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STABILITY OF TWO TIME-INTEGRATORS FOR THE ALIEV-PANFILOV SYSTEM

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Abstract. We propose a second-order accurate method for computing the solutions to the Aliev-Panfilov model of cardiac excitation. This two-variable reaction-diffusion system is due to its simplicity a popular choice for modeling important problems in electrocardiology; e.g. cardiac arrhythmias. The solutions might be very complicated in structure, and hence highly resolved numerical simulations are called for to capture the fine details. Usually the forward Euler time-integrator is applied in these computations; it is very simple to implement and can be effective for coarse grids. For fine-scale simulations, however, the forward Euler method suffers from a severe time-step restriction, rendering it less efficient for simulations where high resolution and accuracy are important.

We analyze the stability of the proposed second-order method and the forward Euler scheme when applied to the Aliev-Panfilov model. Compared to the Euler method the suggested scheme has a much weaker time-step restriction, and promises to be more efficient for computations on finer meshes.

Key Words. reaction-diffusion system, implict Runge-Kutta, electrocardiology

1. Introduction

Pulse propagation in cardiac tissue can adequately be simulated by the use of modern ionic models with diffusive coupling between myocytes. Today's detailed ionic models, however, consist of dozens of ODEs that represent a great numerical challenge to solve at every mesh point for large spatial domains. Such large spatial regions are relevant for the study of for example re-entrant cardiac arrhythmias. If in addition a high spatial resolution is required, it may not be feasible to solve these models on present day computers. The Aliev-Panfilov model [1] was constructed to ameliorate this problem and capture the qualitative behavior of the cardiac tissue in a mathematically and computationally tractable model. It builds upon the FitzHugh-Nagumo model [8, 15] and retains its simplicity while more accurately describing the pulse propagation in collections of heart cells. The Aliev-Panfilov model has been applied in many computationally demanding problems; e.g. spiral wave breakup in coupled cells [17, 23], scroll waves in excitable medium [20].

The bidomain and monodomain models [10, 11, 22] are commonly used to describe the electrical activity in the heart at tissue level. Mathematically, these models are partial differential equations of reaction-diffusion type. Two electrical potentials, the transmembrane and the extracellular, are accounted for in the

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bidomain model. A simplifying assumption reduces the bidomain description to the monodomain model, which only models the transmembrane potential. We will consider the monodomain model description of cardiac tissue with Aliev-Panfilov cell dynamics: On the space-time domain $\Omega_T := \Omega \times (0, T]$ the Aliev-Panfilov model reads

$$\frac{\partial e}{\partial t} = \delta \nabla^2 e - ke(e-a)(e-1) - er, \qquad \text{on } \Omega_T, \tag{1}$$

$$\frac{\partial r}{\partial t} = -\left[\varepsilon + \frac{\mu_1 r}{\mu_2 + e}\right][r + ke(e-b-1)], \qquad \text{on } \Omega_T, \tag{2}$$

and

$$\vec{n} \cdot \delta \nabla e = 0 \text{ on } \partial\Omega, \quad \text{and} \quad (e, r)_{t=0} = (e(0, \cdot), r(0, \cdot)), \tag{3}$$

where $\vec{n}$ is the outer normal vector of the boundary $\partial\Omega$. Here e represents the scaled transmembrane potential, r is the variable responsible for recovery of the tissue and Ω is a two-dimensional domain in the present paper. For the stability analysis of the time-integrators in this paper we will assume that the Aliev-Panfilov parameters μ_1, μ_2, k, ε, b, a and δ are positive. Numerical experiments will be performed in order to investigate the sharpness of the obtained time step restrictions. For these computations we will fix the parameters to the physiological values $\mu_1 = 0.07$, $\mu_2 = 0.3$, $k = 8$, $\varepsilon = 0.01$, $b = 0.1$, $a = 0.1$ and $\delta = 5 \times 10^{-5}$.

A variety of schemes has been applied in numerical electrophysiology. In [4] a finite volume scheme, with explicit Euler time-stepping, for the monodomain model in connection with Aliev-Panfilov or FitzHugh-Nagumo cell kinetics was proven to be first order convergent. Stability properties of several first and second order accurate time-integrators, and even a third order scheme, for the bidomain model with FitzHugh-Nagumo dynamics were studied in [7]. Implicit Euler was used in e.g. [9], where finite element discretization was employed in space. An adaptive method for the Aliev-Panfilov model was recently presented in [2].

The forward Euler method has, however, emerged as the standard approach to solve the Aliev-Panfilov system in time; see e.g [23, 19, 14, 13, 20]. Without doubt this is due to its big advantage of simplicity. Unfortunately, the method becomes less efficient as the spatial resolution is increased because of its very severe time step restriction. Numerical computations on highly resolved meshes are relevant in many important applications; e.g. in fibrillation where a spiral wave pattern needs to be resolved and we want to capture the fine details. These considerations motivate us to consider an alternative scheme for fine-scale computations.

We will present a second-order method for the system (1)-(3) and compare it to the standard forward Euler scheme in terms of stability. The second-order accurate time integration we consider is the Singly Diagonally Implicit Runge-Kutta (SDIRK) method in [3]. To our knowledge the stability of this method when applied to the Aliev-Panfilov system has not been analyzed previously and no time step restriction has been given. We analyze both the forward Euler scheme and the second-order scheme by giving a maximum principle revealing the time step condition needed to keep the solution within the physiologically relevant bounds. The second-order method hinges on a decomposition of the Aliev-Panfilov model into a PDE and two coupled ODEs by an operator-splitting technique in time. The second-order accurate SDIRK method is applied to integrate the ODE system in time. Compared to the forward Euler scheme, we will show that the second-order method has considerably improved stability properties. Although the proposed scheme is more computationally costly than the forward Euler method for coarse