L^{p(\cdot)}(\Omega)}\to\infty$ *as* $n\to\infty \iff \rho_{p(\cdot)}(u_n)\to\infty$ *as* $n\to\infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let $p\in C_+(\overline{\Omega})$, where $C_+(\overline{\Omega}):=\{p\in C(\overline{\Omega});p^->1\}$. For any $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$, we have*

$$\begin{aligned}\int_\Omega |u(x)v(x)|dx &\le\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}\\ &\le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.\end{aligned}$$

Here and from now on, for any $p\in C_+(\overline{\Omega})$, $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$, that is, $p'(x)=p(x)/(p(x)-1)$.

For $p\in C_+(\overline{\Omega})$, define for $x\in\overline{\Omega}$,

$$p^*(x)=\begin{cases}\dfrac{Np(x)}{N-p(x)}, & \text{if} \quad p(x)<N,\\ \infty, & \text{if} \quad p(x)\ge N.\end{cases}$$

Proposition 2.3. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *with* $C^{0,1}$*-boundary and let* $p\in C_+(\overline{\Omega})$. *Then we have the following properties:*

(i) The spaces $L^{p(\cdot)}(\Omega)$ *and* $W^{1,p(\cdot)}(\Omega)$ *are separable, reflexive and uniformly convex Banach spaces.*

(ii) If $q(\cdot)\in C(\overline{\Omega})$ *with* $q^-\geq 1$ *satisfies* $q(x)\leq p(x)$ *for all* $x\in\Omega$, *then* $W^{1,p(\cdot)}(\Omega)\hookrightarrow W^{1,q(\cdot)}(\Omega)$, *where the symbol* $\hookrightarrow$ *means that the embedding is continuous.*

(iii) If $q(x)\in C(\overline{\Omega})$ *with* $q^-\geq 1$ *satisfies that* $q(x)<p^*(x)$ *for all* $x\in\Omega$, *then the embedding* $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ *is compact.*

The following proposition is due to [15, Lemma 2.1].

Proposition 2.4. *Let* $q\in L^\infty(\Omega)$ *and* p *be a measurable function on* Ω *such that* $1\leq p(x)\leq\infty$ *and* $1\leq q(x)p(x)\leq\infty$. *Assume that* $f\in L^{p(\cdot)}(\Omega)$ *with* $f\neq 0$. *Then we have the following:*

(i) $\|f\|_{L^{q(\cdot)p(\cdot)}(\Omega)}\leq 1 \implies \|f\|^{q^+}_{L^{q(\cdot)p(\cdot)}(\Omega)}\leq \||f|^{q(\cdot)}\|_{L^{p(\cdot)}(\Omega)}\leq\|f\|^{q^-}_{L^{q(\cdot)p(\cdot)}(\Omega)}.$

(ii) $\|f\|_{L^{q(\cdot)p(\cdot)}(\Omega)}\geq 1 \implies \|f\|^{q^-}_{L^{q(\cdot)p(\cdot)}(\Omega)}\leq \||f|^{q(\cdot)}\|_{L^{p(\cdot)}(\Omega)}\leq\|f\|^{q^+}_{L^{q(\cdot)p(\cdot)}(\Omega)}.$

In particular, if $q(x)=q=\text{const.}$, *then* $\||f|^q\|_{L^{p(\cdot)}(\Omega)}=\|f\|^q_{L^{qp(\cdot)}(\Omega)}$.

Next we consider the trace (cf. [18]). Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and $p\in C(\overline{\Omega})$ with $p^-\geq 1$. Since $W^{1,p(\cdot)}(\Omega)\subset W^{1,1}(\Omega)$ from Proposition 2.3(ii), the trace $\gamma(u)=u|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1(\Gamma)$. We define

$$(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)=\{f;\, f \text{ is the trace to } \Gamma \text{ of a function } F\in W^{1,p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}=\inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)};\, F\in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F|_\Gamma=f\}$$

for $f\in(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$, where the infimum can be achieved. Then we can see that $(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$ is a Banach space. In the later, we also write $F|_\Gamma=g$ by $F=g$ on Γ. Moreover, for $i=1,2$, we denote

$$(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)=\{f|_{\Gamma_i};\, f\in(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)\}$$

equipped with the norm

$$\|g\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)}=\inf\{\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)};\, f\in(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma) \text{ satisfying } f|_{\Gamma_i}=g\},$$

where the infimum can also be achieved, so for any $g\in(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)$, there exists $F\in W^{1,p(\cdot)}(\Omega)$ such that $F|_{\Gamma_i}=g$ and $\|F\|_{W^{1,p(\cdot)}(\Omega)}=\|g\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)}$.

Let $q\in C_+(\Gamma):=\{q\in C(\Gamma);q^->1\}$ and denote the surface measure on Γ induced from the Lebesgue measure dx on Ω by $d\sigma_x$. We define

$$\begin{aligned}L^{q(\cdot)}(\Gamma)=\Big\{&u;\, u:\Gamma\to\mathbb{R} \text{ is a measurable function with respect to } d\sigma_x\\ &\qquad\qquad \text{satisfying } \int_\Gamma |u(x)|^{q(x)}d\sigma_x<\infty\Big\}\end{aligned}$$

and the norm

$$\|u\|_{L^{q(\cdot)}(\Gamma)}=\inf\left\{\lambda>0;\int_\Gamma\left|\frac{u(x)}{\lambda}\right|^{q(x)}d\sigma_x\le 1\right\}.$$

We also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u)=\int_\Gamma |u(x)|^{q(x)}d\sigma_x.$$

Similar to Proposition 2.1, we have the following proposition.

Proposition 2.5. *Let $q\in C(\Gamma)$ with $q^-\ge 1$, and let $u,u_n\in L^{q(\cdot)}(\Gamma)$ $(n=1,2,\ldots)$. Then we have the following properties:*

(i) $\|u\|_{L^{q(\cdot)}(\Gamma)}<1(=1,>1) \iff \rho_{q(\cdot),\Gamma}(u)<1(=1,>1)$.

(ii) $\|u\|_{L^{q(\cdot)}(\Gamma)}>1 \implies \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}\le\rho_{q(\cdot),\Gamma}(u)\le\|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.

(iii) $\|u\|_{L^{q(\cdot)}(\Gamma)}<1 \implies \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}\le\rho_{q(\cdot),\Gamma}(u)\le\|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.

(iv) $\|u_n\|_{L^{q(\cdot)}(\Gamma)}\to 0 \iff \rho_{q(\cdot),\Gamma}(u_n)\to 0$.

(v) $\|u_n\|_{L^{q(\cdot)}(\Gamma)}\to\infty \iff \rho_{q(\cdot),\Gamma}(u_n)\to\infty$.

For $p\in C_+(\overline{\Omega})$, define for $x\in\overline{\Omega}$,

$$p^\partial(x)=\begin{cases}\dfrac{(N-1)p(x)}{N-p(x)}, & \text{if} \quad p(x)<N,\\ \infty, & \text{if} \quad p(x)\ge N.\end{cases}$$

The following proposition follows from [30, Proposition 2.6].

Proposition 2.6. *Let $p\in C_+(\overline{\Omega})$. Then if $q\in C_+(\Gamma)$ satisfies $q(x)<p^\partial(x)$ for all $x\in\Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Gamma)$ is well-defined and compact.*

Now we consider a weighted variable exponent Lebesgue space. Let $p\in C(\overline{\Omega})$ with $p^-\geq 1$ and let $a(x)$ be a measurable function on Ω with $a(x)>0$ a.e. $x\in\Omega$. We define a modular

$$\rho_{(p(\cdot),a(\cdot))}(u)=\int_\Omega a(x)|u(x)|^{p(x)}dx$$

for any measurable function u in Ω. Then the weighted Lebesgue space is defined by

$$L^{p(\cdot)}_{a(\cdot)}(\Omega)=\Big\{u;\ u \text{ is a measurable function on } \Omega \text{ satisfying } \rho_{(p(\cdot),a(\cdot))}(u)<\infty\Big\}$$

equipped with the norm

$$\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}=\inf\left\{\lambda>0;\int_\Omega a(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\leq 1\right\}.$$

Then $L^{p(\cdot)}_{a(\cdot)}(\Omega)$ is a Banach space.

We have the following proposition (cf. [16, Proposition 2.5]).

Proposition 2.7. *Let $p\in C(\overline{\Omega})$ with $p^-\geq 1$. For $u,u_n\in L^{p(\cdot)}_{a(\cdot)}(\Omega)$ $(n=1,2,\ldots)$, we have the following:*

(i) For $u\neq 0$, $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}=\lambda \iff \rho_{(p(\cdot),a(\cdot))}(u/\lambda)=1$.

