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Error Analysis of *hp* Spectral Element Approximation for Optimal Control Problems with Control Constraint

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Abstract. In this paper, an *hp* spectral element approximation for distributed optimal control problem governed by an elliptic equation is investigated, whose objective functional does not include the control variable. And the constraint set on control variable is stated with L^2-norm. Optimality condition of the continuous and discretized systems are deduced. In order to solve the equivalent systems with high accuracy, *hp* spectral element method is employed to discretize the constrained optimal control systems. Based on the property of some interpolation operators, a posteriori error estimates are also established by using some properties of some interpolation operators carefully. Finally, a projection gradient algorithm and a numerical example are provided, which confirm our analytical results. Such estimators guarantee the construction of reliable adaptive methods for optimal control problems.

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Key words: Optimal control problem, control constraint, *hp* spectral element method, a posteriori error estimate.

1. Introduction

Optimal control problems with partial differential equations (PDEs) constraints have a vast amount of applications and thus have been investigated intensively over the last decades [16,23,34]. In recent years, there is great interest in the extensive theoretical and numerical studies of various optimal control problems. Most of them are solved by finite element methods — cf. [3,6,18,19,24–28,39–42] and references therein.

Nowadays, both spectral and finite element methods are widely applied to optimal control problems and there is a vast literature on this topic. It is well known that spectral

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methods are important numerical tools in solving partial differential equations [2, 32, 33]. They provide a highly accurate simulation if the solutions are smooth enough. Recently, spectral methods for optimal control problems have been successfully investigated [8–10, 12, 17, 21, 22, 43]. Chen *et al.* [10, 12] considered Galerkin spectral approximation for optimal control problems with integral control constraint and proved a priori and a posteriori error estimates rigorously. Lin *et al.* [21] studied Galerkin spectral approximations for elliptic optimal control problem with L^2-norm state constraint and presented a priori and a posteriori error estimates.

Let us briefly review related literature. There are a number of works on optimal control problems where the cost functional does not contain the control variable [5, 13, 14, 31, 36]. In [36], the authors investigated the Tikhonov regularization of control-constrained optimal control problems and developed a parameter choice rule that adaptively selects the Tikhonov regularization parameter depending on a posteriori computable quantities. The solution stability for control problems with the cost functional not involving the usual quadratic term for the control and the sufficient optimality conditions for optimal control problem have been presented in [31]. Sufficient second order optimality conditions for bang-bang control, which guarantee local quadratic growth of the objective functional in L^1 are derived in [7]. Casas *et al.* [4] proved the second order optimality conditions for local strong minimization and presented error estimates for a semilinear parabolic optimal control problem with the box constraint on the control. The cost functional there does not employ a control variable. In [37], simultaneous regularization and discretization of optimal control problem with pointwise control constraint are investigated and the error estimates are derived. Inspired by the works mentioned, we analyze the errors of *hp* spectral element approximations.

To the best of our knowledge, the *hp* method is an important approach for solving partial differential equations [1, 30]. For optimal control problem the method was developed in [10, 15, 29, 38]. Chen and Lin [11] presented *hp* a posteriori error estimators for elliptic optimal control problem with integral control constraint. They can be used for constructing reliable adaptive finite elements methods for optimal control problems. Gong *et al.* [15] investigated residual-based a posteriori error estimates of *hp* finite element method for a distributed convex optimal control problem governed by elliptic PDE with pointwise control constraint.

Note that the spectral accuracy cannot be obtained when the solutions of optimal control problem have a low regularity. However, *hp* spectral element method is a combination of finite element and spectral methods. It can achieve higher order accuracy and geometric flexibility. Therefore, more researchers are attracted by *hp* spectral element methods. In particular, an *hp* spectral element method for elliptic optimal control problem with integral control constraint is discussed in [12]. In [20], *hp* spectral element discretization for elliptic optimal control problem with state constraint is considered and a posteriori error estimates for the optimal control problem are established.

