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Optimal Strong Convergence Rate of Spectral Galerkin Exponential Euler Scheme for Parabolic SPDEs

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Abstract. We provide a new approach to strong error analysis of the spatial-spectral Galerkin and temporal exponential Euler scheme for a family of second-order parabolic stochastic partial differential equations (SPDEs) driven by multiplicative noise. Applying these results to the stochastic advection-diffusion-reaction equation with a gradient term driven by white noise indicates that this scheme achieves optimal strong convergence order exactly $1/2$ in space, which removes an infinitesimal factor in the literature, and $1/4$ in time. Numerical experiments support our theoretical analysis.

AMS subject classifications: 60H35, 60H15

Key words: Parabolic stochastic partial differential equation, multiplicative white noise, spectral Galerkin approximation, exponential integrator, strong convergence rate.

1 Introduction

Consider the semilinear stochastic evolution equation in the form

$$\mathrm{d}X(t)=\big(AX(t)+F\big(X(t)\big)\big)\mathrm{d}t+G\big(X(t)\big)\mathrm{d}W(t) \tag{1.1}$$

on a Hilbert space H. Here A is a generator of an analytic C_0-semigroup on H, and W is

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a **Q**-Wiener process. Since the exact solutions of Eq. (1.1) can not be formulated explicitly in general, there have been numerous works on their numerical approximations; see, e.g. [3–5, 7, 10–12, 14, 15, 20–23, 25, 27] and references therein. One attractive and challenging topic is proving numerical approximations' critical strong convergence rate while relaxing the coefficient conditions.

For second-order parabolic semilinear SPDEs, which can be equivalently rewritten in the form of (1.1) driven by (space-time) white noise, the classical Lipschitz condition (see (2.3)) is not fulfilled in general, even though the coefficients of Eq. (1.1) are Lipschitz continuous functions. In this case, the strong convergence rates of numerical approximations have yet to be studied thoroughly. Based on the framework developed in [16, 17], we present a new approach to deriving optimal strong convergence rates for numerical discretizations under generalized Lipschitz assumptions proposed in combination with the semigroup. There, the smoothing effect of the semigroup is more naturally exploited. Our approach applies to a family of second-order parabolic SPDEs, inclusive of the stochastic heat equation and the stochastic advection-diffusion-reaction equation, where there is a gradient term describing the advection phenomenon [6]. We highlight that the covariance operator of the Wiener process under study does not need to be of trace-class; nonsmooth noise, including white noise, is also available.

We first consider the spatial spectral Galerkin approximations (see Eq. (2.4)) of the Eq. (1.1). We propose the assumptions on the original equation and the spectral Galerkin approximate operator. Then, the spatial strong convergence rate of spectral Galerkin approximations towards Eq. (1.1) is obtained using Gronwall's inequality with a singular kernel given in [17]. We further employ the explicit exponential integrator to construct a fully discrete scheme (see (3.1)), which has implementation advantages over implicit schemes and avoids the CFL-type condition which appears in most explicit schemes for SPDEs such as the Euler-Maruyama scheme (see also [1, 2, 8, 9, 28]). Combining with a temporally uniform Hölder continuity condition on spectral Galerkin approximations, we prove the strong convergence rate in the temporal direction.

To specify the parameters in the proposed assumptions, we interpret our approach by applying it to a second-order parabolic SPDE with a gradient term and multiplicative white noise. A typical example is the stochastic advection-diffusion-reaction equation. We note that in [10], the authors obtained $1/4-\epsilon$ pathwise order of the linearity-implicit Euler scheme, where ϵ denotes an infinitesimal factor. As to smoother noise, [24] analyzes the exponential integrator for finite element discretizations of Eq. (1.1) driven by trace-class noise. Recently, the authors in [3] established spatial $1/4-\epsilon$ and temporal $1/4$ strong orders of the central difference and exponential Euler scheme without a gradient term with regular initial datum, based on a framework of Green functions.

We should also mention another technique based on a framework of Green functions; see, e.g. [3, 12, 14, 15, 27] for strong error estimates of numerical approximations for the stochastic heat equation driven by multiplicative white noise without gradient terms. It should also be noted that obtaining a sharp, strong convergence rate of numerical approximations for SPDEs involving gradient terms would be difficult when estimating the estimates between the continuous and discrete Green functions used in previous literature.

We indicate that the spatial and temporal strong orders spectral Galerkin exponential Euler scheme applied to the 1D stochastic advection-diffusion-reaction equation driven by white noise, achieve exactly $1/2$ and $1/4$, respectively (see Theorem 4.1). Compared with the optimal regularity of the solution (illustrated in Remark 4.2), the spectral Galerkin approximations are superconvergent (which removes the infinitesimal factor in the literature), and the convergence rate of the exponential integrator is optimal.

The rest of this paper is structured as follows. We derive a quantitative continuous dependence result for a sequence of perturbation equations of Eq. (1.1) in the next section, which contains the strong convergence rate of spatial approximations towards Eq. (1.1). In Section 3, we obtain the strong convergence rate of a fully discrete scheme under a temporally uniform Hölder continuity condition for the Galerkin approximated solutions. Concrete examples are given in Section 4, where we verify the key assumptions to derive the strong convergence rate of the scheme under study. In the last section, we will conduct several numerical experiments to confirm our theoretical results.

2 Spectral Galerkin approximations

This section first introduces the setting of the semilinear stochastic evolution equation (1.1). Then, we analyze the strong error estimate between Eq. (1.1) and its spectral Galerkin approximations.

2.1 Setting

Let $(H,\|\cdot\|)$ be a separable Hilbert space with Borel σ-algebra $\mathcal{B}(H)$ and denote by $(\mathcal{L}(H),\|\cdot\|_{\mathcal{L}(H)})$ the space of bounded linear operators on H. We assume that the linear operator $A:D(A)\subseteq H\to H$ is self-adjoint and negative definite and is the infinitesimal generator of an analytic C_0-semigroup $\{S(\cdot)\}$ such that the resolvent set of A contains all $\lambda\in\mathbb{C}$ with $\Re[\lambda]\geq 0$. Then $-A$ possesses an eigensystem $\{(\lambda_j,h_j)\}_{j\in\mathbb{N}_+}$ with $\{\lambda_j\}_{j\in\mathbb{N}_+}$ being an increasing sequence and $\{h_j\}_{j\in\mathbb{N}_+}$ forming an orthonormal basis of H. Then for each $\theta\in\mathbb{R}$, the fractional powers $(-A)^{\theta}$ is well-defined. We denote by $\dot{H}^{\theta}$ the domain of $(-A)^{\theta/2}$ equipped with the norm $\|\cdot\|_{\theta}:=\|(-A)^{\theta/2}(\cdot)\|$. In particular, $\dot{H}^0=H$. It is well known that (see, e.g. [26, Chapter 2.6])

$$\begin{aligned}
&\|(-A)^{\nu}\|_{\mathcal{L}(H)}\leq C,\\
&\|(-A)^{\mu}S(t)\|_{\mathcal{L}(H)}\leq Ct^{-\mu},\\
&\big\|(-A)^{-\theta}\big(S(t)-\mathrm{Id}_H\big)\big\|_{\mathcal{L}(H)}\leq Ct^{\theta}
\end{aligned}\tag{2.1}$$

for any $0<t\leq T$, $\nu\leq 0\leq\mu$ and $0\leq\theta\leq 1$. This paper uses C as a generic positive constant independent of the spatial and temporal discrete parameters.

Let U be another separable Hilbert space and $\mathbf{Q}$ a self-adjoint, nonnegative definite, and bounded linear operator on U. Denote by $\mathcal{L}_2^\theta$ the set of Hilbert-Schmidt operators from $U_0 := \mathbf{Q}^{1/2}(U)$ to $\dot{H}^\theta$. The driving process W in Eq. (1.1) is a U-valued $\mathbf{Q}$-Wiener process with respect to a stochastic basis $(\Omega,\mathscr{F},\mathbb{F}:=(\mathscr{F}_t),\mathbb{P})$. When $\mathbf{Q}$ coincides with the identity operator on U,W is called a cylindrical Wiener process, and its derivative in the distributional sense is called white noise.

Suppose that $F:H\to\dot{H}^{\theta_F}$ with $\theta_F>-2$ and $G:H\to\mathcal{L}(U_0,H)$ are measurable. Throughout the paper, T is a fixed positive time and the initial datum $X_0:=X(0)$ of Eq. (1.1) is assumed to be $\mathscr{F}_0/\mathcal{B}(H)$-measurable. Recall that an $\mathbb{F}$-adapted stochastic process $X:[0,T]\times\Omega\to H$ is called a mild solution of Eq. (1.1) if it holds (almost surely) that $X\in L^p(0,T;H)$ for some $p\geq 1$ and for all $t\in[0,T]$,

$$X(t)=S(t)X_0+\int_0^t S(t-r)F\big(X(r)\big)\mathrm{d}r+\int_0^t S(t-r)G\big(X(r)\big)\mathrm{d}W(r). \tag{2.2}$$

Instead of the classical infinite-dimensional Lipschitz condition

$$\|F(x)-F(y)\|+\|G(x)-G(y)\|_{\mathcal{L}_2^0}\leq C\|x-y\|, \quad \forall x,y\in H \tag{2.3}$$

for Eq. (1.1) with some positive constant C, we consider the following generalized Lipschitz-type condition in the framework of [16,17]. In the sequel, we use θ to denote a positive number in $(0,1]$ that characterizes the spatial regularity for the solution of Eq. (1.1).

