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Error Analysis of hp Spectral Element Approximation for Optimal Control Problems with Control Constraint

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Abstract. In this paper, an hp spectral element approximation for distributed optimal control problem governed by an elliptic equation is investigated, whose objective functional does not include the control variable. And the constraint set on control variable is stated with L^2-norm. Optimality condition of the continuous and discretized systems are deduced. In order to solve the equivalent systems with high accuracy, hp spectral element method is employed to discretize the constrained optimal control systems. Based on the property of some interpolation operators, a posteriori error estimates are also established by using some properties of some interpolation operators carefully. Finally, a projection gradient algorithm and a numerical example are provided, which confirm our analytical results. Such estimators guarantee the construction of reliable adaptive methods for optimal control problems.

AMS subject classifications: 49J20, 65K10, 65N35
Key words: Optimal control problem, control constraint, hp spectral element method, a posteriori error estimate.

1. Introduction

Optimal control problems with partial differential equations (PDEs) constraints have a vast amount of applications and thus have been investigated intensively over the last decades [16,23,34]. In recent years, there is great interest in the extensive theoretical and numerical studies of various optimal control problems. Most of them are solved by finite element methods — cf. [3,6,18,19,24–28,39–42] and references therein.

Nowadays, both spectral and finite element methods are widely applied to optimal control problems and there is a vast literature on this topic. It is well known that spectral

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methods are important numerical tools in solving partial differential equations [2, 32, 33]. They provide a highly accurate simulation if the solutions are smooth enough. Recently, spectral methods for optimal control problems have been successfully investigated [8–10, 12, 17, 21, 22, 43]. Chen *et al.* [10, 12] considered Galerkin spectral approximation for optimal control problems with integral control constraint and proved a priori and a posteriori error estimates rigorously. Lin *et al.* [21] studied Galerkin spectral approximations for elliptic optimal control problem with L^2-norm state constraint and presented a priori and a posteriori error estimates.

Let us briefly review related literature. There are a number of works on optimal control problems where the cost functional does not contain the control variable [5, 13, 14, 31, 36]. In [36], the authors investigated the Tikhonov regularization of control-constrained optimal control problems and developed a parameter choice rule that adaptively selects the Tikhonov regularization parameter depending on a posteriori computable quantities. The solution stability for control problems with the cost functional not involving the usual quadratic term for the control and the sufficient optimality conditions for optimal control problem have been presented in [31]. Sufficient second order optimality conditions for bang-bang control, which guarantee local quadratic growth of the objective functional in L^1 are derived in [7]. Casas *et al.* [4] proved the second order optimality conditions for local strong minimization and presented error estimates for a semilinear parabolic optimal control problem with the box constraint on the control. The cost functional there does not employ a control variable. In [37], simultaneous regularization and discretization of optimal control problem with pointwise control constraint are investigated and the error estimates are derived. Inspired by the works mentioned, we analyze the errors of *hp* spectral element approximations.

To the best of our knowledge, the *hp* method is an important approach for solving partial differential equations [1, 30]. For optimal control problem the method was developed in [10, 15, 29, 38]. Chen and Lin [11] presented *hp* a posteriori error estimators for elliptic optimal control problem with integral control constraint. They can be used for constructing reliable adaptive finite elements methods for optimal control problems. Gong *et al.* [15] investigated residual-based a posteriori error estimates of *hp* finite element method for a distributed convex optimal control problem governed by elliptic PDE with pointwise control constraint.

Note that the spectral accuracy cannot be obtained when the solutions of optimal control problem have a low regularity. However, *hp* spectral element method is a combination of finite element and spectral methods. It can achieve higher order accuracy and geometric flexibility. Therefore, more researchers are attracted by *hp* spectral element methods. In particular, an *hp* spectral element method for elliptic optimal control problem with integral control constraint is discussed in [12]. In [20], *hp* spectral element discretization for elliptic optimal control problem with state constraint is considered and a posteriori error estimates for the optimal control problem are established.

The novelty of this paper is to adopt *hp* spectral element approximation for elliptic optimal control problem and to establish error estimates. Combining the advantages of finite element and spectral methods, we show that the *hp* spectral element method can