(ii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}<1(=1,>1)\Longleftrightarrow\rho_{(p(\cdot),a(\cdot))}(u)<1(=1,>1)$.

(iii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}>1 \implies \|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}\leq\rho_{(p(\cdot),a(\cdot))}(u)\leq\|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(iv) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}<1 \implies \|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}\leq\rho_{(p(\cdot),a(\cdot))}(u)\leq\|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(v) $\lim_{n\to\infty}\|u_n-u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}=0 \iff \lim_{n\to\infty}\rho_{(p(\cdot),a(\cdot))}(u_n-u)=0$.

(vi) $\|u_n\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}\to\infty$ *as* $n\to\infty \iff \rho_{(p(\cdot),a(\cdot))}(u_n)\to\infty$ *as* $n\to\infty$.

Fan [16] also derived the following proposition (cf. [16, Theorem 2.1]).

Proposition 2.8. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *with a* $C^{0,1}$*-boundary and* $p \in C_+(\overline{\Omega})$*. Moreover, let* $a \in L^{\alpha(\cdot)}(\Omega)$ *satisfy* $a(x)>0$ *a.e.* $x \in \Omega$ *and* $\alpha \in C_+(\overline{\Omega})$*. If* $q \in C(\overline{\Omega})$ *satisfies*

$$1 \le q(x) < \frac{\alpha(x)-1}{\alpha(x)} p^*(x) \quad \text{for all} \quad x \in \overline{\Omega},$$

then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Omega)$ *is compact.*

Similarly, let $q \in C(\Gamma)$ with $q^- \ge 1$ and let $b(x)$ be a measurable function with respect to σ on Γ with $b(x)>0$ σ-a.e. $x \in \Gamma$. We define a modular

$$\rho_{(q(\cdot),b(\cdot)),\Gamma}(u) = \int_\Gamma b(x)|u(x)|^{q(x)} d\sigma_x.$$

Then the weighted Lebesgue space on Γ is defined by

$$L^{q(\cdot)}_{b(\cdot)}(\Gamma) = \{u;\ u \text{ is a } \sigma\text{-measurable function on } \Gamma \text{ satisfying } \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) < \infty\}$$

equipped with the norm

$$\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} = \inf\left\{ \lambda > 0; \int_\Gamma b(x) \left| \frac{u(x)}{\lambda} \right|^{q(x)} d\sigma_x \le 1 \right\}.$$

Note that $L^{q(\cdot)}_{b(\cdot)}(\Gamma)$ is a Banach space.

We have the following proposition.

Proposition 2.9. *Let* $q \in C(\Gamma)$ *with* $q^- \ge 1$*. For* $u, u_n \in L^{q(\cdot)}_{b(\cdot)}(\Gamma)$ $(n = 1,2,\ldots)$*, we have the following:*

(i) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} < 1 (=1, >1) \iff \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) < 1 (=1, >1)$.

(ii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} > 1 \implies \|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \le \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} < 1 \implies \|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \le \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} = 0 \iff \lim_{n\to\infty} \rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}\to\infty$ *as* $n\to\infty$ $\iff$ $\rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n)\to\infty$ *as* $n\to\infty$.

The following proposition plays an important role in the present paper.

Proposition 2.10. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *with a* $C^{0,1}$*-boundary* Γ *and let* $p\in C_+(\overline{\Omega})$. *Assume that* $0<b\in L^{\beta(\cdot)}(\Gamma)$, $\beta\in C_+(\Gamma)$. *If* $r\in C(\Gamma)$ *satisfies*

$$1\le r(x)<\frac{\beta(x)-1}{\beta(x)}p^{\partial}(x)\quad \textit{for all}\quad x\in\Gamma,$$

then the embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma)$ *is compact.*

For the proof, see [6, Proposition 2.11].

Now we consider the Nemytskii operator.

Proposition 2.11. *Let* $q\in C(\overline{\Omega})$ *with* $q^-\ge 1$ *and* a *be a measurable function with* $a(x)>0$ *for a.e.* $x\in\Omega$. *Assume that*

(F.1) A function $F(x,t)$ *is a Carathéodory function on* $\Omega\times\mathbb{R}$.

(F.2) The growth condition holds: there exist $c\in L^{q_1(\cdot)}(\Omega)$ *with* $c(x)\ge 0$ *for a.e.* $x\in\Omega$, $q_1\in C(\overline{\Omega})$ *with* $q_1^-\ge 1$, *and a constant* $c_1>0$ *such that*

$$|F(x,t)|\le c(x)+c_1 a(x)^{1/q_1(x)}|t|^{q(x)/q_1(x)}\quad \textit{for a.e.}\quad x\in\Omega\quad \textit{and all}\quad t\in\mathbb{R}.$$

Then the Nemytskii operator $N_F: L^{q(\cdot)}_{a(\cdot)}(\Omega)\ni u\mapsto F(\cdot,u(\cdot))\in L^{q_1(\cdot)}(\Omega)$ *is continuous and there exists a constant* $C>0$ *such that*

$$\rho_{q_1(\cdot)}(N_F(u))\le C(\rho_{q_1(\cdot)}(u)+\rho_{(q(\cdot),a(\cdot))}(u))\quad \textit{for all}\quad u\in L^{q(\cdot)}_{a(\cdot)}(\Omega).$$

In particular, if $q_1(x)\equiv 1$, *then* $N_F: L^{q(\cdot)}_{a(\cdot)}(\Omega)\to L^1(\Omega)$ *is continuous.*

For the proof, see [6, Proposition 2.12].

Similarly we have the following proposition.

Proposition 2.12. *Let* $r\in C(\overline{\Gamma_2})$ *with* $r^-\ge 1$ *and* b *be a* σ*-measurable function with* $b(x)>0$ *for* σ*-a.e.* $x\in\Gamma_2$. *Assume that*

(G.1) A function $G(x,t)$ *is a Carathéodory function on* $\Gamma_2\times\mathbb{R}$.

(G.2) The growth condition holds: there exist $d\in L^{r_1(\cdot)}(\Gamma_2)$ *with* $d(x)\ge 0$ *for* σ*-a.e.* $x\in\Gamma_2$, $r_1\in C(\overline{\Gamma_2})$ *with* $r_1\ge 1$, *and a constant* $d_1>0$ *such that*

$$|G(x,t)|\le d(x)+d_1 b(x)^{1/r_1(x)}|t|^{r(x)/r_1(x)} \quad \text{for} \quad \sigma\text{-a.e.} \quad x\in\Gamma_2 \quad \text{and all} \quad t\in\mathbb{R}.$$

Then the Nemytskii operator $N_G: L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)\ni v\mapsto G(\cdot,v(\cdot))\in L^{r_1(\cdot)}(\Gamma_2)$ *is continuous and there exists a constant* $C>0$ *such that*

$$\rho_{r_1(\cdot),\Gamma_2}(N_G(v))\le C\big(\rho_{r_1(\cdot),\Gamma_2}(d)+\rho_{(r(\cdot),b(\cdot)),\Gamma_2}(v)\big) \quad \text{for all} \quad v\in L^{r(\cdot)}_{b(\cdot)}(\Gamma_2).$$

In particular, if $r_1(x)\equiv 1$, *then* $N_G: L^{r(\cdot)}(\Gamma_2)\to L^1(\Gamma_2)$ *is continuous.*

Define a space by

$$X=\big\{v\in W^{1,p(\cdot)}(\Omega);\ v=0 \text{ on } \Gamma_1\big\}. \tag{2.1}$$

Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable Banach space. We introduce the following Poincaré-type inequality (cf. [11]).

Proposition 2.13. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *with a* $C^{0,1}$*-boundary and let* $p\in C_+(\overline{\Omega})$. *Then there exists a constant* $C=C(\Omega,N,p)>0$ *such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)}\le C\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \text{ for all } u\in X.$$

In particular, the norm $\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$ *is equivalent to the norm* $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ *for* $u\in X$.

For the direct proof, see [2, Lemma 2.5].

Thus, we can define the norm on X so that

$$\|v\|_X=\|\nabla v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \quad \text{for} \quad v\in X, \tag{2.2}$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ from Proposition 2.13.

3 A symmetric mountain pass lemma and its corollary

In this section, we introduce a symmetric mountain pass lemma and its corollary. For this purpose, first we recall the notion of “genus” which is introduced in [28, Chapter 7]. This notion is particularly needed in the Section 5.