Assumption 2.1. There exist four nonnegative, Borel measurable functions K_F,K_{F_θ},K_G and K_{G_θ} on $[0,T]$ with

$$\begin{aligned} K^*(T)&:=\int_0^T \big[K_F(t)+K_G^2(t)\big]\mathrm{d}t<\infty,\\ K_\theta^*(T)&:=\int_0^T \big[K_{F_\theta}(t)+K_{G_\theta}^2(t)\big]\mathrm{d}t<\infty \end{aligned}$$

such that for any $x,y\in H,z\in\dot{H}^\theta$, and $t\in[0,T]$, it holds that

$$\begin{aligned} &\big\|S(t)\big(F(x)-F(y)\big)\big\|\leq K_F(t)\|x-y\|,\\ &\|S(t)F(z)\|_\theta\leq K_{F_\theta}(t)(1+\|z\|_\theta),\\ &\big\|S(t)\big(G(x)-G(y)\big)\big\|_{\mathcal{L}_2^0}\leq K_G(t)\|x-y\|,\\ &\|S(t)G(z)\|_{\mathcal{L}_2^\theta}\leq K_{G_\theta}(t)(1+\|z\|_\theta). \end{aligned}$$

Lemma 2.1 ([17, Theorem 2.1])**.** *Let $X_0\in L^p(\Omega;\dot{H}^\theta)$ with $p\geq 2$ and $\theta>0$ and Assumption 2.1 hold. Eq. (1.1) admits a unique mild solution $X\in\mathcal{C}([0,T];L^p(\Omega;\dot{H}^\theta))$ and there exists a constant $C=C(T,p,K_\theta^*(T))$ such that*

$$\sup_{t\in[0,T]}\mathbb{E}\|X(t)\|_\theta^p\leq C\big(1+\mathbb{E}\|X_0\|_\theta^p\big).$$

2.2 Spectral Galerkin approximation

For an $N\in\mathbb{N}_+$, let V_N be the linear space spanned by the first N eigenvectors of the operator A, i.e. $V_N=\mathrm{span}\{h_1,\cdots,h_N\}$. Let $\mathcal{P}_N:\dot{H}^{-1}\to V_N$ be the generalized orthogonal projection operator and A_N be the Laplacian restricted in V_N. The spectral Galerkin approximation for Eq. (1.1) is to find a sequence of $\mathbb{F}$-adapted V_N-valued processes $\{X^N(t): t\in[0,T]\}$ such that

$$\mathrm{d}X^N(t)=\big(A_NX^N(t)+\mathcal{P}_NF\big(X^N(t)\big)\big)\mathrm{d}t+\mathcal{P}_NG\big(X^N(t)\big)\mathrm{d}W(t),\quad t\in[0,T] \tag{2.4}$$

with $N\in\mathbb{N}_+$ and $X^N(0)=\mathcal{P}_NX_0$.

It is clear that $\mathcal{P}_N$ is a commutative contraction operator with A and the linear operator A_N is the infinitesimal generator of an analytic C_0-semigroup $\{S_N(\cdot)\}$ on V_N. Consequently, under Assumption 2.1, one has

$$\begin{aligned}
&\big\|S_N(t)\mathcal{P}_N\big(F(x)-F(y)\big)\big\|\le K_F(t)\|x-y\|,\\
&\|S_N(t)\mathcal{P}_NF(z)\|_\theta\le K_{F_\theta}(t)(1+\|z\|),\\
&\big\|S_N(t)\mathcal{P}_N\big(G(x)-G(y)\big)\big\|_{\mathcal{L}_2^0}\le K_G(t)\|x-y\|,\\
&\|S_N(t)\mathcal{P}_NG(z)\|_{\mathcal{L}_2^\theta}\le K_{G_\theta}(t)(1+\|z\|)
\end{aligned}$$

for any $x,y,z\in H$, and $t\in[0,T]$, from which we have the well-posedness of the spectral Galerkin approximation (2.4).

Lemma 2.2. *Let $X_0\in L^p(\Omega;\dot{H}^\theta)$ with $p\ge 2$ and $\theta>0$ and Assumption* 2.1 *hold. Then Eq.* (2.4) *admits a unique mild solution $X^N\in\mathcal{C}([0,T];L^p(\Omega;\dot{H}^\theta))$, and there exists a constant $C=C(T,p,K_\theta^*(T))$ such that*

$$\sup_{N\in\mathbb{N}_+}\sup_{t\in[0,T]}\mathbb{E}\big\|X^N(t)\big\|_\theta^p\le C\big(1+\mathbb{E}\|X_0\|_\theta^p\big).$$

To obtain the strong convergence rate of spectral Galerkin approximations (2.4) towards Eq. (1.1), we propose the following assumption, in which the parameters r_F,r_G will be concretized by examples in Section 4.

Assumption 2.2. There exist two nonnegative, Borel measurable functions $R_{F_\theta^N},R_{G_\theta^N}$ in $(0,T]$ and two positive numbers r_F,r_G depending on θ with

$$\int_0^T R_{F_\theta^N}(t)\mathrm{d}t\le CN^{-r_F},\quad \int_0^T R^2_{G_\theta^N}(t)\mathrm{d}t\le CN^{-2r_G}$$

such that for any $z\in H^\theta$, it holds that

$$\begin{aligned}
&\big\|[S_N(t)\mathcal{P}_N-S(t)]F(z)\big\|\le R_{F_\theta^N}(t)(1+\|z\|_\theta),\\
&\big\|[S_N(t)\mathcal{P}_N-S(t)]G(z)\big\|_{\mathcal{L}_2^0}\le R_{G_\theta^N}(t)(1+\|z\|_\theta).
\end{aligned}$$

This section's main result is the following strong error estimate between spectral Galerkin approximations (2.4) and Eq. (1.1).

Theorem 2.1. *Let* $p\geq 2,\theta>0$*, Assumptions* 2.1-2.2 *hold, and* $X_0\in L^p(\Omega;\dot{H}^\theta)$ *such that*

$$\sup_{t\in[0,T]}\big\|[S_N(t)\mathcal{P}_N-S(t)]X_0\big\|_{L^p(\Omega;H)}\leq CN^{-r_0}\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}. \tag{2.5}$$

Then there exists a constant $C=C(T,p,K^*(T),K^*_\theta(T))$ *such that*

$$\sup_{t\in[0,T]}\big\|X^N(t)-X(t)\big\|_{L^p(\Omega;H)}\leq C\left(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\right)N^{-(r_0\wedge r_F\wedge r_G)}. \tag{2.6}$$

Proof. For any $t\in[0,T]$, we decompose the error between $X^N(t)$ and $X(t)$ as

$$\begin{aligned}
&\big\|X^N(t)-X(t)\big\|_{L^p(\Omega;H)}\\
\leq&\big\|[S_N(t)\mathcal{P}_N-S(t)]X_0\big\|_{L^p(\Omega;H)}+\left\|\int_0^t\big[S_N(t-r)F\big(X^N(r)\big)-S(t-r)F\big(X(r)\big)\big]\mathrm{d}r\right\|_{L^p(\Omega;H)}\\
&+\left\|\int_0^t\big[S_N(t-r)G\big(X^N(r)\big)-S(t-r)G\big(X(r)\big)\big]\mathrm{d}W(r)\right\|_{L^p(\Omega;H)}\\
=:&I_1(t)+I_2(t)+I_3(t).
\end{aligned}$$

For the first term, it follows from the condition (2.5) that

$$I_1(t)\leq CN^{-\theta}\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}.$$

We deduce by the Minkowski inequality, Assumption 2.1, and (2.7) that

$$\begin{aligned}
I_2(t)\leq&\int_0^t\big\|S_N(t-r)\big[F\big(X^N(r)\big)-F\big(X(r)\big)\big]\big\|_{L^p(\Omega;H)}\mathrm{d}r\\
&+\int_0^t\big\|[S_N(t-r)\mathcal{P}_N-S(t-r)]F\big(X(r)\big)\big\|_{L^p(\Omega;H)}\mathrm{d}r\\
\leq&\int_0^t K_F(t-r)\big\|X^N(r)-X(r)\big\|_{L^p(\Omega;H)}\mathrm{d}r\\
&+C\int_0^t R_{F^N_\theta}(t-r)(1+\|X(r)\|_{L^p(\Omega;H)})\mathrm{d}r.
\end{aligned}$$

For the last term, Burkholder-Davis-Gundy (BDG) inequality, Assumption 2.1, and (2.8) yield

$$\begin{aligned}
I_3^2(t)\leq&2\int_0^t\big\|S_N(t-r)\big[G\big(X^N(r)\big)-G\big(X(r)\big)\big]\big\|^2_{L^p(\Omega;\mathcal{L}_2^0)}\mathrm{d}r\\
&+2\int_0^t\big\|[S_N(t-r)-S(t-r)]G\big(X(r)\big)\big\|^2_{L^p(\Omega;\mathcal{L}_2^0)}\mathrm{d}r\\
\leq&2\int_0^t K_G^2(t-r)\big\|X^N(r)-X(r)\big\|^2_{L^p(\Omega;H)}\mathrm{d}r\\
&+C\int_0^t R_{G^N_\theta}(t-r)\Big(1+\|X(r)\|_{L^p(\Omega;H)}\Big)^2\mathrm{d}r.
\end{aligned}$$