The novelty of this paper is to adopt *hp* spectral element approximation for elliptic optimal control problem and to establish error estimates. Combining the advantages of finite element and spectral methods, we show that the *hp* spectral element method can provide a high accuracy. Furthermore, we derive optimality conditions. Under suitable assumptions, we establish a posteriori error estimates in L^2-L^2 and L^2-H^1 norms.

The rest of this paper is organized as follows. In Section 2, we introduce the optimal control model and investigate optimality conditions for continuous problem. In Section 3, *hp* spectral element approximations for the corresponding optimal control problem are considered and the weak formulation of the optimal control problem is presented. Besides, a posteriori error estimates for optimal control problem are determined by using two important projection operators. In Section 4, we use the projection gradient algorithm, to verify the theoretical results. Finally, a brief conclusion and discussion are presented in the last section.

2. Optimal Control Problems

In this section, we consider an elliptic optimal control problem and construct an *hp* spectral element discretization scheme.

More exactly, we let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz continuous boundary $\partial\Omega$ and denote by $W^{m,q}(\Omega)$ the Sobolev space on Ω equipped with the norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and semi-norm $|\cdot|_{W^{m,q}(\Omega)}$. Besides, we set

$$W_0^{m,q}(\Omega) = \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\},$$

and respectively write $H^m(\Omega)$ and $H_0^m(\Omega)$ for $W^{m,2}$ and (Ω) and $W_0^{m,2}(\Omega)$. In addition, c and C refer to positive constants independent of discrete parameter and different in different occasions.

2.1. Optimal control problems

For the sake of simplicity, we take the state space $Y = H_0^1(\Omega)$ and the control space $U = L^2(\Omega)$. Consider the following elliptic optimal control problem with the L^2-norm control constraint:

$$\min_{q\in Q} J(q,y) = \frac{1}{2}\|y - y_d\|_{0,\Omega}^2 \tag{2.1}$$

subject to the elliptic equation

$$\begin{aligned} -\Delta y &= f + q \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

and with the control constraint we set

$$Q = \{w \in L^2(\Omega) : \|w\|_{L^2(\Omega)} \leq \gamma\},$$

where $q \in Q \subset U$ is the control variable, $y \in Y$ the state variable, and y_d a desired state. Note that in this paper, we always assume that $\gamma = 1$ and $y_d, f \in L^2(\Omega)$ are infinitely differentiable functions.

Consider a weak formulation of the state equation (2.2). For this, let

$$\mathscr{A}(u,v) = \int_\Omega \nabla u \cdot \nabla v \quad \text{for all} \quad u,v \in Y,$$
$$(u,v) = \int_\Omega uv \qquad \text{for all} \quad u,v \in U.$$

It is well known that there are constants $c > 0$ and $C > 0$ such that

$$|\mathscr{A}(u,v)| \le C\|u\|_Y\|v\|_Y \quad \text{for all} \quad u,v \in Y,$$
$$\mathscr{A}(u,u) \ge c\|u\|_Y^2 \qquad \text{for all} \quad u \in Y.$$

The standard weak formulation of the state equation (2.2) reads: Given f, find $y(q) \in Y$ such that

$$\mathscr{A}\big(y(q),w\big) = (f+q,w) \quad \text{for all} \quad w \in Y. \tag{2.3}$$

Using the weak formulation (2.3), we write the optimal control problem(2.1)-(2.2) as: Find (q,y) such that

$$\begin{aligned} &\min_{q\in Q} J(q,y) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} \\ &\text{s.t.} \quad \mathscr{A}\big(y(q),w\big) = (f+q,w) \quad \text{for all} \quad w \in Y. \end{aligned} \tag{2.4}$$