Let E be a real Banach space and let $\mathcal{E}$ denote the family of subsets $A\subset E\setminus\{0\}$ such that A is closed in E and symmetric with respect to 0, that is, $u\in A$ implies $-u\in A$. For $A\in\mathcal{E}$, define the genus of A to be n (denoted by $\gamma(A)=n$) if there is a map $\varphi\in C(A,\mathbb{R}^n\setminus\{\mathbf{0}\})$ with φ odd and n is the smallest integer with this property. When there does not exist a finite such n, set $\gamma(A)=\infty$. Finally set $\gamma(\emptyset)=0$.

The main properties of genus will be listed in the next proposition.

Proposition 3.1. *Let $A,B\in\mathcal{E}$. Then the following properties hold:*

(i) If there exists an odd continuous map from A to B, then $\gamma(A)\le\gamma(B)$. In particular, if $A\subset B$, then $\gamma(A)\le\gamma(B)$.

(ii) If there exists an odd homeomorphism from A onto B, then $\gamma(A)=\gamma(B)$.

(iii) $\gamma(A\cup B)\le\gamma(A)+\gamma(B)$.

(iv) If $\gamma(B)<\infty$, then $\gamma(\overline{A\setminus B})\ge\gamma(A)-\gamma(B)$.

(v) If A is compact, then $\gamma(A)<\infty$ and there exists $\delta>0$ such that if we put $N_\delta(A)=\{u\in E;\|u-A\|:=\inf\{\|u-v\|;v\in A\}\le\delta\}$, then $N_\delta(A)\in\mathcal{E}$ and $\gamma(N_\delta(A))=\gamma(A)$.

(vi) If $\widehat{\Omega}$ is a bounded neighborhood of $\mathbf{0}$ in $\mathbb{R}^n$, and there exists a map $h\in C(A,\partial\widehat{\Omega})$ with h an odd homeomorphism, then $\gamma(A)=n$.

For the proof, see [28, Lemma 7.5 and Proposition 7.7]. We note that it can be easily seen that when $A\in\mathcal{E}$, $A\neq\emptyset$ if and only if $\gamma(A)\ge1$.

Definition 3.1. *When a functional I on a real Banach space E belongs to $C^1(E,\mathbb{R})$, we say that I satisfies the Palais-Smale condition ((PS)-condition), if a sequence $\{u_n\}_{n=1}^\infty\subset E$ satisfies that*

$$\lim_{n\to\infty} I(u_n)=d \quad \text{exists in } \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty}\|I'(u_n)\|_{E^*}=0,$$

then the sequence $\{u_n\}_{n=1}^\infty$ has a convergent subsequence.

Define

$$\widehat{\Gamma}_k=\{A\in\mathcal{E};\ \gamma(A)\ge k\} \quad \text{for} \quad k=1,2,\dots, \tag{3.1}$$

$$d_k=\inf_{A\in\widehat{\Gamma}_k}\sup_{u\in A} I(u) \qquad \text{for} \quad k=1,2,\dots \tag{3.2}$$

and for $d\in\mathbb{R}$,

$$K_d=\{u\in E;\ I(u)=d, I'(u)=0\}. \tag{3.3}$$

If $K_d\neq\emptyset$, we call d a critical value of I and $u\in K_d$ a critical point of I.

The following theorem is a slight improvement of a symmetric mountain pass lemma (cf. [1,12,24]).

Theorem 3.1 (A Symmetric Mountain Pass Lemma). *Let $(E,\|\cdot\|_E)$ be an infinite-dimensional real Banach space and $I\in C^1(E,\mathbb{R})$ satisfying (PS)-condition. Furthermore, assume that the following two conditions hold:*

(I.1) $I(0)=0$, I is an even functional, that is $I(-u)=I(u)$ for all $u\in E$, and bounded from below.

(I.2) For each $k=1,2,\ldots$, there exists $A_k\in\widehat{\Gamma}_k$ such that $\sup_{u\in A_k}I(u)<0$.

Then d_k defined by (3.2) *is a critical value of I, $d_k\le d_{k+1}<0$ for $k=1,2,\ldots$ and the sequence $\{d_k\}_{k=1}^\infty$ converges to zero as $k\to\infty$. Moreover, if $d_k=d_{k+1}=\cdots=d_{k+p}=:d$, then $\gamma(K_d)\ge p+1$.*

For the proof, see [1,12,24].

Corollary 3.1 (A Variant of the Symmetric Mountain Pass Lemma). *Addition to the hypotheses of Theorem* 3.1, *we assume that*

(I.3) If $I(u)=0$ and $I'(u)=0$, then $u=0$.

Then the sequence $\{u_k\}_{k=1}^\infty$, where $u_k\in K_{d_k}$ for $k=1,2,\ldots$ converges to zero as $k\to\infty$.

Proof. Let $u_k\in K_{d_k}$ for $k=1,2,\ldots$. Then $I(u_k)=d_k$ and $I'(u_k)=0$ for $k=1,2,\ldots$. Since $d_k\to 0$ as $k\to\infty$ and I satisfies the (PS)-condition, there exists a subsequence $\{u_{k'}\}_{k'=1}^\infty$ of $\{u_k\}_{k=1}^\infty$ and $u\in E$ such that $u_{k'}\to u$ as $k'\to\infty$. Hence, since $I\in C^1(E,\mathbb{R})$, $I(u_{k'})=d_{k'}\to I(u)=0$ and $I'(u)=0$. From (I.3), we have $u=0$, that is, $u_{k'}\to 0$ as $k'\to\infty$. By the convergent principle [31, Theorem 10.13(1)], for full sequence $\{u_k\}_{k=1}^\infty$, we have $u_k\to 0$ as $k\to\infty$. □

Remark 3.1. If (I.3) does not hold, the sequence $\{u_k\}_{k=1}^\infty$ of the critical points does not necessarily converge to zero. Kajikiya [24] constructed an example to demonstrate this.

4 Assumptions on the problem (1.1) and some properties

In this section, we state some assumptions on the problem (1.1) and some properties.

Let $p\in C_+(\overline{\Omega})$ be fixed. Assume that the following (A.1)-(A.4) hold.

(A.1) Let $A:\Omega\times\mathbb{R}^N\to\mathbb{R}$ be a function satisfying that for a.e. $x\in\Omega$ the function $A(x,\cdot):\mathbb{R}^N\ni\boldsymbol{\xi}\mapsto A(x,\boldsymbol{\xi})$ is of C^1-class, and for all $\boldsymbol{\xi}\in\mathbb{R}^N$ the function $A(\cdot,\boldsymbol{\xi})$: $\Omega\ni x\mapsto A(x,\boldsymbol{\xi})$ is measurable. Moreover, suppose that $A(x,\mathbf{0})=0$, $A(x,-\boldsymbol{\xi})=A(x,\boldsymbol{\xi})$ for a.e. $x\in\Omega$ and all $\boldsymbol{\xi}\in\mathbb{R}^N$, and put $\boldsymbol{a}(x,\boldsymbol{\xi})=\boldsymbol{\nabla}_{\xi}A(x,\boldsymbol{\xi})$. Then $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$.

Moreover, we assume the following structure conditions. There exist constants $C_0,k_0>0$, nonnegative functions $h_0\in L^{p'(\cdot)}(\Omega)$ and $h_1\in L^1(\Omega)$ with $h_1(x)\geq 1$ for a.e. $x\in\Omega$ such that the following conditions hold.

(A.2) $|\boldsymbol{a}(x,\boldsymbol{\xi})|\leq C_0(h_0(x)+h_1(x)|\boldsymbol{\xi}|^{p(x)-1})$ for all $\boldsymbol{\xi}\in\mathbb{R}^N$ and a.e. $x\in\Omega$.

(A.3) $\boldsymbol{a}(x,\mathbf{0})=\mathbf{0}$ for a.e. $x\in\Omega$ and

$$\big(\boldsymbol{a}(x,\boldsymbol{\xi})-\boldsymbol{a}(x,\boldsymbol{\eta})\big)\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\geq\begin{cases}k_0h_1(x)|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)}, & \text{if} \quad p(x)\geq 2,\\ k_0h_1(x)(1+|\boldsymbol{\xi}|+|\boldsymbol{\eta}|)^{p(x)-2}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2, & \text{if} \quad p(x)<2\end{cases}$$

for a.e. $x\in\Omega$ and all $\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^N$.

(A.4) $A(x,\boldsymbol{\xi})$ and $\boldsymbol{a}(x,\boldsymbol{\xi})$ satisfy that

$$\boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi}\leq\begin{cases}p(x)A(x,\boldsymbol{\xi}), & \text{if} \quad p(x)\geq 2,\\ p(x)A(x,\boldsymbol{\xi})+h_1(x), & \text{if} \quad p(x)<2\end{cases}$$

for all $\boldsymbol{\xi}\in\mathbb{R}^N$ and a.e. $x\in\Omega$.