We combine the previous four estimates with Assumption 2.1, Lemma 2.1 to derive

$$\begin{aligned}&\left\|X^N(t)-X(t)\right\|_{L^p(\Omega;H)}^2\\ \le& CN^{-2(r_0\wedge r_F\wedge r_G)}\left(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\right)^2\\ &+C\int_0^t\left[K_F(t-r)+K_G^2(t-r)\right]\left\|X^N(r)-X(r)\right\|_{L^p(\Omega;H)}^2\mathrm{d}r,\end{aligned}$$

where C above depends on $K_\theta^*(T)$. According to Assumption 2.1, $K_F+K_G^2$ is uniformly integrable on $[0,T], N\in\mathbb{N}_+$. Then we conclude (2.6) by the Gronwall lemma with a singular kernel given in [17, Lemma 3.1]. $\square$

Remark 2.1. It follows from the condition (2.5) and Assumption 2.1 that

$$\begin{aligned}&\left\|[S_N(t)\mathcal{P}_N-S(t)]F(z)\right\|\\ =&\left\|\left[S_N\left(\frac{t}{2}\right)\mathcal{P}_N-S\left(\frac{t}{2}\right)\right]S\left(\frac{t}{2}\right)F(z)\right\|\\ \le& CN^{-r_0}K_{F_\theta}(t)(1+\|z\|_\theta)\end{aligned}\tag{2.7}$$

for all $t>0$ and $z\in H$. Similarly, for all $t>0$ and $z\in H$,

$$\left\|[S_N(t)\mathcal{P}_N-S(t)]G(z)\right\|_{\mathcal{L}_2^0}\le CN^{-r_0}K_{G_\theta}(t)(1+\|z\|\theta).\tag{2.8}$$

We will find in Section 4 that the orders r_F and r_G in Assumption 2.2 are better than those in (2.7) and (2.8), respectively.

3 Fully discrete scheme

To construct a full discretization of Eq. (1.1), we apply the explicit exponential integrator to spectral Galerkin approximations (2.4).

Let $M\in\mathbb{N}_+$ and denote $\mathbb{Z}_M=\{0,1,\ldots,M\}$. Divide the temporal interval $(0,T]$ into K equidistant subinterval $\{(t_i,t_{i+1}]: i\in\mathbb{Z}_M\}$, i.e. $t_i=i\tau$ for $i\in\mathbb{Z}_M$ with time step $\tau=T/M$. Then we define the spectral Galerkin exponential Euler scheme $\{X_k^N : k=1,\ldots,M\}$ of Eq. (1.1) by the recursion

$$X_k^N=S_N(\tau)X_{k-1}^N+\tau S_N(\tau)\mathcal{P}_NF\left(X_{k-1}^N\right)+S_N(\tau)\mathcal{P}_NG\left(X_{k-1}^N\right)\Delta W_k\tag{3.1}$$

with the same initial datum as Eq. (2.4), where $\Delta W_k=W(t_k)-W(t_{k-1})$. Equivalently,

$$\begin{aligned}X_k^N=&S_N(t_k)X_0^N+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}S_N(t_k-t_i)\mathcal{P}_NF\left(X_i^N\right)\mathrm{d}r\\ &+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}S_N(t_k-t_i)\mathcal{P}_NG\left(X_i^N\right)\mathrm{d}W(r),\quad k=1,\ldots,M.\end{aligned}$$

To derive a strong convergence rate of the scheme (3.1), we propose the following local error assumption on S_N.

Assumption 3.1. There exist two nonnegative, Borel measurable functions $R_{F_\theta^N,\tau}$ and $R_{G_\theta^N,\tau}$ in $(0,T]$ and two positive numbers η_F,η_G depending on θ with

$$\sup_{N\in\mathbb{N}_+}\int_0^T R_{F_\theta^N,\tau}(t)\mathrm{d}t\le C\tau^{\eta_F},\quad \sup_{N\in\mathbb{N}_+}\int_0^T R^2_{G_\theta^N,\tau}(t)\mathrm{d}t\le C\tau^{2\eta_G}$$

such that for any $z\in\dot{H}^\theta$ and $t\in(t_i,t_{i+1}]$, $i\in\mathbb{Z}_{M-1}$, it holds that

$$\begin{aligned}&\big\|[S_N(t)-S_N(t_{i+1})]\mathcal{P}_NF(z)\big\|\le R_{F_\theta^N,\tau}(t)(1+\|z\|_\theta),\\&\big\|[S_N(t)-S_N(t_{i+1})]\mathcal{P}_NG(z)\big\|_{\mathcal{L}_2^0}\le R_{G_\theta^N,\tau}(t)(1+\|z\|_\theta).\end{aligned}$$

Remark 3.1. As illustrated in Remark 2.1, the orders η_F and η_G in Assumption 3.1 are better than those followed from Assumption 2.1 and the regularity (2.1).

In addition to the above assumption on the data of spectral Galerkin approximations (2.4), we need a uniform Hölder regularity of their solutions.

Assumption 3.2. The uniform Hölder exponent of X^N in $L^p(\Omega;H)$ is γ for some $\gamma\in(0,1/2]$ depending on θ, i.e. there exists a constant C such that

$$\sup_{N\in\mathbb{N}_+}\big\|X^N(t)-X^N(s)\big\|_{L^p(\Omega;H)}\le C\Big(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\Big)(t-s)^\gamma. \tag{3.2}$$

Remark 3.2. The Hölder exponent in Assumption 3.2 is not larger than $1/2$ due to the temporal regularity of **Q**-Wiener processes. In the general multiplicative noise case, the uniform Hölder continuity assumption is essential in our error analysis. We also note that in the additive noise case, for instance, if there exists a constant operator G such that $G(z)\equiv G$ for all $z\in H$, then Assumption 3.2 is not necessary for the strong error analysis.

We give a sufficient condition for Assumption 3.2.

Lemma 3.1. *Let $p\ge 2,\theta\in(0,2)$ and $X_0\in L^p(\Omega;\dot{H}^\theta)$. Assume in addition to Assumption* 2.1 *that there exist two positive constants $\gamma_1\le 1$ and $\gamma_2\le 1/2$ such that*

$$\int_0^t K_F(r)\mathrm{d}r\le Ct^{\gamma_1},\quad \int_0^t K_G^2(r)\mathrm{d}r\le Ct^{2\gamma_2},\quad t\in(0,T]. \tag{3.3}$$

Then X is uniformly Hölder continuous with exponent $\gamma=(\theta/2)\wedge\gamma_1\wedge\gamma_2$ in Assumption 3.2.

Proof. Let $p\ge 2$ and $0\le s<t\le T$. By Minkowski and BDG inequalities, we get

$$\begin{aligned}&\big\|X^N(t)-X^N(s)\big\|_{L^p(\Omega;H)}\\\le&\big\|\big(S(t)-S(s)\big)\mathcal{P}_NX_0\big\|_{L^p(\Omega;H)}\\&+\int_s^t\big\|S(t-r)\mathcal{P}_NF\big(X^N(r)\big)\big\|_{L^p(\Omega;H)}\mathrm{d}r\end{aligned}$$

$$
\begin{aligned}
&+\int_0^s \big\|\big(S(t-s)-\mathrm{Id}_H\big)S(s-r)\mathcal{P}_N F\big(X^N(r)\big)\big\|_{L^p(\Omega;H)}\mathrm{d}r \\
&+C\left(\int_s^t \big\|S(t-r)\mathcal{P}_N G\big(X^N(r)\big)\big\|^2_{L^p(\Omega;\mathcal{L}_2^0)}\mathrm{d}r\right)^{\frac{1}{2}} \\
&+C\left(\int_0^s \big\|\big(S(t-s)-\mathrm{Id}_H\big)S(s-r)\mathcal{P}_N G\big(X^N(r)\big)\big\|^2_{L^p(\Omega;\mathcal{L}_2^0)}\mathrm{d}r\right)^{\frac{1}{2}} \\
=:\,& I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

The smoothing effect (2.1) of the semigroup S yields that

$$
\begin{aligned}
I_1 &\le C\big\|(-A)^{-(\frac{\theta}{2}\wedge 1)}\big(S(t-s)-\mathrm{Id}_H\big)\big\|_{\mathcal{L}(H)}\|S(s)\|_{\mathcal{L}(H)}\|X_0\|_{L^p(\Omega;\dot{H}^\theta)} \\
&\le C(t-s)^{\frac{\theta}{2}\wedge 1}\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}.
\end{aligned}
$$

By Lemma 2.2 and condition (3.3), we get

$$
\begin{aligned}
I_2+I_4 &\le \left[\int_0^{t-s}K_F(r)\mathrm{d}r+\left(\int_0^{t-s}K_G^2(r)\mathrm{d}r\right)^{\frac{1}{2}}\right]\left(1+\sup_{t\in[0,T]}\big\|X^N(t)\big\|_{L^p(\Omega;H)}\right) \\
&\le C\Big(1+\|X_0\|_{L^p(\Omega;H)}\Big)(t-s)^{\gamma_1\wedge\gamma_2}.
\end{aligned}
$$

For the third term I_3 and the last term I_5, we derive by the smoothing property (2.1) and Assumption 2.1 that

$$
\begin{aligned}
I_3+I_5 &\le C\big\|(-A)^{-(\frac{\theta}{2}\wedge 1)}\big(S(t-s)-\mathrm{Id}_H\big)\big\|_{\mathcal{L}(H)} \\
&\quad\times\left[\int_0^s K_{F_\theta}\mathrm{d}r+\left(\int_0^s K_{G_\theta}^2\mathrm{d}r\right)^{\frac{1}{2}}\right]\left(1+\sup_{t\in[0,T]}\big\|X^N(t)\big\|_{L^p(\Omega;\dot{H}^\theta)}\right) \\
&\le C\Big(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\Big)(t-s)^{\frac{\theta}{2}\wedge 1}.
\end{aligned}
$$

Combining the above five estimations, Assumption 3.2 holds with $\gamma=(\theta/2)\wedge\gamma_1\wedge\gamma_2$. □

We also need a discrete version of Gronwall's inequality with a singular kernel. For the corresponding continuous version, we refer to [17, Lemma 3.1].