In order to analyze the optimal control problem, we need to investigate the optimality conditions. It follows from [23] that if a pair $(q,y) \in U \times Y$ is an optimal solution of the problem (2.4), then there exists a pair $(z,\lambda) \in Y \times \mathbb{R}^-$ where $\mathbb{R}^- = \{c \in \mathbb{R}; c \le 0\}$ such that (q,y,z,λ) satisfies the following optimality conditions:

$$\mathscr{A}(y,w) = (f+q,w) \quad \text{for all} \quad w \in Y, \tag{2.5a}$$
$$\mathscr{A}(\phi,z) = (y-y_d,\phi) \quad \text{for all} \quad \phi \in Y, \tag{2.5b}$$
$$(z,v-q) \ge 0 \quad \text{for all} \quad v \in Q, \tag{2.5c}$$

where

$$z = \lambda q,$$

and

$$\lambda = \begin{cases} \text{constant} \le 0, & \text{if} \quad \|q\|_{L^2(\Omega)} = \gamma, \\ 0, & \text{if} \quad \|q\|_{L^2(\Omega)} < \gamma. \end{cases} \tag{2.6}$$

Note that we need to show that

$$z = \lambda q,$$

where λ satisfies (2.6). Our considerations can be divided into two cases — viz. $\|q\|_{L^2(\Omega)} < \gamma$ and $\|q\|_{L^2(\Omega)} = \gamma$.

If $\|q\|_{L^2(\Omega)} < \gamma$, then the Lagrange method gives

$$(z,w-q) \ge 0 \quad \text{for all} \quad w \in K.$$

Since $(\gamma-\|q\|_{L^2(\Omega)})\varphi+q\in K$, for all $\varphi\in L^2(\Omega)$ and $\|\varphi\|_{L^2(\Omega)}=1$, we have

$$(z,\varphi)=\frac{1}{\gamma-\|q\|_{L^2(\Omega)}}\big(z,(\gamma-\|q\|_{L^2(\Omega)})\varphi+q-q\big)\geq 0. \tag{2.7}$$

Analogously, since $(\gamma-\|q\|_{L^2(\Omega)})(-\varphi)+q\in K$, we obtain

$$(z,-\varphi)\geq 0. \tag{2.8}$$

It follows from (2.7) and (2.8) that $z=0$.

If $\|q\|_{L^2(\Omega)}=\gamma$, we have

$$\begin{aligned}\|-z\|_{L^2(\Omega)}=&\sup_{\text{for all } v\in L^2(\Omega)}\frac{(-z,v)}{\|v\|_{L^2(\Omega)}}=\sup_{\text{for all } v\in L^2(\Omega)}\frac{(-z,v)}{\gamma}\\ &\leq\frac{1}{\gamma}(-z,q)\leq\frac{1}{\gamma}\|-z\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}=\|-z\|_{L^2(\Omega)}.\end{aligned}$$

The inequality (2.5c) and the Cauchy-Schwarz inequality give

$$(-z,q)=\|-z\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}.$$

Therefore, $z=\lambda q$.

2.2. hp spectral element discretization

We apply the hp spectral element method to discretize the optimal control problem. First, the domain is divided into N_τ nonoverlapping subdomains (elements) $\tau_i, 1\leq i\leq N_\tau$, i.e.

$$\overline{\Omega}=\bigcup_{i=0}^{N_\tau}\bar{\tau}_i,\quad \tau_i\bigcap\tau_j=\emptyset,\quad i\neq j,\quad 1\leq i,j\leq N_\tau.$$

Let $\mathscr{T}=\{\tau\}$ be a local quasi-uniform partition of Ω by regular elements τ. Let $\widehat{\tau}=(-1,1)^2$ be the reference element, $\varepsilon(\mathscr{T})$ the set of all edges, and $\varepsilon_0(\mathscr{T})$ the set of all edges located outside of the boundary $\partial\Omega$. Each element τ can be the image of the reference element $\widehat{\tau}$ under an affine map $F_\tau:\hat{\tau}\to\tau$. By $\widehat{\tau}$ we denote the diameter of τ and assume that the triangulation is χ-shape regular, — i.e.

$$h_\tau^{-1}\left\|F_\tau'\right\|+h_\tau\left\|(F_\tau')^{-1}\right\|\leq\chi. \tag{2.9}$$

This implies that the sizes of the neighboring elements are comparable.