Remark 4.1. The condition (A.1) is more general than that of Mashiyev *et al.* [26] who considered the case $h_1(x)\equiv 1$. In our case, to overcome this we have to consider the space Y defined by (4.1) later as a basic space rather than the space X defined by (2.1).

Lemma 4.1. *Under (A.1) and (A.3), there exists a constant $c>0$ such that*

$$\begin{aligned}&\frac{1}{2}A(x,\boldsymbol{\xi})+\frac{1}{2}A(x,\boldsymbol{\eta})-A\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right)\\ &\geq\begin{cases}ch_1(x)|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)}, & \text{if} \quad p(x)\geq 2,\\ ch_1(x)(1+|\boldsymbol{\xi}|+|\boldsymbol{\eta}|)^{p(x)-2}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2, & \text{if} \quad p(x)<2\end{cases}\end{aligned}$$

for a.e. $x\in\Omega$ and all $\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^N$. In particular, $A(x,\boldsymbol{\xi})$ is convex with respect to $\boldsymbol{\xi}$.

For the proof, see [7, Lemma 3.2].

Example 4.1.

(i) $A(x,\boldsymbol{\xi})=(h(x)/p(x))|\boldsymbol{\xi}|^{p(x)}$ with $p\in C_+(\overline{\Omega})$, $h\in L^1(\Omega)$ satisfying $h(x)\geq 1$ for a.e. $x\in\Omega$.

(ii) $A(x,\boldsymbol{\xi})=(h(x)/p(x))((1+|\boldsymbol{\xi}|^2)^{p(x)/2}-1)$ with $p\in C_+(\overline{\Omega})$, $h\in L^{p'(\cdot)}(\Omega)$ satisfying $h(x)\geq 1$ for a.e. $x\in\Omega$.

Then $A(x,\boldsymbol{\xi})$ and $\boldsymbol{a}(x,\boldsymbol{\xi})=\boldsymbol{\nabla}_{\boldsymbol{\xi}}A(x,\boldsymbol{\xi})$ of (i) and (ii) satisfy (A.1)-(A.4).

Remark 4.2. In Example 4.1, when $h(x)\equiv 1$, the operator $u\mapsto\operatorname{div}[\boldsymbol{a}(\cdot,\boldsymbol{\nabla}u(\cdot))]$ of (i) corresponds to the $p(\cdot)$-Laplacian and that of (ii) corresponds to the prescribed mean curvature operator for nonparametric surface.

Lemma 4.2. *Under (A.1)-(A.3), we have the following:*

(i) $|A(x,\boldsymbol{\xi})|\leq C_0(h_0(x)|\boldsymbol{\xi}|+h_1(x)|\boldsymbol{\xi}|^{p(x)})$ *for a.e. $x\in\Omega$ and all $\boldsymbol{\xi}\in\mathbb{R}^N$.*

(ii) There exist constants $c>0$ and $C\geq 0$ such that

$$\boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi}\geq ch_1(x)|\boldsymbol{\xi}|^{p(x)}-Ch_1(x) \quad \textit{for a.e.} \quad x\in\Omega \quad \textit{and all} \quad \boldsymbol{\xi}\in\mathbb{R}^N.$$

In particular, if $p^-\geq 2$, then we can take $C=0$.

For the proof, see [7, Lemma 3.2].

For the function $h_1\in L^1(\Omega)$ with $h_1(x)\geq 1$ for a.e. $x\in\Omega$, we define a modular on X by

$$\rho_{p(\cdot),h_1(\cdot)}(v)=\rho_{p(\cdot),h_1(\cdot),\Omega}(v)=\int_\Omega h_1(x)|\boldsymbol{\nabla}v(x)|^{p(x)}dx \quad \text{for} \quad v\in X,$$

where the space X is defined by (2.1). Define our basic space

$$Y=\left\{v\in X;\ \rho_{p(\cdot),h_1(\cdot)}(v)<\infty\right\} \tag{4.1}$$

equipped with the norm

$$\|v\|_Y=\inf\left\{\tau>0;\ \rho_{p(\cdot),h_1(\cdot)}\left(\frac{v}{\tau}\right)\leq 1\right\}.$$

Proposition 4.1. *The space* $(Y,\|\cdot\|_Y)$ *is a separable and reflexive real Banach space.*

For the proof, see [5, Proposition 3.4].

We note that $C_0^\infty(\Omega)\subset Y$. Since $h_1(x)\geq 1$ a.e. $x\in\Omega$, it follows that

$$\rho_{p(\cdot),h_1(\cdot)}(v)=\rho_{p(\cdot)}\big(h_1^{1/p(\cdot)}|\nabla v|\big)\geq\rho_{p(\cdot)}(|\nabla v|)\quad\text{for}\quad v\in Y$$

and

$$\|v\|_Y=\big\|h_1^{1/p(\cdot)}\nabla v\big\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}\geq\|\nabla v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}=\|v\|_X\quad\text{for}\quad v\in Y. \tag{4.2}$$

From (4.2) and Proposition 2.1, we have the following proposition.

Proposition 4.2. *Let* $p\in C_+(\overline{\Omega})$ *and let* $u,u_n\in Y$ $(n=1,2,\ldots)$. *Then the following properties hold:*

(i) $Y\hookrightarrow X$ *and* $\|u\|_X\leq\|u\|_Y$.

(ii) $\|u\|_Y>1(=1,<1)\iff\rho_{p(\cdot),h_1(\cdot)}(u)>1(=1,<1)$.

(iii) $\|u\|_Y>1\implies\|u\|_Y^{p^-}\leq\rho_{p(\cdot),h_1(\cdot)}(u)\leq\|u\|_Y^{p^+}$.

(iv) $\|u\|_Y<1\implies\|u\|_Y^{p^+}\leq\rho_{p(\cdot),h_1(\cdot)}(u)\leq\|u\|_Y^{p^-}$.

(v) $\lim_{n\to\infty}\|u_n-u\|_Y=0\iff\lim_{n\to\infty}\rho_{p(\cdot),h_1(\cdot)}(u_n-u)=0$.

(vi) $\|u_n\|_Y\to\infty$ *as* $n\to\infty\iff\rho_{p(\cdot),h_1(\cdot)}(u_n)\to\infty$ *as* $n\to\infty$.

Define a functional on Y by

$$\Phi(u)=\int_\Omega A\big(x,\nabla u(x)\big)dx\quad\text{for}\quad u\in Y. \tag{4.3}$$

Then we have the following proposition which fulfills an important role in this paper. In particular, (v) in the following proposition is first derived by Aramaki [9, Proposition 3.5].

Proposition 4.3. *Under the hypotheses (A.1)-(A.4), the functional* Φ *has the following properties:*

(i) $\Phi\in C^1(Y,\mathbb{R})$, Φ *is an even functional and its Fréchet derivative* Φ' *satisfies*

$$\langle\Phi'(u),v\rangle=\int_\Omega \boldsymbol{a}\big(x,\nabla u(x)\big)\cdot\nabla v(x)dx\quad\textit{for}\quad u,v\in Y,$$

and we have

$$\Phi(u)-\Phi(v)\geq\langle\Phi'(v),u-v\rangle\quad\textit{for}\quad u,v\in Y. \tag{4.4}$$

Here and hereafter, we denote $\langle\cdot,\cdot\rangle_{Y^*,Y}$ *by* $\langle\cdot,\cdot\rangle$ *for brevity of notation.*

(ii) *Φ is coercive, that is, $\Phi(u)\to\infty$ as $\|u\|_Y\to\infty$.*

(iii) *Φ is sequentially weakly lower-semicontinuous in Y.*

(iv) *Φ is bounded on every bounded subset of Y.*

(v) *Let $\Omega_1=\{x\in\Omega;p(x)\geq 2\}$ and $\Omega_2=\{x\in\Omega;p(x)<2\}$. Then Φ' is uniformly monotone in the sense of*

$$\begin{aligned}&\langle\Phi'(u)-\Phi'(v),u-v\rangle\\ &\geq c\rho_{p(\cdot),h_1(\cdot),\Omega_1}(u-v)\\ &\quad+\Big\{c(C+\|u\|_Y+\|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2}\rho_{p(\cdot),h_1(\cdot),\Omega_2}(u-v)\Big\}^{2/p^+(\Omega_2)}\\ &\quad\wedge\Big\{c(C+\|u\|_Y+\|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2}\rho_{p(\cdot),h_1(\cdot),\Omega_2}(u-v)\Big\}^{2/p^-(\Omega_2)}\end{aligned}$$

for all $u,v\in Y$ with some positive constants c and C. Here and from now on, we denote $a\vee b=\max\{a,b\}$ and $a\wedge b=\min\{a,b\}$ for real numbers a and b.