Lemma 3.2. *Let $m>0$ and $\Psi^N:[0,T]\to\mathbb{R}$, $N\in\mathbb{N}_+$, be a sequence of nonnegative, Borel measurable functions such that*

$$
\sup_{N\in\mathbb{N}_+}\sup_{K\in\mathbb{N}_+}\sum_{i=1}^K\Psi^N(t_i)\tau<\infty. \tag{3.4}
$$

For any $N\in\mathbb{N}_+$, assume that $\{F(k)\}_{k=0}^K$ is a nonnegative sequence such that

$$
F(k)\le m+\sum_{i=0}^{k-1}\Psi^N(t_k-t_i)F(i)\tau,
$$

then there exists a constant μ independent of K and N such that

$$\sup_{N\in\mathbb{N}_+}\sup_{K\in\mathbb{N}_+}\sup_{0\le k\le K}F(k)\le 2me^{\mu T}.$$

Proof. For any $\mu\ge 0$, it holds that

$$e^{-\mu t_k}F(k)\le m+\sum_{i=0}^{k-1}e^{-\mu(t_k-t_i)}\Psi^N(t_k-t_i)e^{-\mu t_i}F(i)\tau.$$

Denoting $F_\mu(k):=e^{-\mu t_k}F(k)$ and $\Psi_\mu^N(t):=e^{-\mu t}\Psi^N(t)$, we have

$$F_\mu(k)\le m+\sum_{i=0}^{k-1}\Psi_\mu^N(t_k-t_i)F_\mu(i)\tau.$$

From the condition (3.4), we get

$$\alpha_\mu(T):=\sup_{N\in\mathbb{N}_+}\sup_{K\in\mathbb{N}_+}\sum_{i=1}^{K}\Psi_\mu^N(t_i)\tau$$

decreases concerning μ. Since

$$\lim_{\mu\to 0}\alpha_\mu(T)\le\sup_{N\in\mathbb{N}_+}\sup_{K\in\mathbb{N}_+}\sum_{i=1}^{K}\Psi^N(t_i)\tau\quad\text{and}\quad\lim_{\mu\to\infty}\alpha_\mu(T)=0,$$

there exists a constant μ_0 independent of K and N such that $\alpha_{\mu_0}(T)\le 1/2$. As a consequence,

$$F_{\mu_0}(k)\le m+\alpha_{\mu_0}(T)\sup_{0\le i\le k-1}F_{\mu_0}(i)\le m+\frac{1}{2}\sup_{0\le i\le K}F_{\mu_0}(i),$$

from which we obtain $\sup_{0\le k\le K}F_{\mu_0}(k)\le 2m$. This completes the proof. □

The following theorem shows the strong convergence rate of the fully discrete scheme for Eq. (1.1).

Theorem 3.1. *Let $p\ge 2,\theta>0,X_0\in L^p(\Omega;\dot{H}^\theta)$ such that* (2.5) *and Assumptions* 2.1, 2.2, *and* 3.1 *hold with certain positive constants $r_0,r_F,r_G,\eta_F,\eta_G$, and γ. Then there exists a constant $C=C(T,p,K^*(T),K_\theta^*(T))$ such that*

$$\sup_{k\in\mathbb{Z}_M}\left\|X(t_k)-X_k^N\right\|_{L^p(\Omega;H)}\le C\Big(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\Big)\big(N^{-(r_0\wedge r_F\wedge r_G)}+\tau^{\gamma\wedge\eta_F\wedge\eta_G}\big).\tag{3.5}$$

Proof. In terms of Theorem 2.1 and the triangle inequality, it suffices to show that

$$\sup_{k\in\mathbb{Z}_M}\left\|X^N(t_k)-X_k^N\right\|_{L^p(\Omega;H)}\le C\Big(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\Big)\tau^{\gamma\wedge\eta_F\wedge\eta_G}.\tag{3.6}$$

For any $k=1,\dots,K$, the error satisfies

$$
\begin{aligned}
&\left\|X^N(t_k)-X_k^N\right\|_{L^p(\Omega;H)}\\
\leq&\left\|\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\Big(S_N(t_k-r)\mathcal{P}_NF\big(X^N(r)\big)-S_N(t_k-t_i)\mathcal{P}_NF\big(X_i^N\big)\Big)\mathrm{d}r\right\|_{L^p(\Omega;H)}\\
&+\left\|\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\Big(S_N(t_k-r)\mathcal{P}_NG\big(X^N(r)\big)-S_N(t_k-t_i)\mathcal{P}_NG\big(X_i^N\big)\Big)\mathrm{d}W(r)\right\|_{L^p(\Omega;H)}\\
=:&J_1+J_2.
\end{aligned}
$$

Using the Minkowski inequality, we have

$$
\begin{aligned}
J_1\leq&\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left\|[S_N(t_k-r)-S_N(t_k-t_i)]\mathcal{P}_NF\big(X^N(r)\big)\right\|_{L^p(\Omega;H)}\mathrm{d}r\\
&+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left\|S_N(t_k-t_i)\mathcal{P}_N\big[F\big(X^N(r)\big)-F\big(X^N(t_i)\big)\big]\right\|_{L^p(\Omega;H)}\mathrm{d}r\\
&+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left\|S_N(t_k-t_i)\mathcal{P}_N\big[F\big(X^N(t_i)\big)-F\big(X_i^N\big)\big]\right\|_{L^p(\Omega;H)}\mathrm{d}r\\
=:&J_{11}+J_{12}+J_{13}.
\end{aligned}
$$

For the term J_{11}, we use Assumption 3.1 and Lemma 2.2 to obtain

$$
\begin{aligned}
J_{11}=&\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left\|[S_N(t_k-r)-S_N(t_k-t_i)]\mathcal{P}_NF\big(X^N(r)\big)\right\|_{L^p(\Omega;H)}\mathrm{d}r\\
\leq& C\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}R_{F_\theta^N,\tau}(t_k-r)\Big(1+\left\|X^N(r)\right\|_{L^p(\Omega;\dot{H}^\theta)}\Big)\mathrm{d}r\\
\leq& C\Big(1+\sup_{t\in[0,T]}\|X^N(t)\|_{L^p(\Omega;\dot{H}^\theta)}\Big)\int_0^T R_{F_\theta^N,\tau}(t)\mathrm{d}t\\
\leq& C\Big(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\Big)\tau^{\eta_F}.
\end{aligned}
$$

Assumptions 2.1 and 3.2 and the integrability of K_F over $(0,T)$ imply

$$
\begin{aligned}
J_{12}\leq&\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}K_F(t_k-t_i)\left\|X^N(r)-X^N(t_i)\right\|_{L^p(\Omega;H)}\mathrm{d}r\\
\leq& C\tau^\gamma\Big(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\Big)\sum_{i=1}^{k}K_F(t_i)\tau\\
\leq& C\Big(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\Big)\tau^\gamma.
\end{aligned}
$$

According to Cauchy-Schwarz inequality and Assumption 2.1, we get

$$
\begin{aligned}
J_{13}^2 &\leq \left(\sum_{i=0}^{k-1} K_F(t_k-t_i)\left\|X^N(t_i)-X_i^N\right\|_{L^p(\Omega;H)}\tau\right)^2 \\
&\leq \sum_{i=1}^{k} K_F(t_i)\tau \sum_{i=0}^{k-1} K_F(t_k-t_i)\left\|X^N(t_i)-X_i^N\right\|_{L^p(\Omega;H)}^2\tau \\
&\leq C\sum_{i=0}^{k-1} K_F(t_k-t_i)\left\|X^N(t_i)-X_i^N\right\|_{L^p(\Omega;H)}^2\tau.
\end{aligned}
$$

Collecting the above estimates for J_{11}, J_{12}, and J_{13} together, we have that

$$
\begin{aligned}
J_1^2 \leq & C\tau^{2(\eta_F\wedge\gamma)}\left(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\right)^2 \\
&+C\sum_{i=0}^{k-1} K_F(t_k-t_i)\left\|X^N(t_i)-X_i^N\right\|_{L^p(\Omega;H)}^2\tau. \qquad (3.7)
\end{aligned}
$$

