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Higher-Order Rogue Waves of the Nonlocal Nonlinear Schrödinger Equation in the Defocusing Regime

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Abstract. The previous studies have shown that the defocusing nonlinear Schrödinger equation (NLSE) has no the modulational instability, and was not found to admit the rogue wave phenomenon so far. In this paper, we address the question of the higher-order rogue wave solutions of the nonlocal $\mathscr{PT}$-symmetric NLSE in the defocusing regime. Based on Darboux transformation and iterations, we derive an explicit solution for the higher-order rogue waves by adopting a variable separation and Taylor expansion technique. The higher-order rogue wave solutions are expressed in separation-of-variables form. Furthermore, in order to understand these solutions better, patterns of the rogue waves for lowest three order are explored clearly and conveniently. The reported results may be useful for the design of experiments for observation of rogue waves in the defocusing nonlinear physical systems.

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Key words: Defocusing nonlocal nonlinear Schrödinger equation, variable separation technique, parity-time symmetric, rogue waves.

1. Introduction

It is well known that integrable nonlinear systems play a pivotal role in mathematical physics. Most of these integrable nonlinear systems are local equations. In other words, the solutions evolution relies only on local solution values. In recent years, integrable nonlocal nonlinear systems have attracted a lot of attention and are extensively studied. This type of equation is Parity-time ($\mathscr{PT}$) symmetric — i.e. it is invariant under complex conjugation and joint transformations. The first such system introduced by Ablowitz and Musslimani — c.f., e.g. Refs. [3–5, 20], has the form

$$\mathrm{i}u_t(x,t)+\frac{1}{2}u_{xx}(x,t)\pm u^2(x,t)\bar{u}(-x,t)=0, \tag{1.1}$$

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where the bar denotes the complex conjugation and the sign ± determines whether the Eq. (1.1) is focused or defocused. It is worth mentioning that $\mathcal{PT}$ symmetric equations play a vital role in optics and other physical fields [26]. Inspired by the above nonlocal $\mathcal{PT}$ symmetric nonlinear equation, new nonlocal nonlinear integrable equations have been proposed and studied over past few years [1, 2, 5, 8, 21, 27, 29, 30, 32, 33, 41, 59].

Rogue waves originally attracted a lot of attention due to mysterious and severely destructive oceanic surface waves [25, 38]. This types of waves are spontaneous large waves that "appear out of nowhere, disappear without a trace" [7]. The first analytical expression of rogue waves was derived for the cubic nonlinear Schrödinger equation by Peregrine [39]. Thereafter, higher-order rogue waves in the local NLSE were found, and their interesting dynamical patterns were also discussed [6, 9, 10, 12, 15–17, 22–24, 35–37, 40, 45, 50]. For instance, Kedziora *et al.* [24] studied circular rogue wave clusters of the local NLSE by adopting Darboux transformation (DT). Soon after Guo *et al.* [22] derived N-th order rogue wave solutions of the local NLSE using a generalized DT. In addition, Ohta and Yang [37] investigated N-th order rogue wave solutions of the local NLSE by using the Hirota bilinear method. A variable separation technique presented by Mu and Qin [12] is used to construct N-th order explicit rogue wave solutions of the local NLSE. In particular, Mu *et al.* [35] extend above own work to study general higher-order rogue waves of a vector NLSE using a DT with an asymptotic expansion method. Nowadays, rogue waves have been rapidly overspread to many research fields encompassing oceanography [18], nonlinear optics [43], Bose-Einstein condensation [14], superfluid helium [19], plasmas [34] and even finance [52, 53], quantum droplet [28], etc. As an unexplored and interesting subject, rogue waves in nonlocal integrable systems have received much attention recently [41, 42, 46–48, 55, 56].

Motivated by the works of Baronio *et al.* [12] and Mu *et al.* [35], we next attempt to study rogue wave solutions in several reverse-time integrable nonlocal nonlinear equations using the variable separation technique. As a typical example, we consider the scalar reverse-time nonlocal NLSE

$$iq_t(x,t) + \frac{1}{2}q_{xx}(x,t) + \sigma q^2(x,t)\widetilde{q}(x,t) = 0, \quad \sigma = \pm 1, \tag{1.2}$$

cf. [5, 44, 51, 54, 57, 58]. Here we define $\widetilde{q}(x,t) = q(x,-t)$. If $\sigma = 1$, the Eq. (1.2) describes the focusing regime while the case $\sigma = -1$ is related to the defocusing regime. This nonlocal equation which also called $\mathcal{PT}$-symmetric NLSE is a special reduction from the famous AKNS system. Due to this $\mathcal{PT}$-symmetry, it is related to a cutting research area of contemporary physics [13, 26]. Therefore, this system have attracted a lot of attention in optics and other physical fields in recent years. Bounded multi-soliton solutions and their asymptotic analysis for the Eq. (1.2) had been explored [44]. Yang [57] had derived general multi-soliton solutions in the Eq. (1.2). In addition, there are also many other researchers who have made their own contributions to the study of reverse-time nonlocal NLSE. For instance, Ye and Zhang [58] had constructed the general soliton solutions with zero and non-zero background to a reverse-time nonlocal NLSE via binary DT. The Eq. (1.2) was generalized by Ma [31] to a multi-component one and construct its multi-soliton solutions. In this work, we mainly consider system (1.2) with the defocusing case.