(vi) *Φ' is bounded on every bounded subset of Y.*

(vii) *Φ' is coercive, that is,*

$$\lim_{\|u\|_Y\to\infty}\frac{\langle\Phi'(u),u\rangle}{\|u\|_Y}=\infty.$$

(viii) *Φ' is of (S_+)-type, that is, if $u_n\rightharpoonup u$ in Y as $n\to\infty$ and*

$$\limsup_{n\to\infty}\langle\Phi'(u_n),u_n-u\rangle\leq 0,$$

then $u_n\to u$ in Y. Here and from now on, the symbol $\rightharpoonup$ means weak convergence, in contrast to this, the symbol $\to$ means strong convergence.

(ix) *The mapping $\Phi':Y\to Y^*$ is a homeomorphism.*

For the proof, see [7, Proposition 3.3] and [4, Proposition 4.4].

Proposition 4.4. *Assume that (A.1)-(A.4) hold. Then there exist positive constants c_1 and C_1 such that*

$$\begin{aligned}&\frac{1}{2}\Phi(u)+\frac{1}{2}\Phi(v)-\Phi\left(\frac{u+v}{2}\right)\\ &\geq c_1\rho_{p(\cdot),h_1(\cdot),\Omega_2}(u-v)\\ &\quad+c_1\Big\{(C_1+\|u\|_Y+\|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2}\rho_{p(\cdot),h_1(\cdot),\Omega_2}(u-v)\Big\}^{2/p^+(\Omega_2)}\end{aligned}$$

$$\wedge c_1\left\{(C_1+\|u\|_Y+\|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2}\rho_{p(\cdot),h_1(\cdot),\Omega_2}(u-v)\right\}^{2/p^-(\Omega_2)}$$

for $u,v\in Y$.

Proof. Let $u,v\in X$. We have

$$\begin{aligned}&\frac{1}{2}\Phi(u)+\frac{1}{2}\Phi(v)-\psi\left(\frac{u+v}{2}\right)\\ =&\frac{1}{2}\left(\Phi(u)-\Phi\left(\frac{u+v}{2}\right)\right)+\frac{1}{2}\left(\Phi(v)-\Phi\left(\frac{u+v}{2}\right)\right)\\ =&\frac{1}{4}\int_0^1\frac{1}{\theta}\left\langle\Phi'\left(\frac{u+v}{2}+\theta\frac{u-v}{2}\right)-\Phi'\left(\frac{u+v}{2}+\theta\frac{v-u}{2}\right),\theta(u-v)\right\rangle d\theta.\end{aligned}$$

Therefore, we can prove this proposition by using Proposition 4.3(v). $\square$

Corollary 4.1. *In* $u_n\rightharpoonup u$ *in* Y *as* $n\to\infty$ *and* $\lim_{n\to\infty}\Phi(u_n)=\Phi(u)$, *then* $u_n\to u$ *in* Y *as* $n\to\infty$.

Proof. Let $u_n\rightharpoonup u$ in Y as $n\to\infty$ and $\lim_{n\to\infty}\Phi(u_n)=\Phi(u)$. Since Φ is sequentially weakly lower semi-continuous from Proposition 4.3(iii) and $(u_n+u)/2\rightharpoonup u$ as $n\to\infty$, we have

$$\Phi(u)\le\liminf_{n\to\infty}\Phi\left(\frac{u_n+u}{2}\right).$$

Hence, we see that

$$\begin{aligned}0&\le\liminf_{n\to\infty}\left(\frac{1}{2}\Phi(u_n)+\frac{1}{2}\Phi(u)-\Phi\left(\frac{u_n+u}{2}\right)\right)\\ &\le\limsup_{n\to\infty}\left(\frac{1}{2}\Phi(u_n)+\frac{1}{2}\Phi(u)-\Phi\left(\frac{u_n+u}{2}\right)\right)\\ &=\Phi(u)-\liminf_{n\to\infty}\Phi\left(\frac{u_n+u}{2}\right)\le\Phi(u)-\Phi(u)=0,\end{aligned}$$

so we have

$$\lim_{n\to\infty}\left(\frac{1}{2}\Phi(u_n)+\frac{1}{2}\Phi(u)-\Phi\left(\frac{u_n+u}{2}\right)\right)=0.$$

Since $u_n\rightharpoonup u$ in Y as $n\to\infty$, the sequence $\{\|u_n\|_Y\}_{n=1}^{\infty}$ is bounded. Therefore, it follows from Proposition 4.4 that $\rho_{p(\cdot),h_1(\cdot),\Omega}(u_n-u)\to 0$ as $n\to\infty$, so $u_n\to u$ in Y as $n\to\infty$ from Proposition 4.2(v). $\square$

We consider the hypotheses on the data f and g in (1.1).

(f.1) Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying

$$|f(x,t)|\le c_1\big(1+a(x)|t|^{q(x)-1}\big)\quad\text{for all}\quad t\in\mathbb{R}\quad\text{and a.e.}\quad x\in\Omega,$$

where c_1 is a positive constant, $1\le a\in L^{\alpha(\cdot)}(\Omega)$, $\alpha\in C_+(\overline{\Omega})$ and $q\in C(\overline{\Omega})$ such that

$$1\le q(x)<\frac{\alpha(x)-1}{\alpha(x)}p^*(x)\quad\text{for all}\quad x\in\overline{\Omega}.$$

(g.1) Let $g:\Gamma_2\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying

$$|g(x,t)|\le c_2\big(1+b(x)|t|^{r(x)-1}\big)\quad\text{for all}\quad t\in\mathbb{R}\quad\text{and}\quad\sigma\text{-a.e.}\quad x\in\Gamma_2,$$

where c_2 is a positive constant, $1\le b\in L^{\beta(\cdot)}(\Gamma_2)$, $\beta\in C_+(\overline{\Gamma_2})$ and $r\in C(\overline{\Gamma_2})$ such that

$$1\le r(x)<\frac{\beta(x)-1}{\beta(x)}p^\partial(x)\quad\text{for all}\quad x\in\overline{\Gamma_2}.$$

Define functions by

$$F(x,t)=\int_0^t f(x,s)ds\quad\text{for a.e}\quad x\in\Omega\quad\text{and}\quad t\in\mathbb{R},\tag{4.5}$$

$$G(x,t)=\int_0^t g(x,s)ds\quad\text{for}\quad\sigma\text{-a.e.}\quad x\in\Gamma_2\quad\text{and}\quad t\in\mathbb{R}.\tag{4.6}$$

Moreover, define functionals on Y by

$$J(u)=\int_\Omega F\big(x,u(x)\big)dx,\quad K(u)=\int_{\Gamma_2}G\big(x,u(x)\big)d\sigma_x,\quad u\in Y.\tag{4.7}$$

Then we have the following proposition.

Proposition 4.5. *Assume that (f.1) and (g.1) hold. Then the following properties are derived:*

(i) $J,K\in C^1(Y,\mathbb{R})$ and their Fréchet derivatives J' and K' satisfy

$$\begin{aligned}\langle J'(u),v\rangle&=\int_\Omega f\big(x,u(x)\big)v(x)dx,\\ \langle K'(u),v\rangle&=\int_{\Gamma_2}g\big(x,u(x)\big)v(x)d\sigma\quad\textit{for all}\quad u,v\in Y.\end{aligned}$$

(ii) J and K are sequentially weakly continuous on Y.

For the proof, see [4, Propositions 4.2 and 4.4].

Define a functional on Y by

$$I(u)=\Phi(u)-J(u)-K(u) \text{ for } u\in Y. \tag{4.8}$$

Furthermore, we assume the following conditions:

(f.2) For a.e. $x\in\Omega$, $f(x,t)$ is an odd function with respect to t.

(g.2) For σ-a.e. $x\in\Gamma_2$, $g(x,t)$ is an odd function with respect to t.

We note that under (A.1), (f.2) and (g.2), I is an even functional. We also assume that the following Ambrosetti-Rabinowitz conditions hold: there exist $\theta>p^+$ and $t_0>1$ such that

(f.3) $0<\theta F(x,t)\le f(x,t)t$ for a.e. $x\in\Omega$ and all $t\in\mathbb{R}\setminus(-t_0,t_0)$, where the function $F(x,t)$ is defined by (4.5),

(g.3) $0<\theta G(x,t)\le g(x,t)t$ for σ-a.e. $x\in\Gamma_2$ and all $t\in\mathbb{R}\setminus(-t_0,t_0)$, where the function $G(x,t)$ is defined by (4.6).