With each element τ of a χ-shape regular mesh $\mathscr{T}$ of the domain Ω, we associate an integer $p_\tau\in\mathbb{N}_0$. These integers $\{p_\tau\}$ are collected in the vector $\boldsymbol{p}=\{p_\tau\}$, called the polynomial degree vector. After that, we define the spaces of hp spectral element approximations $U^{\boldsymbol{p}}(\mathscr{T},\Omega)$, $Y^{\boldsymbol{p}}(\mathscr{T},\Omega)$, $Y_0^{\boldsymbol{p}}(\mathscr{T},\Omega)$ by

$$\begin{aligned}U^{\boldsymbol{p}}(\mathscr{T},\Omega)&:=\left\{u\in L^2(\Omega):u|_\tau\circ F_\tau\in\mathscr{P}_{p_\tau}(\widehat{\tau})\right\},\\ Y^{\boldsymbol{p}}(\mathscr{T},\Omega)&:=\left\{u\in H^1(\Omega):u|_\tau\circ F_\tau\in\mathscr{P}_{p_\tau}(\widehat{\tau})\right\},\\ Y_0^{\boldsymbol{p}}(\mathscr{T},\Omega)&:=Y^{\boldsymbol{p}}(\mathscr{T},\Omega)\cap H_0^1(\Omega),\end{aligned}$$

where $\mathscr{P}_{p_\tau}(\widehat{\tau})$ denotes the space of polynomials in $\widehat{\tau}$ of the degree at most $\le p_\tau$ in each variable. For the polynomial degree distribution $\boldsymbol{p}$, it similar to (2.9). We assume that the polynomial degrees of neighboring elements are comparable — i.e. there exists a constant $C > 0$ such that

$$C^{-1}(p_\tau + 1) \le p'_\tau + 1 \le C(p_\tau + 1) \quad \text{for all} \quad \tau, \tau' \in \mathscr{T}, \quad \overline{\tau} \cap \overline{\tau}' \neq \emptyset.$$

Convergence is obtained either by increasing the degree of the polynomials or by increasing the number of nonoverlapping regular elements.

Let $Y_0^{\boldsymbol{p}}(\mathscr{T}, \Omega)$ denote the approximation space of the state and adjoint state, and let $Q^{\boldsymbol{p}} := Q \cap U^{\boldsymbol{p}}(\mathscr{T}, \Omega)$ denote the approximation space of the control. Consider the hp spectral element method for the Eq. (2.3), viz.

$$\mathscr{A}(y_{hp}, w_{hp}) = (f + q_{hp}, w_{hp}) \quad \text{for all} \quad w_{hp} \in Y_0^{\boldsymbol{p}}(\mathscr{T}, \Omega).$$

Thus, the hp spectral element approximation of the problem (2.4) can be restated as follows:

$$\begin{aligned} &\min_{q_{hp} \in Q^{\boldsymbol{p}}} J(q_{hp}, y_{hp}) = \frac{1}{2} \|y_{hp} - y_d\|^2_{L^2(\Omega)} \\ &\text{s.t.} \quad \mathscr{A}(y_{hp}, w_{hp}) = (f + q_{hp}, w_{hp}) \quad \text{for all} \quad w_{hp} \in Y_0^{\boldsymbol{p}}(\mathscr{T}, \Omega). \end{aligned} \tag{2.10}$$

Similarly, the discrete first order optimality conditions are also presented: If a pair $(q_{hp}, y_{hp}) \in U^{\boldsymbol{p}}(\mathscr{T}, \Omega) \times Y_0^{\boldsymbol{p}}(\mathscr{T}, \Omega)$ is the optimal solution of the optimal control problem (2.10), and there exists a pair $(z_{hp}, \lambda_{hp}) \in Y_0^{\boldsymbol{p}}(\mathscr{T}, \Omega) \times R^-$ where $R^- = \{c \in R; c \le 0\}$ such that $(q_{hp}, y_{hp}, z_{hp}, \lambda_{hp})$ satisfies the optimality conditions

