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The Shock Formation for 2D Isentropic Compressible Euler Equations with Damping

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Abstract. In this paper, we investigate the blowup mechanism for the 2D isentropic compressible Euler equations with a damping term. Generally, damped Euler equations have a global classical solution for small smooth initial data. However, we identify a class of large initial data for the damped Euler equations, under the condition of azimuthal symmetry and by choosing a suitable self-similar transformation, resulting in the solution to the Cauchy problem blowing up in a finite time. Additionally, we introduce modulation variables to accurately describe the blowup time and location. Furthermore, the blowup profile exhibits a cusp singularity with Hölder $C^{1/3}$ regularity at the blowup point.

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Key words: Compressible Euler equations, damping, modulation variant, bootstrap estimation, shock formation.

1. Introduction

1.1. Model problem

In this paper, we study the shock formation for 2D isentropic compressible Euler equations with damping term under azimuthal symmetry. We consider the 2D isentropic compressible Euler equations with the form

$$\partial_t\rho + \partial_1(\rho u_1) + \partial_2(\rho u_2) = 0, \tag{1.1a}$$

$$\partial_t(\rho u_1) + \partial_1\left(\rho u_1^2\right) + \partial_2(\rho u_1 u_2) + \partial_1 p(\rho) = -\rho u_1, \tag{1.1b}$$

$$\partial_t(\rho u_2) + \partial_1(\rho u_1 u_2) + \partial_2\left(\rho u_2^2\right) + \partial_2 p(\rho) = -\rho u_2, \tag{1.1c}$$

where $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ denotes the velocity of the fluid, $\rho : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}_+$ the density, and

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$p:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}_+$ the pressure defined by the ideal gas law

$$p(\rho)=\frac{1}{\gamma}\rho^{\gamma},\quad \gamma>1.$$

We also note that $\sigma(\rho)=\sqrt{\partial p/\partial\rho}=\rho^{\alpha}$ and $\alpha=(\gamma-1)/2$ is the speed of sound.

In order to study perturbations of purely azimuthal waves, we need transform Euler system into polar coordinates. Setting

$$e_r=\frac{x}{r},\quad e_\theta=\frac{x^\perp}{r},\quad u_r=u\cdot\frac{x}{r}=u\cdot e_r,\quad u_\theta=u\cdot\frac{x^\perp}{r}=u\cdot e_\theta,$$

we have

$$u=u_re_r+u_\theta e_\theta,\quad \nabla=(\partial_1,\partial_2)=e_r\partial_r+\frac{e_\theta}{r}\partial_\theta. \tag{1.2}$$

Using the above transformation, we write the system of compressible Euler equations with damping term in polar coordinates as

$$\begin{aligned}&\left(\partial_t+u_r\partial_r+\frac{1}{r}u_\theta\partial_\theta\right)u_r-\frac{u_\theta^2}{r}+\rho^{\gamma-2}\partial_r\rho+u_r=0,\\&\left(\partial_t+u_r\partial_r+\frac{1}{r}u_\theta\partial_\theta\right)u_\theta+\frac{1}{r}u_\theta u_r+\frac{1}{r}\rho^{\gamma-2}\partial_\theta\rho+u_\theta=0,\\&\left(\partial_t+u_r\partial_r+\frac{1}{r}u_\theta\partial_\theta\right)\rho+\left(\partial_ru_r+\frac{1}{r}\partial_\theta u_\theta+\frac{u_r}{r}\right)\rho=0,\end{aligned} \tag{1.3}$$

where $\theta\in\mathbb{T}=[-\pi,\pi]$, $r>0$, and $t\geqslant t_0$. Similar to [5], we also introduce the following new variables:

$$u_r=ra(\theta,t),\quad u_\theta=rb(\theta,t),\quad \rho=r^{2/(\gamma-1)}P(\theta,t). \tag{1.4}$$

Further, the system (1.3) is changed to

$$\begin{aligned}&(\partial_t+b\partial_\theta)a+a^2-b^2+a+\frac{1}{\alpha}P^{2\alpha}=0,\\&(\partial_t+b\partial_\theta)b+2ab+b+P^{\gamma-2}\partial_\theta P=0,\\&(\partial_t+b\partial_\theta)P+\frac{\gamma}{\alpha}aP+(\partial_\theta b)P=0.\end{aligned} \tag{1.5}$$

Next we analyze the vorticity $\omega=\partial_1u_2-\partial_2u_1=2b-\partial_\theta a$. Firstly we need to derive the equation for vorticity. Taking the derivative with respect to x_2, x_1 at the both sides of Eqs. (1.1b) and (1.1c), then subtracting them yields

$$\partial_t\omega+u\cdot\nabla\omega+(\operatorname{div}u)\omega+\omega=0.$$

Eliminating $\operatorname{div}u$ by the Eq. (1.1a), one has

$$\partial_t\frac{\omega}{\rho}+u\cdot\nabla\frac{\omega}{\rho}+\frac{\omega}{\rho}=0. \tag{1.6}$$

In terms of (1.2) and (1.4), direct calculation yields $\omega/\rho=(2b-\partial_\theta a)/(r^{1/\alpha}P)$. For consequence, we set $\varpi=\omega/P=(2b-\partial_\theta a)/P=r^{1/\alpha}\omega/\rho$, then (1.6) is changed to