Proposition 4.6. *Assume that (A.1)-(A.4), (f.1), (f.3), (g.1) and (g.3) hold. Then the functional I defined by* (4.8) *satisfies (PS)-condition.*

Proof. Assume that a sequence $\{u_n\}_{n=1}^\infty\subset Y$ satisfies that $\lim_{n\to\infty}I(u_n)=c$ exists in $\mathbb{R}$ and $\lim_{n\to\infty}\|I'(u_n)\|_{Y^*}=0$.

Step 1. The sequence $\{u_n\}_{n=1}^\infty$ is bounded in Y. Indeed, for large n, we have $I(u_n)\le|c|+1$ and $|\langle I'(u_n),u_n\rangle|/\theta\le\|u_n\|_Y$. Hence, using (A.3),

$$\begin{aligned}
|c|+1+\|u_n\|_Y\ge{}& I(u_n)-\frac{1}{\theta}\langle I'(u_n),u_n\rangle\\
={}&\Phi(u_n)-\int_\Omega F\big(x,u_n(x)\big)dx-\int_{\Gamma_2}G\big(x,u_n(x)\big)d\sigma_x\\
&-\frac{1}{\theta}\langle\Phi'(u_n),u_n\rangle+\frac{1}{\theta}\int_\Omega f\big(x,u_n(x)\big)u_n(x)dx\\
&+\frac{1}{\theta}\int_{\Gamma_2}g\big(x,u_n(x)\big)u_n(x)d\sigma_x\\
={}&I_1+I_2+I_3,
\end{aligned}$$

where

$$I_1=\int_\Omega A\big(x,\nabla u_n(x)\big)dx-\frac{1}{\theta}\int_\Omega \boldsymbol{a}\big(x,\nabla u_n(x)\big)\cdot\nabla u_n(x)dx,$$

$$
\begin{aligned}
I_2 &= \int_\Omega \left(\frac{1}{\theta} f(x,u_n(x))u_n(x) - F(x,u_n(x))\right)dx,\\
I_3 &= \int_{\Gamma_2} \left(\frac{1}{\theta} g(x,u_n(x))u_n(x) - G(x,u_n(x))\right)d\sigma_x.
\end{aligned}
$$

It follows from (f.1) and (f.3) that

$$
\begin{aligned}
I_2 = &\int_{\{x\in\Omega;|u_n(x)|\geq t_0\}} \left(\frac{1}{\theta} f(x,u_n(x))u_n(x) - F(x,u_n(x))\right)dx\\
&+\int_{\{x\in\Omega;|u_n(x)|< t_0\}} \left(\frac{1}{\theta} f(x,u_n(x))u_n(x) - F(x,u_n(x))\right)dx\\
\geq &\int_{\{x\in\Omega;|u_n(x)|< t_0\}} \left(\frac{1}{\theta} f(x,u_n(x))u_n(x) - F(x,u_n(x))\right)dx\\
\geq &-\int_\Omega \left(\frac{1}{\theta} c_1(1+a(x)t_0^{q^+-1})t_0 + c_1(t_0+a(x))t_0^{q^+}\right)dx = -C_1,
\end{aligned}
$$

where

$$
C_1 = \left(\frac{1}{\theta}+1\right)c_1\left(t_0|\Omega| + t_0^{q^+}\|a\|_{L^1(\Omega)}\right)
$$

is a constant independent of n. Similarly, from (g.1) and (g.3), we have $I_3 \geq -C_2$, where C_2 is a constant independent of n.

On the other hand, it follows from (A.4) and Lemma 4.2(ii) that

$$
\begin{aligned}
I_1 &\geq \left(\frac{1}{p^+}-\frac{1}{\theta}\right)\int_\Omega \boldsymbol{a}(x,\boldsymbol{\nabla} u_n(x))\cdot\boldsymbol{\nabla} u_n(x)dx - \frac{1}{p^+}\|h_1\|_{L^1(\Omega)} - \frac{1}{\theta}\|h_1\|_{L^1(\Omega)}\\
&\geq \left(\frac{1}{p^+}-\frac{1}{\theta}\right)c\|u_n\|_Y^{p^-} - C_3
\end{aligned}
$$

for large n and $\|u_n\|_Y \geq 1$, where c and C_3 are positive constants independent of n. Therefore, if n is large and $\|u_n\|_Y \geq 1$, then we have

$$
|c|+1+\|u_n\|_Y \geq \left(\frac{1}{p^+}-\frac{1}{\theta}\right)\|u_n\|_Y^{p^-} - C_4,
$$

where C_4 is a constant independent of n. Since $1/p^+ - 1/\theta > 0$ and $p^- > 1$, the above inequality implies that $\{u_n\}_{n=1}^\infty$ is bounded.

Step 2. Since $\{u_n\}_{n=1}^\infty$ is bounded on the reflexive Banach space Y, there exist a subsequence $\{u_{n'}\}_{n'=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and $u\in Y$ such that $u_{n'} \rightharpoonup u$ as $n'\to\infty$. Since $\lim_{n'\to\infty}\|I'(u_{n'})\|_{Y^*}=0$ and $\{u_{n'}\}_{n'=1}^\infty$ is bounded in Y, we see that

$$
\langle I'(u_{n'}), u - u_{n'}\rangle \to 0 \quad \text{as} \quad n'\to\infty. \tag{4.9}
$$

Moreover, since the embeddings $Y \hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Omega), L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)$ are compact by Propositions 2.8 and 2.10, we see that $u_{n'} \to u$ in $L^{q(\cdot)}_{a(\cdot)}(\Omega)$ and $L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)$. From the Hölder inequality (Proposition 2.2), we have

$$\begin{aligned}
&|\langle J'(u_{n'}), u_{n'} - u\rangle| \\
=&\left|\int_\Omega f\big(x, u_{n'}(x)\big)\big(u_{n'}(x) - u(x)\big)dx\right| \\
=&\left|\int_\Omega a(x)^{-1/q(x)} f\big(x, u_{n'}(x)\big)\big(a(x)^{1/q(x)}u_{n'}(x) - a(x)^{1/q(x)}u(x)\big)dx\right| \\
\le& 2\|a^{-1/q(\cdot)} f(\cdot, u_{n'}(\cdot))\|_{L^{q'(\cdot)}(\Omega)} \|a^{1/q(\cdot)}u_{n'}(\cdot) - a^{1/q(\cdot)}u(\cdot)\|_{L^{q(\cdot)}(\Omega)}.
\end{aligned}$$

Here, since $a(x) \ge 1$ a.e. $x \in \Omega$, we have

$$\begin{aligned}
&\rho_{q'(\cdot)}\big(a^{-1/q(\cdot)} f(\cdot, u_n(\cdot))\big) \\
=&\int_\Omega a(x)^{-q'(x)/q(x)} |f(x, u_{n'}(x))|^{q'(x)} dx \\
\le& c_1' \int_\Omega \big(1 + a(x)\big)|u_{n'}(x)|^{q(x)} dx \le C,
\end{aligned}$$

where c_1' and C are constants independent of n, because $u_{n'} \to u$ in $L^{q(\cdot)}_{a(\cdot)}(\Omega)$. Hence, we have

$$\lim_{n' \to \infty} \langle J'(u_{n'}), u_{n'} - u\rangle = 0. \tag{4.10}$$

Similarly, we have

$$\lim_{n' \to \infty} \langle K'(u_{n'}), u_{n'} - u\rangle = 0. \tag{4.11}$$

Thus, it follows from (4.9)-(4.11) that

$$\begin{aligned}
&\lim_{n' \to \infty} \langle \Phi'(u_{n'}), u_{n'} - u\rangle \\
=&\lim_{n' \to \infty} \big(\langle I'(u_{n'}), u_{n'} - u\rangle + \langle J'(u_{n'}), u_{n'} - u\rangle \\
&\qquad + \langle K'(u_{n'}), u_{n'} - u\rangle\big) = 0.
\end{aligned} \tag{4.12}$$

By (4.4) and (4.12), we have

$$\begin{aligned}
\Phi(u) &\le \liminf_{n' \to \infty} \Phi(u_{n'}) \le \limsup_{n' \to \infty} \Phi(u_{n'}) \\
&\le \limsup_{n' \to \infty} \big(\Phi(u) - \langle \Phi'(u_{n'}), u - u_{n'}\rangle\big) = \Phi(u).
\end{aligned}$$