To estimate J_2, we utilize BDG inequality, similar arguments as in the estimates of J_{11}, J_{12} and J_{13}, the square integrability of K_G and K_{G_θ} over $(0,T)$, and Assumption 3.1 to show

$$
\begin{aligned}
J_2^2 \leq & C\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left\|[S_N(t_k-r)-S_N(t_k-t_i)]\mathcal{P}_N G\big(X^N(r)\big)\right\|_{L^p(\Omega;\mathcal{L}_2^0)}^2 \mathrm{d}r \\
&+C\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left\|S_N(t_k-t_i)\mathcal{P}_N\big[G\big(X^N(r)\big)-G\big(X^N(t_i)\big)\big]\right\|_{L^p(\Omega;\mathcal{L}_2^0)}^2 \mathrm{d}r \\
&+C\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left\|S_N(t_k-t_i)\mathcal{P}_N\big[G\big(X^N(t_i)\big)-G\big(X_i^N\big)\big]\right\|_{L^p(\Omega;\mathcal{L}_2^0)}^2 \mathrm{d}r \\
\leq & C\tau^{2(\eta_G\wedge\gamma)}\left(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\right)^2+C\sum_{i=0}^{k-1}K_G^2(t_k-t_i)\left\|X^N(t_i)-X_i^N\right\|_{L^p(\Omega;H)}^2\tau. \qquad (3.8)
\end{aligned}
$$

Combining (3.7)-(3.8), we obtain

$$
\begin{aligned}
&\left\|X^N(t_k)-X_k^N\right\|_{L^p(\Omega;H)}^2 \\
\leq & C\tau^{2(\gamma\wedge\eta_F\wedge\eta_G)}\left(1+\|X_0\|_{L^p(\Omega;\dot{H}^\theta)}\right)^2 \\
&+C\sum_{i=0}^{k-1}\left[K_F(t_k-t_i)+K_G^2(t_k-t_i)\right]\left\|X^N(t_i)-X_i^N\right\|_{L^p(\Omega;H)}^2\tau.
\end{aligned}
$$

As a consequence of Lemma 3.2, we conclude (3.6). □

4 Application to stochastic advection-diffusion-reaction equation

We illustrate our results by the following second-order semilinear parabolic SPDE with gradient term and multiplicative noise under homogeneous Dirichlet boundary condition

$$\begin{aligned}&\mathrm{d}X(t,\xi)=\big(\Delta X(t,\xi)+\nabla f\big(X(t,\xi)\big)+\widetilde{f}\big(X(t,\xi)\big)\big)\mathrm{d}t+g\big(X(t,\xi)\big)\mathrm{d}W(t,\xi),\\ &X(t,\xi)=0, \qquad (t,\xi)\in[0,T]\times\partial\mathscr{O},\\ &X(0,\xi)=X_0(\xi), \quad \xi\in\mathscr{O}\subset\mathbb{R}^d,\end{aligned} \tag{4.1}$$

where $\mathscr{O}\subset\mathbb{R}^d$ is a bounded open set with regular boundary. Assume that $f,\widetilde{f},g:\mathbb{R}\to\mathbb{R}$ are Lipschitz continuous functions with Lipschitz constants $L_f,L_{\widetilde{f}},L_g>0$, i.e.

$$\begin{aligned}&|f(\xi_1)-f(\xi_2)|\le L_f|\xi_1-\xi_2|,\\ &|\widetilde{f}(\xi_1)-\widetilde{f}(\xi_2)|\le L_{\widetilde{f}}|\xi_1-\xi_2|,\\ &|g(\xi_1)-g(\xi_2)|\le L_g|\xi_1-\xi_2|\end{aligned} \tag{4.2}$$

for any ξ_1, $\xi_2\in\mathbb{R}$.

Let $H=U=L^2(\mathscr{O})$ and define $A=\Delta$ with domain $\mathrm{Dom}(A)=H_0^1(\mathscr{O})\cap H^2(\mathscr{O})$, where $H_0^1(\mathscr{O}):=\{f\in H^1(\mathscr{O}):\ f|_{\partial\mathscr{O}}=0\}$. It is known from Weyl's law (see, e.g. [13]) that

$$\lambda_j\simeq j^{\frac{2}{d}}, \quad \|h_j\|_{L^\infty(0,1)}\le C\lambda_j^{\frac{d-1}{2}}\le Cj^{\frac{d-1}{d}}. \tag{4.3}$$

Define the operators $F_f:H\to\dot{H}^{-1},F_{\widetilde{f}}:H\to H$ and $G:H\to\mathcal{L}(H)$ by Nemytskii operators associated with $\nabla f,\widetilde{f}$ and g, respectively

$$F_f(z)(\xi):=\nabla f\big(z(\xi)\big), \quad F_{\widetilde{f}}(z)(\xi):=\widetilde{f}\big(z(\xi)\big), \quad G(z)u(\xi):=g\big(z(\xi)\big)u(\xi),$$

where $z\in H,\xi\in\mathscr{O}$ and $u\in H$. Denoting $F(z):=F_f(z)+F_{\widetilde{f}}(z)$, we fit Eq. (4.1) into the framework considered in previous two sections with Assumption 2.1 for $\theta\in(0,1)$.

Indeed, fix $x,y\in H$, $z\in\dot{H}^\theta$, and $t>0$, by the Lipschitz continuity (and thus linear growth) of f and $\widetilde{f}$, we have

$$\begin{aligned}&\big\|S(t)\big(F(x)-F(y)\big)\big\|\\ &\le\big\|(-A)^{\frac12}S(t)(-A)^{-\frac12}\big(F_f(x)-F_f(y)\big)\big\|+\frac12\big\|S(t)\big(F_{\widetilde{f}}(x)-F_{\widetilde{f}}(y)\big)\big\|\\ &\le\big\|(-A)^{\frac12}S(t)\big\|_{\mathcal{L}(H)}\|F_f(x)-F_f(y)\|_{-1}+\|S(t)\|_{\mathcal{L}(H)}\big\|F_{\widetilde{f}}(x)-F_{\widetilde{f}}(y)\big\|\\ &\le\Big(L_f\big\|(-A)^{\frac12}S(t)\big\|_{\mathcal{L}(H)}+L_{\widetilde{f}}\|S(t)\|_{\mathcal{L}(H)}\Big)\|x-y\|\\ &\le C\big(t^{-\frac12}+1\big)\|x-y\|=:K_F(t)\|x-y\|.\end{aligned} \tag{4.4}$$

Similarly, we obtain that for any $z\in H$,

$$\begin{aligned}\|S(t)F(z)\|_\theta &\le \big\|(-A)^{\frac{1+\theta}{2}}S(t)(-A)^{-\frac{1}{2}}F_f(z)\big\|+\big\|(-A)^{\frac{\theta}{2}}S(t)F_{\tilde{f}}(z)\big\| \\ &\le C\big(t^{-\frac{1+\theta}{2}}+t^{-\frac{\theta}{2}}\big)(1+\|z\|)=:K_{F_\theta}(t)(1+\|z\|). \end{aligned} \tag{4.5}$$

Then the functions K_F and K_{F_θ} are integrable on $[0,T]$ for $\theta\in(0,1)$.

We aim to verify the assumptions in Section 3 in the following two types of frequently used noise.

4.1 White noise

In the white noise case, $\mathbf{Q}=\mathrm{Id}_H$ corresponds to the cylindrical Wiener process.

Concerning the diffusion term, Weyl's law (4.3) and the Lipschitz continuity of g yield that

$$\begin{aligned}&\big\|S(t)\big(G(x)-G(y)\big)\big\|^2_{\mathcal{L}_2^0} \\ =&\sum_{j=1}^\infty \big\|S(t)\big(G(x)-G(y)\big)h_j\big\|^2 \\ \le& C\sum_{j=1}^\infty \lambda_j^{\frac{d-1}{2}}e^{-2\lambda_j t}\|x-y\|^2,\end{aligned}$$

and

$$\|S(t)G(z)\|^2_{\mathcal{L}^\theta}=\sum_{j=1}^\infty \lambda_j^{\theta+\frac{d-1}{2}}e^{-2\lambda_j t}(1+\|z\|)^2.$$

Define

$$K_G^2(t):=C\sum_{j=1}^\infty \lambda_j^{\frac{d-1}{2}}e^{-2\lambda_j t},\quad K^2_{G_\theta}(t):=C\sum_{j=1}^\infty \lambda_j^{\theta+\frac{d-1}{2}}e^{-2\lambda_j t}. \tag{4.6}$$

It follows that

$$\begin{aligned}&\int_0^T K_G^2(t)\mathrm{d}t\le C\sum_{j=1}^\infty \lambda_j^{\frac{d-1}{2}}\int_0^T e^{-2\lambda_j t}\mathrm{d}t\le C\sum_{j=1}^\infty \lambda_j^{\frac{d-3}{2}}\le C,\\ &\int_0^T K_{G_\theta}^2(t)\mathrm{d}t\le C\sum_{j=1}^\infty \lambda_j^{\theta+\frac{d-1}{2}}\int_0^T e^{-2\lambda_j t}\mathrm{d}t\le C\sum_{j=1}^\infty \lambda_j^{\theta+\frac{d-3}{2}},\end{aligned}$$

then K_G and K_{G_θ} are square integrable on $[0,T]$ for $d=1$ and any $\theta\in(0,1/2)$. This verifies Assumption 2.1 for Eq. (4.1) with $d=1$. Thus, we focus on the 1D case in this part. By Lemma 2.1, Eq. (4.1) driven by white noise possesses a unique mild solution $X\in\mathcal{C}([0,T];L^p(\Omega;\dot{H}^\theta))$ as soon as $X_0\in L^p(\Omega;\dot{H}^\theta)$ with $p\ge 2$ and $\theta\in(0,1/2)$.

