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DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR QUASI-STATIC POROELASTICITY PROBLEMS

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Abstract. This paper is devoted to a discontinuous Galerkin (DG) method for nonlinear quasi-static poroelasticity problems. The fully implicit nonlinear numerical scheme is constructed by utilizing DG method for the spatial approximation and the backward Euler method for the temporal discretization. The existence and uniqueness of the numerical solution is proved. Then we derive the optimal convergence order estimates in a discrete H^1 norm for the displacement and in H^1 and L^2 norms for the pressure. Finally, numerical experiments are supplied to validate the theoretical error estimates of our proposed method.

Key words. Nonlinear quasi-static poroelasticity problem, discontinuous Galerkin method, fully implicit nonlinear numerical scheme, optimal convergence order estimate.

1. Introduction

Poroelasticity refers to the movement of Darcy flow within a deformable and porous medium. It is widely used in many practical problems, such as materials science [34], biomechanics [30], and reservoir engineering [16]. Poroelasticity theory is also called Biot's consolidation model when the porous media is linear elastic, homogeneous, isotropic and saturated by incompressible Newtonian fluid. The original Biot's model can retrospect to the contribution of Terzaghi and Biot. Terzaghi [33] analyzes the one-dimensional case and then finds the relevant theory based on the consolidation of a soil column. Biot [7] generalizes Terzaghi's theory and research to the three-dimensional situation. Up to now, many complex mathematical models based on the Biot's model have been proposed and studied, including various nonlinear models, where, for example, the permeability is taken as a nonlinear function of the fluid content [18, 19, 17].

In this paper, we are concerned with a nonlinear quasi-static poroelasticity problem. The linear type of this problem (the permeability is a constant) is studied by Showalter [29] on the well-posedness of solutions of porous elastic systems. And various numerical methods have been applied to the linear model, such as finite volume method [23, 24], finite difference method [12], finite element method [22, 21]. The nonlinear model, where the permeability depending nonlinearly on the dilatation of the medium, is firstly introduced in [15] for the simulation of paper production. In [9], Cao et al. establish the variational formulation of the nonlinear model and firstly discuss the existence and uniqueness of the solution by the modified Rothe's method. In [8], Bociu et al. extend the theoretical results of [9] to the case of more general boundary conditions. Compared with the development of numerical methods of the linear model, the existing results mainly make use of finite element based methods for spatial discretization of the nonlinear model. Cao et al. [9] respectively adopt the linear conforming finite element approximation to the displacement and pressure of the nonlinear model, and derive optimal error estimates. Subsequently, they utilize the conforming finite element method [10] and the hybrid

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finite element method [11] for the static case of the nonlinear problem and obtain a priori error estimates. In [39], Zhang et al. investigate a variant formulation of the steady nonlinear problem, use weak Galerkin-finite element method for the spatial discretization of the displacement, pressure and volumetric stress and derive the optimal convergence order estimates.

DG method is one of the important numerical methods for solving partial differential equations. The method is first proposed by Reed and Hill [27] for solving the neutron transport equation. DG method solves the differential equations by piecewise polynomial functions over a finite element space without any requirement on inter-element continuity. Continuity on inter-element boundaries together with boundary conditions is weakly enforced through the bilinear form. DG method has many advantages that make it very attractive for practical numerical simulations, such as good mesh flexibility, local mass conservation, convenience for *hp*-adaptivity. At present, DG method has been widely used to solve various partial differential equations [3, 32, 25, 36, 37, 38, 20, 26]. For the nonlinear poroelasticity model, Wen et al. [35] consider the four-field mixed formulation by introducing two additional variables, construct a linearized fully discrete DG scheme and analyze a priori error esimates.

In this paper, we propose an interior penalty DG method for solving nonlinear quasi-static poroelasticity problems. It is well-known that the fully implicit nonlinear numerical scheme is stable and can preserve the physical properties of the original problem. Based on this, we consider the original two-field model and establish the fully implicit nonlinear DG scheme by the backward Euler method for time discretization, which is different from [35]. The existence and uniqueness of the numerical solution is proved and the optimal error estimates for the displacement and pressure are obtained. Finally, numerical experiments are given to verify the theoretical results of our proposed method.

The outline of this paper goes as follows. In Section 2, we present the nonlinear quasi-static poroelastic model and the corresponding variational formulation. The fully discrete DG scheme is provided in Section 3, and we prove the existence and uniqueness of the numerical solution. In Section 4, the optimal convergence order estimates for the DG scheme are derived. We supply numerical experiments to validate our theoretical findings in Section 5. And finally, some conclusions are made in Section 6.

2. Mathematical model and variational formulation

In this section, we firstly present the mathematical model of nonlinear quasi-static poroelasticity problems. Then we establish the corresponding variational formulation after introducing some necessary notations and definitions.

Let Ω be a convex polygonal or polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary $\partial\Omega = \Gamma_{p,D} \cup \Gamma_{p,N}$ with $\Gamma_{p,D}$ nonempty, and $T > 0$ is the final time. In this paper, we concentrate on the following nonlinear quasi-static poroelasticity problems [9]: Seeking the displacement of porous solid media $\boldsymbol{u}(t) : \Omega \to \mathbb{R}^d$ and the pore pressure of fluid $p(t) : \Omega \to \mathbb{R}$, such that

$$-(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) - \mu\triangle\boldsymbol{u} + \alpha\nabla p = \boldsymbol{f}, \quad \text{in } \Omega,\ t \in (0, T], \tag{1}$$

$$\frac{\partial}{\partial t}(c_0 p + \alpha\nabla \cdot \boldsymbol{u}) - \nabla \cdot (\kappa(\nabla \cdot \boldsymbol{u})\nabla p) = g, \quad \text{in } \Omega,\ t \in (0, T], \tag{2}$$