Thereby we have

$$\lim_{n'\to\infty}\Phi(u_{n'})=\Phi(u). \tag{4.13}$$

Thus, it follows from Corollary 4.1 that $u_{n'}\to u$ in Y as $n'\to\infty$. □

Lemma 4.3. *Under the hypotheses (f.1)-(f.3) and (g.1)-(g.3), we have the following:*

- *(i) There exists $\gamma\in L^1(\Omega)$ such that $\gamma(x)>0$ a.e. $x\in\Omega$ and $F(x,t)\ge\gamma(x)t^\theta$ for all $t\in[t_0,\infty)$ and a.e. $x\in\Omega$.*
- *(ii) There exists $\delta\in L^1(\Gamma_2)$ such that $\delta(x)>0$ σ-a.e. $x\in\Gamma_2$ and $G(x,t)\ge\delta(x)t^\theta$ for all $t\in[t_0,\infty)$ and σ-a.e. $x\in\Gamma_2$.*

Proof. (i) From (f.3), for $t\ge t_0$, we have

$$0<\theta F(x,t)\le f(x,t)t \quad \text{for a.e.} \quad x\in\Omega. \tag{4.14}$$

Put $\gamma(x)=F(x,t_0)t_0^{-\theta}$. Then $\gamma(x)>0$ a.e. $x\in\Omega$ and it follows that

$$\gamma(x)\le c_1\big(t_0+a(x)t_0^{q(x)}\big)t_0^{-\theta}\le c_1\big(t_0+a(x)t_0^{q^+}\big)t_0^{-\theta}.$$

So $\gamma\in L^{\alpha(\cdot)}(\Omega)\subset L^1(\Omega)$. From (4.14),

$$\frac{\theta}{\tau}\le\frac{f(x,\tau)}{F(x,\tau)}=\frac{\frac{\partial F}{\partial\tau}(x,\tau)}{F(x,\tau)} \quad \text{for} \quad \tau\ge t_0.$$

Integrating this inequality over (t_0,t), we have

$$\theta\log\frac{t}{t_0}\le\log\frac{F(x,t)}{F(x,t_0)} \quad \text{for all} \quad t\ge t_0.$$

This implies $F(x,t)\ge\gamma(x)t^\theta$ for all $t\ge t_0$.

(ii) follows from the same argument as (i). □

5 Main theorems and their proofs

First we introduce the notion of weak solutions for problem (1.1).

Definition 5.1. *We say $u\in Y$ is a weak solution of* (1.1), *if u satisfies that*

$$\int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla v(x)dx=\int_\Omega f(x,u(x))v(x)dx+\int_{\Gamma_2}g(x,u(x))v(x)d\sigma_x \tag{5.1}$$

for all $v\in Y$.

It is easy to see that if $u\in Y$ is a weak solution of the problem (1.1), then u satisfies (1.1) in the distribution sense, and $u\in Y$ being a weak solution of the problem (1.1) is equivalent to u being critical point of the functional I defined by (4.6), that is, $I'(u)=0$.

Now that we are ready, we can state the main theorem.

Theorem 5.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ $(N\geq 2)$ with a $C^{0,1}$-boundary Γ satisfying* (1.2). *Assume that (A.1)-(A.4), (f.1)-(f.3) and (g.1)-(g.3) hold. Furthermore, suppose that*

(f.4) $p^-\leq q^-$ *and* $f(x,t)=o(t^{p^--1})$ *uniformly as* $t\to\infty$,

(g.4) $p^-\leq r^-$ *and* $g(x,t)=o(t^{p^--1})$ *uniformly as* $t\to\infty$.

We define $\widehat{\Gamma}_k,d_k$ *and* K_d *by* (3.1)-(3.3) *with* $(E,\|\cdot\|_E)=(Y,\|\cdot\|_Y)$. *Let* $u_k\in K_{d_k}$ $(k=1,2,\ldots)$.

Then $\{u_k\}_{k=1}^\infty$ *is the set of infinitely many weak solutions of the problem* (1.1). *Moreover,* $\{u_k\}_{k=1}^\infty$ *is bounded in* Y.

Proof. We apply Theorem 3.1 with $(E,\|\cdot\|_E)=(Y,\|\cdot\|_Y)$. By Propositions 4.3(i), 4.5(i) and the definition of the functional I defined by (4.8), we can see that $I\in C^1(Y,\mathbb{R})$. It follows from Proposition 4.6 that I satisfies (PS)-condition. From (A.1), (f.2) and (g.2), we see that the functional I is even.

We show that I is bounded from below. From (f.4), for any $\varepsilon>0$, there exists $t_1>1$ such that $|f(x,t)|\leq\varepsilon t^{p^--1}$ for $t\geq t_1$ and a.e. $x\in\Omega$. Hence, for $t>t_1$,

$$|F(x,t)|\leq\int_0^t|f(x,s)|ds=\int_0^{t_1}|f(x,s)|ds+\int_{t_1}^t|f(x,s)|ds\leq c_1\big(t_1+a(x)t_1^{q^+}\big)+\frac{\varepsilon}{p^-}t^{p^-}.$$

For $0\leq t\leq t_1$, $|F(x,t)|\leq c_1(t_1+a(x)t_1^{q^+})$. Since $F(x,t)$ is even with respect to t from (f.2), we see that

$$|F(x,t)|\leq c_1\big(t_1+a(x)t_1^{q^+}\big)+\frac{\varepsilon}{p^-}|t|^{p^-}\quad\text{for a.e.}\quad x\in\Omega\quad\text{and}\quad t\in\mathbb{R}.$$

Hence, we see that

$$|J(u)|\leq c_1\int_\Omega\big(t_1+a(x)t_1^{q^+}\big)dx+\frac{\varepsilon}{p^-}\int_\Omega|u(x)|^{p^-}dx\leq C_1+\frac{\varepsilon}{p^-}\|u\|^{p^-}_{L^{p^-}(\Omega)},$$

where $C_1=c_1(t_1|\Omega|+t_1^{q^+}\|a\|_{L^1(\Omega)})$. Since

$$p^-\leq q^-<\frac{\alpha(x)-1}{\alpha(x)}p^*(x)<p^*(x)$$

for all $x\in\overline{\Omega}$, we have $Y\hookrightarrow L^{p^-}(\Omega)$, so $\|u\|_{L^{p^-}(\Omega)}\le C_2\|u\|_Y$ for $u\in Y$ with some constant C_2. Hence, we have

$$|J(u)|\le C_1+\frac{\varepsilon}{p^-}C_2^{p^-}\|u\|_Y \quad \text{for} \quad u\in Y.$$

Similarly, we have

$$|K(u)|\le C_3+\frac{\varepsilon}{p^-}C_4^{p^-}\|u\|_Y^{p^-} \quad \text{for} \quad u\in Y$$

for some constants C_3 and C_4.

On the other hand, it follows from (A.4) and Lemma 4.2(ii) that

$$\begin{aligned}\Phi(u)&=\int_\Omega A\big(x,\nabla u(x)\big)dx\\&\ge\int_\Omega\frac{1}{p(x)}\boldsymbol{a}\big(x,\nabla u(x)\big)\cdot\nabla u(x))dx-\int_\Omega\frac{h_1(x)}{p(x)}dx\\&\ge\frac{c}{p^+}\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}dx-(C+1)\int_\Omega\frac{h_1(x)}{p(x)}dx\\&\ge\frac{c}{p^+}\|u\|_Y^{p^+}\wedge\|u\|_Y^{p^-}-(C+1)\|h_1/p\|_{L^1(\Omega)}\end{aligned}$$

for any $u\in Y$. If we choose $\varepsilon>0$ so that

$$c_1:=\frac{c}{p^+}-\frac{\varepsilon}{p^-}C_2^{p^-}-\frac{\varepsilon}{p^-}C_4^{p^-}>0,$$

then

$$\begin{aligned}I(u)&\ge\Phi(u)-|J(u)|-|K(u)|\\&\ge c_1\|u\|_Y^{p^+}\wedge\|u\|_Y^{p^-}-C_1-C_3-(C+1)\|h_1/p\|_{L^1(\Omega)}\\&\ge-C_1-C_3-(C+1)\|h_1/p\|_{L^1(\Omega)}\end{aligned}\tag{5.2}$$

for all $u\in Y$. Thus, the functional I is bounded from below. Therefore, I satisfies Theorem 3.1(I.1).