$$\begin{aligned} &\mathscr{A}(y_{hp}, w_{hp}) = (f + q_{hp}, w_{hp}) && \text{for all} \quad w_{hp} \in Y_0^{\boldsymbol{p}}(\mathscr{T}, \Omega), \\ &\mathscr{A}(\phi_{hp}, z_{hp}) = (y_{hp} - y_d, \phi_{hp}) && \text{for all} \quad \phi_{hp} \in Y_0^{\boldsymbol{p}}(\mathscr{T}, \Omega), \\ &(z_{hp}, w_{hp} - q_{hp}) \ge 0 && \text{for all} \quad w_{hp} \in Q^{\boldsymbol{p}}, \end{aligned} \tag{2.11}$$

and

$$z_{hp} = \lambda_{hp} q_{hp},$$

where

$$\lambda_{hp} = \begin{cases} \text{constant} \le 0, & \|q_{hp}\|_{L^2(\Omega)} = \gamma, \\ 0, & \|q_{hp}\|_{L^2(\Omega)} < \gamma. \end{cases}$$

Remark 2.1. In what follows, we only consider the errors of the problem under the condition $\lambda \neq 0$. If $\lambda = 0$, then we can obtain $z = 0$. This is trivial case. The condition $\lambda_{hp} \neq 0$ is considered in this paper.

3. A Posteriori Error Estimates

Let us prove a posteriori error estimates of hp spectral element approximation for control problem. For this, we have to recall lemmas generalizing the well-known Scott-Zhang-type quasi interpolation to the hp context — cf. [29,35].

Lemma 3.1 (Scott-Zhang-Type Quasi Interpolation). *Let $\mathscr{T}$ be a χ-shape regular triangulation of a domain $\Omega \subset \mathbb{R}^2$, and $\boldsymbol{p}$ be a comparable polynomial degree distribution. Then there exists a bounded linear operator $\widehat{\Pi} : H_0^1(\Omega) \to Y_0^{\boldsymbol{p}}(\mathscr{T},\Omega)$, and a constant $C > 0$ which depends only on χ such that for every $u \in H_0^1(\Omega)$, all elements $\tau \in \mathscr{T}$, and all edges $e \in \varepsilon(\tau)$ the following estimates holds:*

$$
\begin{aligned}
&\|u-\widehat{\Pi}u\|_{L^2(\tau)} + \frac{h_\tau}{p_\tau}\|\nabla(u-\widehat{\Pi}u)\|_{L^2(\tau)} \le C\frac{h_\tau}{p_\tau}\|\nabla u\|_{L^2(\omega_\tau)},\\
&\|u-\widehat{\Pi}u\|_{L^2(e)} \le C\sqrt{\frac{h_e}{p_e}}\|\nabla u\|_{L^2(\omega_e)},
\end{aligned}
$$

where h_e is the length of the edge e, $p_e = \max(p_\tau, p_{\tau'})$, τ, τ' are elements sharing the edge e, and w_τ, w_e are patches covering τ and e with a few layers, respectively.

Lemma 3.2 (New Scott-Zhang-Type Quasi Interpolation). *Let $\mathscr{T}$ be a χ-shape regular triangulation of a domain $\Omega \subset \mathbb{R}^2$, and let $\boldsymbol{p}$ be a polynomial degree distribution which is comparable. Then there exists a bounded linear operator $\Lambda : H_0^1(\Omega)\cap H^2(\Omega) \to Y_0^{\boldsymbol{p}}(\mathscr{T},\Omega)\cap H_0^1(\Omega)$, and there exists a constant $C > 0$, which depends only on χ such that for every $u \in H_0^1(\Omega)\cap H^2(\Omega)$, all elements $\tau \in \mathscr{T}$, and all edges $e \in \varepsilon(\tau)$,*

$$
\begin{aligned}
&\|u-\Lambda u\|_{L^2(\tau)} + \frac{h_\tau}{p_\tau}\|\nabla(u-\Lambda u)\|_{L^2(\tau)} \le C\left(\frac{h_\tau}{p_\tau}\right)^2 |u|_{H^2(\omega_\tau)},\\
&\|u-\Lambda u\|_{L^2(e)} \le C\left(\frac{h_e}{p_e}\right)^{3/2} |u|_{H^2(\omega_e)},
\end{aligned}
$$

where h_e is the length of the edge e, $p_e = \max(p_\tau, p_{\tau'})$, τ, τ' are the elements sharing the edge e, and w_τ, w_e is the patch covering τ and e with a few layers.