To obtain the optimal convergence rate of the fully discrete scheme (3.1), it remains to give the maximal values of r_0 in (2.5), r_F,r_G in Assumption 2.2, η_F,η_G in Assumption 3.1 and γ in Assumption 3.2.

Lemma 4.1. *Let $p\geq 2$ and $\beta\in(0,1/2]$. Assume that $X_0\in L^p(\Omega;\dot{H}^\beta)$, $f,\widetilde{f},g:\mathbb{R}\to\mathbb{R}$ are Lipschitz continuous functions and $\mathbf{Q}=\mathrm{Id}_H$. Then* (2.5) *and Assumptions* 2.2 *and* 3.1-3.2 *hold for Eq.* (4.1) *with*

$$r_0=\beta,\quad r_F=1-\epsilon,\quad r_G=\frac{1}{2},\quad \eta_F=\frac{1}{2}-\epsilon,\quad \eta_G=\frac{1}{4},\quad \gamma=\frac{\beta}{2}, \tag{4.7}$$

where ϵ can be taken as an arbitrary sufficiently small positive number.

Proof. To show (2.5) and Assumption 2.2, we mainly use the standard estimation (see, e.g. [19, Lemma 5.3])

$$\big\|[S_N(t)\mathcal{P}_N-S(t)]z\big\|\leq \lambda_N^{-\frac{\mu}{2}}\|z\|_\mu,\quad \forall z\in\dot{H}^\mu,\quad \mu\geq 0. \tag{4.8}$$

For $X_0\in L^p(\Omega;\dot{H}^\beta)$, we have

$$\sup_{t\in[0,T]}\big\|[S_N(t)\mathcal{P}_N-S(t)]X_0\big\|_{L^p(\Omega;H)}\leq CN^{-\beta}\|X_0\|_{L^p(\Omega;\dot{H}^\beta)},$$

i.e. $r_0=\beta$ in (2.5) (with $\theta=\beta$).

The estimates (4.5) and (4.8) imply that for any $\tilde{\theta}\in(0,1)$,

$$\begin{aligned}&\big\|[S_N(t)\mathcal{P}_N-S(t)]F(z)\big\|\\ =&\big\|(\mathcal{P}_N-\mathrm{Id}_H)S(t)F(z)\big\|\\ \leq& C\lambda_{N+1}^{-\frac{\tilde{\theta}}{2}}\|S(t)F(z)\|_{\tilde{\theta}}\\ \leq& CN^{-\tilde{\theta}}\big(t^{-\frac{1+\tilde{\theta}}{2}}+t^{-\frac{\tilde{\theta}}{2}}\big)(1+\|z\|).\end{aligned}$$

Take $R_{F_\theta^N}(t)=CN^{-\tilde{\theta}}(t^{-(1+\tilde{\theta})/2}+t^{-\tilde{\theta}/2})$ and we have

$$\int_0^T R_{F_\theta^N}(t)\mathrm{d}t\leq CN^{-\tilde{\theta}}.$$

This shows one can take r_F in Assumption 2.2 as any positive number less than 1. Analogously, for the diffusion term, we have

$$\big\|[S_N(t)\mathcal{P}_N-S(t)]G(z)\big\|_{\mathcal{L}_2^0}^2=\sum_{j=N+1}^\infty e^{-2\lambda_j t}\|G(z)e_j\|^2\leq C\sum_{j=N+1}^\infty e^{-2\lambda_j t}(1+\|z\|)^2.$$

Take $R^2_{G_\theta^N}(t)=C\sum_{j=N+1}^\infty e^{-2\lambda_j t}$ and we have

$$\int_0^T R^2_{G_\theta^N}(t)\mathrm{d}t=C\int_0^T\sum_{j=N+1}^\infty e^{-2\lambda_j t}\mathrm{d}t\leq C\sum_{j=N+1}^\infty j^{-2}\leq CN^{-1},$$

i.e. one can take $r_G=1/2$ in Assumption 2.2. In particular, Assumption 2.2 holds with $r_F=1-\epsilon$ for any $\epsilon\in(0,1]$ and $r_G=1/2$.

For Assumption 3.1, let $t_i < t \le t_{i+1}$ for some $i \in \mathbb{Z}_{M-1}$. It follows from (2.1) and (4.5) that

$$\begin{aligned}&\big\|[S_N(t)-S_N(t_{i+1})]\mathcal{P}_N F(z)\big\|\\=&\big\|\big(\mathrm{Id}_H - S(t_{i+1}-t)\big)\mathcal{P}_N S(t)F(z)\big\|\\\le& C\tau^{\frac{\tilde{\theta}}{2}}K_{F_{\tilde{\theta}}}(t)(1+\|z\|)\end{aligned}$$

for any $\tilde{\theta}\in(0,1)$ and thus $\eta_F=\tilde{\theta}/2$. Similarly,

$$\begin{aligned}&\big\|[S_N(t)-S_N(t_{i+1})]\mathcal{P}_N G(z)\big\|^2_{\mathcal{L}_2^0}\\=&\sum_{j=1}^N\big(e^{-\lambda_j t}-e^{-\lambda_j t_{i+1}}\big)^2\|G(z)e_l\|^2\\\le& C\sum_{j=1}^\infty\big(e^{-\lambda_j t}-e^{-\lambda_j t_{i+1}}\big)^2(1+\|z\|)^2.\end{aligned}$$

Since

$$\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}\sum_{j=1}^\infty\big(e^{-\lambda_j t}-e^{-\lambda_j t_{i+1}}\big)^2\mathrm{d}t\le C\tau^{\frac{1}{2}},$$

we have $\eta_G=1/4$. Thus, Assumption 3.1 holds with $\eta_F=1/2-\epsilon$ for any $\epsilon\in(0,1/2]$ and $\eta_G=1/4$.

Finally, we prove the uniform $\gamma=\beta/2$-Hölder regularity of $\{X^N\}$ in $L^p(\Omega;H)$ as required in Assumption 3.2. In this case, K_F and K_G are given by (4.4) and (4.6), respectively. Together with representations (4.5) and (4.6) for K_{F_θ} and K_{G_θ}, respectively, the arguments in the proof of Lemma 3.1 are available for $\gamma_1=1/2$ and $\gamma_2=1/4$ in condition (3.3). As a consequence, it suffices to refine the term I_5 there.

By the Lipschitz continuity of g, we obtain

$$\begin{aligned}I_5^2\le& C\int_0^s\big\|\big(S(t-r)-S(s-r)\big)G\big(X^N(r)\big)\big\|^2_{L^p(\Omega;\mathcal{L}_2^0)}\mathrm{d}r\\\le& C\int_0^s\sum_{j=1}^\infty\big(e^{-\lambda_j(t-r)}-e^{-\lambda_j(s-r)}\big)^2\big\|g\big(X^N(r)\big)e_j\big\|^2_{L^p(\Omega;H)}\mathrm{d}r\\\le& C\Big(1+\sup_{t\in[0,T]}\big\|X^N(t)\big\|_{L^p(\Omega;H)}\Big)^2\int_0^s\sum_{j=1}^\infty\big(e^{-\lambda_j(t-r)}-e^{-\lambda_j(s-r)}\big)^2\mathrm{d}r\\\le& C\Big(1+\|X_0\|_{L^p(\Omega;H)}\Big)(t-s)^{\frac{1}{4}}.\end{aligned}$$

Thus, we conclude Assumption 3.2 with $\gamma=\beta/2(\le 1/4)$. □

Remark 4.1. If we use the approach in Lemma 3.1 to deal with the term I_5, we can only derive the uniform $(1/4-\epsilon)\wedge(\beta/2)$-Hölder continuity of X^N in $L^p(\Omega;H)$ with an infinitesimal factor ϵ when $X_0\in L^p(\Omega;\dot{H}^\beta)$ with $\beta\in(0,1/2]$. The estimate in Lemma 4.1 is more accurate using the eigenmodes and accurate integration so that we can obtain the optimal strong convergence orders in the white noise case.

Combining Lemma 4.1 and Theorem 3.1, we derive our main result on a sharp, strong convergence rate for the scheme (3.1) applied to Eq. (4.1) driven by white noise.

Theorem 4.1. *Let $p\geq 2,\beta\in(0,1/2],X_0\in L^p(\Omega;\dot{H}^\beta)$, $f,\widetilde{f},g:\mathbb{R}\to\mathbb{R}$ are Lipschitz continuous functions and $\mathbf{Q}=\mathrm{Id}_H$ in Eq.* (4.1)*. There exists a constant C such that*

$$\max_{k\in\mathbb{Z}_M}\big\|X(t_k)-X_k^N\big\|_{L^p(\Omega;H)}\leq C\Big(1+\|X_0\|_{L^p(\Omega;\dot{H}^\beta)}\Big)\big(N^{-\beta}+\tau^{\frac{\beta}{2}}\big).$$

Remark 4.2. (1) Compared with the optimal spatial regularity for the solution of Eq. (4.1), the spectral Galerkin approximation is superconvergent. In the case $\beta=1/2$, our strong convergence order is $1/2$. Moreover, the convergence rate in the temporal direction is consistent with the optimal Hölder regularity under the $L^p(\Omega;H)$-norm in Lemma 4.1 and thus sharp.