We show that the functional I satisfies Theorem 3.1(I.2). By Lemma 4.2(i), we have

$$\Phi(u)\le\int_\Omega|A(x,\nabla u(x))|dx\le C_0\int_\Omega\big(h_0(x)|\nabla u(x)|+h_1(x)|\nabla u(x)|^{p(x)}\big)dx.$$

Here since it follows from the Hölder inequality (Proposition 2.2) that

$$\int_\Omega h_0(x)|\nabla u(x)|dx \le 2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \le 2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u\|_Y^{p^+}$$

for $u\in Y$ with $\|u\|_Y>1$, so there exists a constant $C>0$ such that we have $\Phi(u)\le C\|u\|_Y^{p^+}$ for $u\in Y$ with $\|u\|_Y>1$.

Choose $0\ne e_1\in Y$, and inductively choose $e_{k+1}\notin \operatorname{span}\{e_1,\dots,e_k\}=:Y_k$. For any $k=1,2,\dots$, define

$$A_k^\rho=\left\{\sum_{i=1}^k x_ie_i;\ x_i\in\mathbb{R}\ \text{ for }\ i=1,\dots,k\ \text{ and }\ \sum_{i=1}^k x_i^2=\rho^2\right\}$$

for $\rho>0$. Then $A_k^\rho\in\mathcal{E}$. If we define a map

$$h:\ A_k^\rho\ni\sum_{i=1}^k x_ie_i\ \mapsto\ (x_1,\dots,x_k)\in\mathbb{R}^k$$

and let

$$\widehat{\Omega}=\left\{(x_1,\dots,x_k)\in\mathbb{R}^k;\ \sum_{i=1}^k x_i^2<\rho^2\right\},$$

then $h\in C(A_k^\rho,\partial\widehat{\Omega})$ and h is an odd homeomorphism from A_k^ρ onto $\partial\widehat{\Omega}$. Thus, it follows from Proposition 3.1(vi) that $\gamma(A_k^\rho)=k$, so $A_k^\rho\in\widehat{\Gamma}_k$. Let $u\in Y$ with $\|u\|_Y>1$. Then

$$I(u)=\Phi(u)-J(u)-K(u)\le C\|u\|_Y^{p^+}-\int_\Omega F\big(x,u(x)\big)dx-\int_{\Gamma_2}G\big(x,u(x)\big)d\sigma_x.$$

By Lemma 4.3(i), we have

$$F(x,t)\ge\gamma(x)|t|^\theta\quad\text{for a.e.}\quad x\in\Omega\quad\text{and}\quad |t|\ge t_0.$$

If we define $\Omega_{t_0}=\{x\in\Omega;|u(x)|\ge t_0\}$, we have

$$\begin{aligned}\int_\Omega F\big(x,u(x)\big)dx&=\int_{\Omega_{t_0}}F\big(x,u(x)\big)dx+\int_{\Omega\setminus\Omega_{t_0}}F\big(x,u(x)\big)dx\\&\ge\int_{\Omega_{t_0}}\gamma(x)|u(x)|^\theta dx-\int_{\Omega\setminus\Omega_{t_0}}c_1\big(t_0+a(x)t_0^{q^+}\big)dx\\&\ge\int_\Omega\gamma(x)|u(x)|^\theta dx-\int_{\Omega\setminus\Omega_{t_0}}\gamma(x)|u(x)|^\theta dx-C_5\\&\ge\int_\Omega\gamma(x)|u(x)|^\theta dx-C_6\end{aligned}$$

for some constant C_5 and C_6.

Similarly, we have

$$\int_{\Gamma_2} G(x,u(x))d\sigma_x \geq \int_{\Gamma_2} \delta(x)|u(x)|^\theta d\sigma_x - C_7$$

for some constant C_7. Here

$$\left(\int_\Omega \gamma(x)|u(x)|^\theta dx + \int_{\Gamma_2} \delta(x)|u(x)|^\theta d\sigma_x\right)^{1/\theta}$$

is a norm in Y. Since Y_k is of finite-dimensional, all the norms in Y_k are equivalent, so there exist positive constants c_1,c_1 and c_3 such that for $u=\sum_{i=1}^k x_i e_i \in Y_k$,

$$c_1\|u\|_Y^\theta \leq \int_\Omega \gamma(x)|u(x)|^\theta dx + \int_{\Gamma_2} \delta(x)|u(x)|^\theta d\sigma_x$$

and

$$c_2\left(\sum_{i=1}^k x_i^2\right)^{1/2} \leq \|u\|_Y \leq c_3\left(\sum_{i=1}^k x_i^2\right)^{1/2}.$$

Since $\theta > p^+$, if we choose $\rho > 1$ large enough so that $C\rho^{p^+} - c_1\rho^\theta + C_2 + C_3 < 0$, then we have

$$\sup_{u\in A_k^\rho} I(u) < 0.$$

Thus, Theorem 3.1(I.2) holds. According to Theorem 3.1, the functional I has critical points u_k with critical value d_k and $d_k \to 0$ as $k \to \infty$. Thus, the problem (1.1) has infinitely many weak solutions u_k $(k=1,2,\dots)$.

By (5.2), we have

$$d_k = I(u_k) \geq c_1\|u_k\|_Y^{p^+} \wedge \|u_k\|_Y^{p^-} - C_8$$

for some constant C_8. Since $d_k < 0$, we can see that $\{\|u_k\|_Y\}_{k=1}^\infty$ is bounded. □

Finally, we derive the following theorem.

Theorem 5.2. *Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ satisfying* (1.2)*. Assume that (A.1)-(A.4) with $p^- \geq 2$, (f.1), (f.2), (f.4), (g.1), (g.2) and (g.4) hold. Furthermore, suppose the following (f.3 ′) and (g.3 ′): there exists a constant $\theta > p^+$ such that*

(f.3′) $0 < \theta F(x,t) \leq f(x,t)t$ *for a.e.* $x \in \Omega$ *and* $t \neq 0$,

(g.3′) $0<\theta G(x,t)\le g(x,t)t$ *for* σ*-a.e.* $x\in\Gamma_2$ *and* $t\ne 0$.

Then if $u_k\in K_{d_k}$ *for* $k=1,2,\dots$, *then the sequence* $\{u_k\}_{k=1}^{\infty}$ *converges to zero.*

Proof. It follows from (f.3′) and (g.3′) that

$$\begin{aligned}I(u)-\frac{1}{\theta}\langle I'(u),u\rangle=&\,\Phi(u)-\frac{1}{\theta}\langle\Phi'(u),u\rangle\\&+\int_\Omega\left(\frac{1}{\theta}f(x,u(x))u(x)-F(x,u(x)\right)dx\\&+\int_\Omega\left(\frac{1}{\theta}g(x,u(x))u(x)-G(x,u(x)\right)d\sigma_x\\\ge&\,\Phi(u)-\frac{1}{\theta}\langle\Phi'(u),u\rangle\quad\text{for}\quad u\in Y.\end{aligned}$$

Here, since $p^-\ge 2$, using (A.4) and Lemma 4.2(ii), we have

$$\begin{aligned}\Phi(u)-\frac{1}{\theta}\langle\Phi'(u),u\rangle\ge&\int_\Omega\frac{1}{p(x)}\boldsymbol{a}\big(x,\boldsymbol{\nabla}u(x)\big)\cdot\boldsymbol{\nabla}u(x)dx\\&-\int_\Omega\frac{1}{\theta}\boldsymbol{a}\big(x,\boldsymbol{\nabla}u(x)\big)\cdot\boldsymbol{\nabla}u(x)dx\\\ge&\left(\frac{1}{p^+}-\frac{1}{\theta}\right)\int_\Omega\boldsymbol{a}\big(x,\boldsymbol{\nabla}u(x)\big)\cdot\boldsymbol{\nabla}u(x)dx\\\ge&\left(\frac{1}{p^+}-\frac{1}{\theta}\right)c\int_\Omega h_1(x)|\boldsymbol{\nabla}u(x)|^{p(x)}dx\\\ge&\left(\frac{1}{p^+}-\frac{1}{\theta}\right)c\|u\|_Y^{p^+}\wedge\|u\|_Y^{p^-}.\end{aligned}$$

Thus, we have

$$I(u)-\frac{1}{\theta}\langle I'(u),u\rangle\ge\left(\frac{1}{p^+}-\frac{1}{\theta}\right)c\|u\|_Y^{p^+}\wedge\|u\|_Y^{p^-}.\tag{5.3}$$

Since $\theta>p^+$, this inequality (5.3) implies that if $I(u)=0$ and $I'(u)=0$, then $u=0$. Hence, Corollary 3.1(I.3) holds. Therefore, for the sequence $\{u_k\}_{k=1}^{\infty}$, where $u_k\in K_{d_k}$ $(k=1,2,\dots)$, we have $u_k\to 0$ in Y as $k\to\infty$. □

Acknowledgements

The author would like to thank the anonymous referee(s) for useful comments and suggestions.

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