(2) To our knowledge, Theorem 4.1 is the first optimal strong convergence rate of numerical approximations applied to Eq. (4.1) with gradient term driven by white noise.

4.2 Equation driven by general Q-Wiener process

As remarked in the previous subsection, Eq. (4.1) driven by white noise possesses a mild solution if and only if $d=1$. In this part, we assume that the diffusion operator G and the covariance operator $\mathbf{Q}$ of the driven Wiener process are as general as possible such that Assumption 2.1 holds for some $\theta>0$. In this case, one can handle both 1D SPDEs driven by noises rougher than white noise and higher-dimensional SPDEs driven by colored noises, particularly trace-class noises.

Examples such as Assumption 2.1 hold in the case of trace-class noise are given in Remark 4.4. Under this assumption, the inequality (2.5) holds with $r_0=\theta/d$ by Weyl's law (4.3) and a similar argument in the previous subsection. For the drift and diffusion terms, we have

$$\begin{aligned}
&\big\|[S_N(t)\mathcal{P}_N-S(t)]F(z)\big\|\leq\lambda_{N+1}^{-\frac{\theta}{2}}K_{F_\theta}(t)(1+\|z\|_\theta)=:R_{F_\theta^N}(t)(1+\|z\|_\theta),\\
&\big\|[S_N(t)\mathcal{P}_N-S(t)]G(z)\big\|_{\mathcal{L}_2^0}\leq\lambda_{N+1}^{-\frac{\theta}{2}}K_{G_\theta}(t)(1+\|z\|_\theta)=:R_{G_\theta^N}(t)(1+\|z\|_\theta),
\end{aligned}$$

which show $r_F=r_G=\theta/d$ in Assumption 2.2.

To obtain the rates η_F and η_G in Assumption 3.1, let us fix $z\in\dot{H}^\theta$ and $t\in(t_i,t_{i+1}]$ for some $i\in\mathbb{Z}_{M-1}$. It follows from (2.1) that

$$\begin{aligned}
&\big\|[S_N(t)-S_N(t_{i+1})]\mathcal{P}_NF(z)\big\|\\
\leq&\big\|(-A)^{-\frac{\theta}{2}}\big(\mathrm{Id}_H-S(t_{i+1}-t)\big)(-A)^{\frac{\theta}{2}}S(t)F(z)\big\|\\
\leq& C\tau^{\frac{\theta}{2}\wedge 1}K_{F_\theta}(t)(1+\|z\|_\theta):=R_{F_\theta^N,\tau}(t)(1+\|z\|_\theta),
\end{aligned}$$

$$
\begin{aligned}
&\big\|[S_N(t)-S_N(t_{i+1})]\mathcal{P}_N G(z)\big\|_{\mathcal{L}_2^0}\\
\le&\big\|(-A)^{-\frac{\theta}{2}}\big(\mathrm{Id}_H-S(t_{i+1}-t)\big)(-A)^{\frac{\theta}{2}}S(t)G(z)\big\|_{\mathcal{L}_2^0}\\
\le& C\tau^{\frac{\theta}{2}\wedge 1}K_{G_\theta}(t)(1+\|z\|_\theta):=R_{G_\theta^N,\tau}(t)(1+\|z\|_\theta).
\end{aligned}
$$

Therefore,

$$
\sup_N\int_0^T R_{F_\theta^N,\tau}(t)\mathrm{d}t\le C\tau^{\frac{\theta}{2}\wedge 1},\quad \sup_N\int_0^T R^2_{G_\theta^N,\tau}(t)\mathrm{d}t\le C\tau^{\theta\wedge 2},
$$

which shows $\eta_F=\eta_G=(\theta/2)\wedge 1$.

Applying Theorem 3.1, we derive our main result on the strong convergence rate for the scheme (3.1) of Eq. (4.1) driven by a general **Q**-Wiener process.

Theorem 4.2. *Let $p\ge 2,\theta\in(0,2),0<\gamma\le 1/2,X_0\in L^p(\Omega;\dot H^\theta)$, and Assumptions* 2.1 *and* 3.2 *hold. Then, there exists a constant C such that*

$$
\max_{k\in\mathbb{Z}_M}\|X(t_k)-X_k^N\|_{L^p(\Omega;H)}\le C\Big(1+\|X_0\|_{L^p(\Omega;\dot H^\theta)}\Big)\big(N^{-\frac{\theta}{d}}+\tau^{\frac{\theta}{2}\wedge\gamma}\big).
$$

Remark 4.3. An example of SPDE driven by a noise rougher than white noise is given by (5.3) with $\nu\in(0,1/2)$ in Section 5. It is clear that Assumptions 2.1 and 3.2 hold by using the previous approach.

Remark 4.4. Several examples in the case of trace-class noise are given in [18, Section 4], under the following classical conditions on F and G:

$$
\begin{aligned}
&\|F(x)-F(y)\|\le C\|x-y\|, &&\|G(x)-G(y)\|_{\mathcal{L}_2^0}\le C\|x-y\|,\\
&\|F(z)\|_\nu\le C(1+\|z\|_\nu), &&\|G(z)\|_{\mathcal{L}_2^\nu}\le C(1+\|z\|_\nu)
\end{aligned}
\tag{4.9}
$$

for some $\nu>0$. In this case, Assumption 2.1 is satisfied with $\theta=\nu+1-\epsilon$ for any $\epsilon\in(0,1]$. Indeed, the uniform boundedness (2.1) of the semigroup leads to

$$
\begin{aligned}
&\|S(t)F(z)\|_{\nu+1-\epsilon}\le Ct^{-\frac{1-\epsilon}{2}}\|F(z)\|_\nu\le Ct^{-\frac{1-\epsilon}{2}}(1+\|z\|_{\nu+1-\epsilon}),\\
&\|S(t)G(z)\|_{\mathcal{L}_2^{\nu+1-\epsilon}}\le Ct^{-\frac{1-\epsilon}{2}}\|G(z)\|_{\mathcal{L}_2^\nu}\le Ct^{-\frac{1-\epsilon}{2}}(1+\|z\|_{\nu+1-\epsilon})
\end{aligned}
$$

for any $z\in H^{\nu+1-\epsilon}$. By Lemma 3.1 with $\gamma_1=1,\gamma_2=1/2$, we obtain that scheme (3.1) applied to Eq. (4.1) with initial datum $X_0\in\dot H^\theta$, whose drift and diffusion operators satisfy (4.9), possesses the strong convergence rate $\mathcal{O}(N^{-\theta/d}+\tau^{(\theta\wedge 1)/2})$. In addition, notice that there is a technique assumption $\nu\in(0,\theta/10)$ in [24, Theorem 2.8] for similar results.

Remark 4.5. The main result, Theorem 3.1, can also be applied to other SPDEs, such as fourth-order parabolic SPDEs, with Lipschitz continuous drift/diffusion functions.

5 Numerical experiments

In this section, we give several numerical tests to verify our main result, Theorem 3.1, on the convergence rate of the fully discrete scheme (3.1). More precisely, we apply scheme (3.1) to the following one-dimensional stochastic advection-diffusion-reaction equation (where we take $f(v)=-v, \widetilde{f}(v)=-v/(1+|v|)$ and $g(v)=(1+v)/8$ for $v\in\mathbb{R}$):

$$\mathrm{d}X(t,\xi)=\left(\Delta X(t,\xi)-\nabla X(t,\xi)-\frac{X(t,\xi)}{1+|X(t,\xi)|}\right)\mathrm{d}t+\frac{1+X(t,\xi)}{8}\mathrm{d}W(t,\xi) \tag{5.1}$$

with homogeneous Dirichlet boundary condition, in the time-space domain $(0,T]\times\mathscr{O}=(0,1]\times(0,1)$. We refer to [6] and references therein for relevant applications.

To illustrate the effect of the regularity of the initial datum X_0 and covariance operator $\mathbf{Q}$ on the strong convergence rate of the scheme (3.1), we take the initial datum and the $\mathbf{Q}$-Wiener process as

$$X(0,\xi)=X_0(\xi)=\sum_{j=1}^{\infty}\frac{e_j(\xi)}{j^{\theta+1/2}},\quad \xi\in(0,1), \tag{5.2}$$

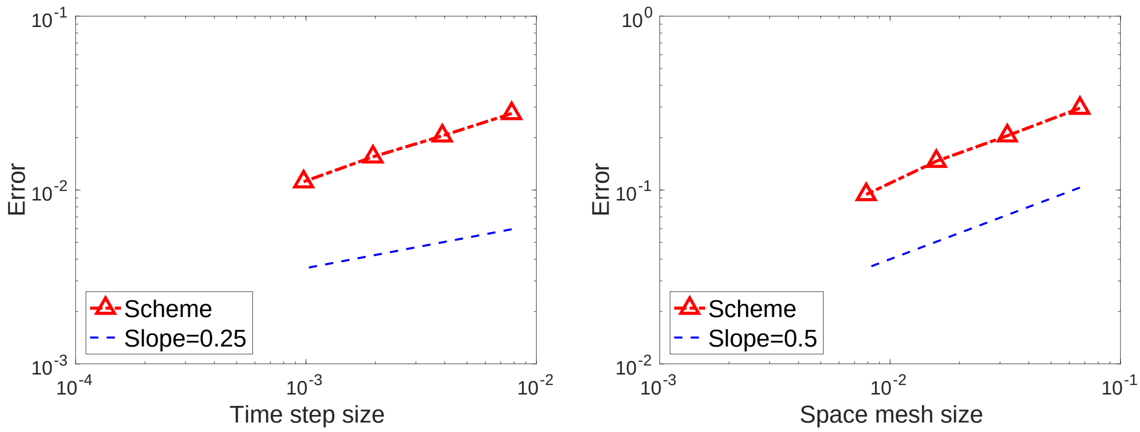
$$W(t,\xi)=\sum_{j=1}^{\infty}\frac{e_j(\xi)}{j^{\nu-1/2}}h_j(t),\qquad (t,\xi)\in(0,1]\times(0,1) \tag{5.3}$$

for some $\theta>0$ and $\nu>0$. Here $\{\theta_j\}_{j\in\mathbb{N}_+}$ is a sequence of independent standard $\mathscr{F}_t$-Brownian motions and $\{h_j(\cdot)=\sqrt{2}\sin(j\pi\cdot)\}_{j\in\mathbb{N}_+}$ is a sequence of eigenfunctions of the Laplacian operator Δ which forms an orthonormal basis of $L^2(0,1)$. Then the solution of Eq. (5.1) with initial datum (5.2) and driven by the $\mathbf{Q}$-Wiener process (5.3) possess a unique mild solution in $\mathcal{C}([0,T];L^p(\Omega;\dot{H}^\theta))$ for any $p\geq 2$ and $\theta<\theta\wedge\nu$. In our numerical tests, we always take $\nu=\theta$ for simplicity.

When θ=0.5, the proposed $\mathbf{Q}$-Wiener process is the $L^2(0,1)$-valued cylindrical Wiener process, corresponding to white noise. By Theorem 4.1, the spatial and temporal strong orders are 0.5 and 0.25, respectively. These mean-square convergence rates (i.e. $p=2$) are shown numerically in Fig. 1.

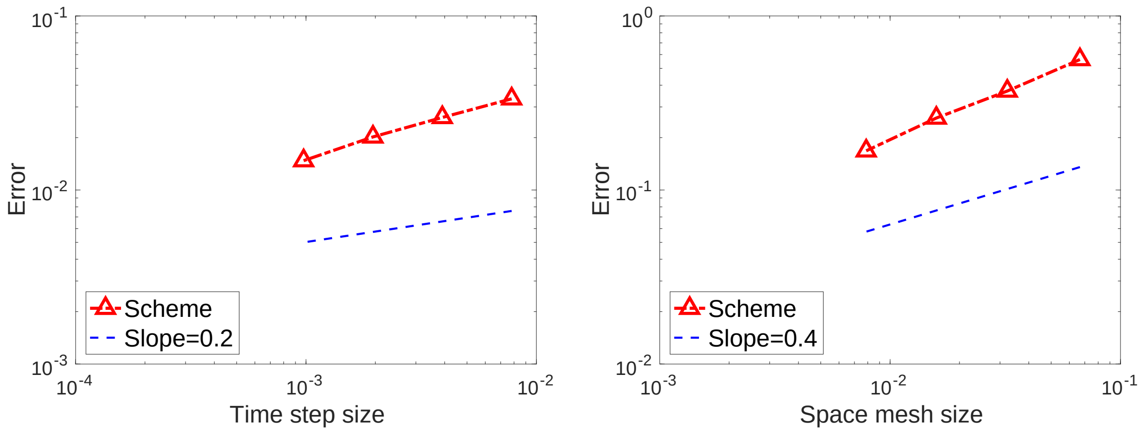
In the following three tests, we take three $\mathbf{Q}$-Wiener processes, with the first one weaker than the $L^2(0,1)$-valued cylindrical Wiener process and the last two smoother than the $L^2(0,1)$-valued cylindrical Wiener process. They correspond to (5.3) with $\theta=0.4,1$ and 1.6, respectively. By Theorem 4.2 and Remark 4.3, the proposed scheme (3.1) possesses the sharp strong convergence rate $\mathcal{O}(N^{-0.4}+\tau^{0.2})$, $\mathcal{O}(N^{-1}+\tau^{0.5})$ and $\mathcal{O}(N^{-1.6}+\tau^{0.5})$, respectively. The numerical results for these three cases are presented in Figs. 2-4.

To simulate the "exact" solution, we perform the fully discrete scheme by $N=2^9$ for the dimension of spectral Galerkin approximation and $\tau=2^{-13}$ for the time step size of the exponential integrator, where the solution is obtained from the fast Fourier transform



(a) Temporal error (b) Spatial error

Figure 1: Temporal (left) and spatial (right) convergence rates with $\theta=0.5$.



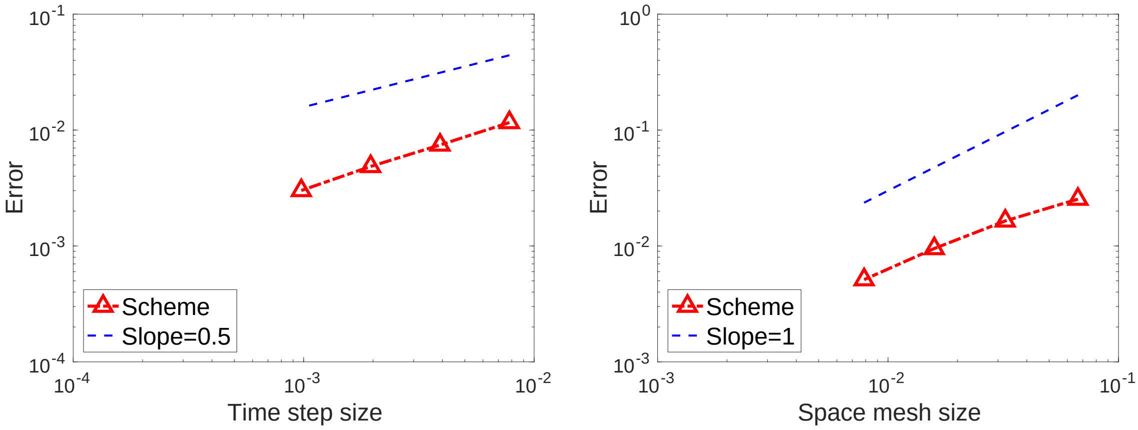
(a) Temporal error (b) Spatial error

Figure 2: Temporal (left) and spatial (right) convergence rates with $\theta=0.4$.

algorithm with space mesh size $h=2^{-9}$. The expectation is approximated from the average of 200 sample paths, and the series of (5.3) is truncated as a finite summation up to 2^9-1 terms.

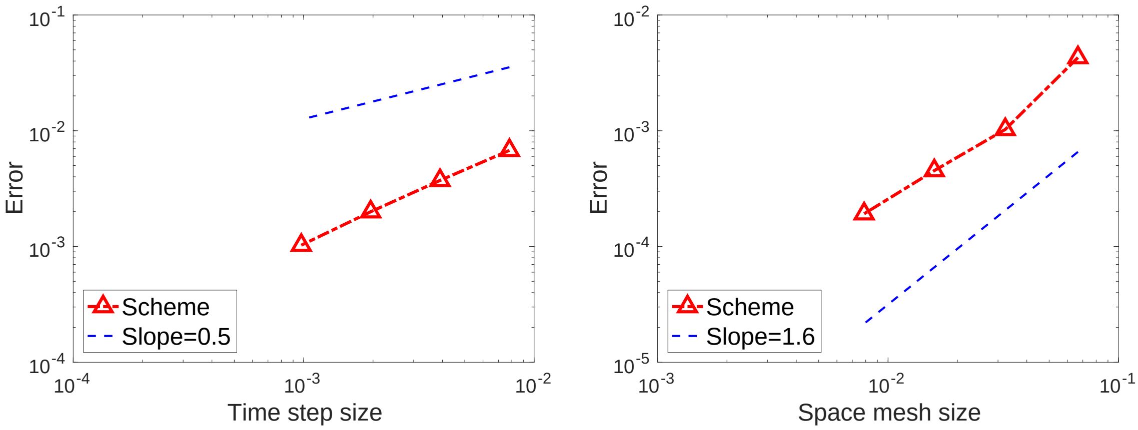
Taking $\tau=2^i$ with $i=7,8,9,10$ and $N=2^9-1$, we have the mean-square convergence rates in the temporal direction shown in the left pictures of Figs. 1-4. Choosing four different spatial dimensions $N=2^i-1$ with $i=4,5,6,7$ and fixing $\tau=2^{-13}$, we get the convergence rates in the spatial direction, which are presented in the right pictures of Figs. 1-4.

From these numerical experiments, it is clear that the spatial and temporal strong convergence orders of the scheme (3.1) applied to Eq. (5.1) are θ and $(1\wedge\theta)/2$, respectively. This confirms the theoretical result of Theorem 3.1.



(a) Temporal error (b) Spatial error

Figure 3: Temporal (left) and spatial (right) convergence rates with $\theta=1$.



(a) Temporal error (b) Spatial error

Figure 4: Temporal (left) and spatial (right) convergence rates with $\theta=1.6$.

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