We first apply the L^2-norm to estimate the control approximation error and the H^1-norm to estimate the state and costate approximation error — viz. $L^2 - H^1$ a posteriori error estimates.

We introduce an auxiliary system to find $(y(q_{hp}), z(q_{hp})) \in Y \times Y$, viz.

$$
\mathscr{A}\big(y(q_{hp}), w\big) = (f + q_{hp}, w) \qquad \text{for all} \quad w \in Y, \tag{3.1a}
$$

$$
\mathscr{A}\big(\varphi, z(q_{hp})\big) = \big(y(q_{hp}) - y_d, \varphi\big) \quad \text{for all} \quad \varphi \in Y. \tag{3.1b}
$$

Using this system, we establish intermediate error estimates.

Lemma 3.3. *Let (q, y, z) and (q_{hp}, y_{hp}, z_{hp}) be the optimal solutions of optimality conditions* (2.5) *and* (2.11)*, respectively. If $(y(q_{hp}), z(q_{hp}))$ is the solution of the auxiliary system* (3.1)*, then*

$$
\|y - y(q_{hp})\|_{H^1(\Omega)} + \|z - z(q_{hp})\|_{H^1(\Omega)} \le C\|q - q_{hp}\|_{L^2(\Omega)}.
$$

Proof. It follows from the continuous optimality conditions (2.5) and the auxiliary system (3.1) that

$$\mathscr{A}\big(y-y(q_{hp}),w\big)=(q-q_{hp},w) \qquad \text{for all} \quad w\in Y, \tag{3.2a}$$
$$\mathscr{A}\big(\varphi,z-z(q_{hp})\big)=\big(y-y(q_{hp}),\varphi\big) \quad \text{for all} \quad \varphi\in Y. \tag{3.2b}$$

Setting $w=y-y(q_{hp})$ in (3.2a) gives

$$\begin{aligned} c\|y-y(q_{hp})\|^2_{H^1(\Omega)} &\le \mathscr{A}\big(y-y(q_{hp}),y-y(q_{hp})\big) \\ &=\big(q-q_{hp},y-y(q_{hp})\big) \\ &\le C_1\|q-q_{hp}\|_{L^2(\Omega)}\|y-y(q_{hp})\|_{L^2(\Omega)}, \end{aligned}$$

which implies

$$\|y-y(q_{hp})\|_{H^1(\Omega)}\le C\|q-q_{hp}\|_{L^2(\Omega)}.$$

Similarly, choosing $\varphi=z-z(q_{hp})$ in (3.2b) yields

$$\begin{aligned} c\|z-z(q_{hp})\|^2_{H^1(\Omega)} &\le \mathscr{A}\big(z-z(q_{hp}),z-z(q_{hp})\big) \\ &=\big(z-z(q_{hp}),y-y(q_{hp})\big) \\ &\le C_2\|z-z(q_{hp})\|_{L^2(\Omega)}\|y-y(q_{hp})\|_{L^2(\Omega)}, \end{aligned}$$

and

$$\|z-z(q_{hp})\|_{H^1(\Omega)}\le C\|y-y(q_{hp})\|_{L^2(\Omega)}\le C\|q-q_{hp}\|_{L^2(\Omega)}.$$

This completes the proof. □

Lemma 3.4. *Let* (q,y,z) *and* (q_{hp},y_{hp},z_{hp}) *be the solutions of optimality conditions* (2.5) *and* (2.11)*, respectively. Then*

$$\|q-q_{hp}\|_{L^2(\Omega)}\le C\|z(q_{hp})-z_{hp}\|_{L^2(\Omega)}.$$

Proof. It is clear that

$$\begin{aligned} (z-z_{hp},\varphi) &=(\lambda q-\lambda_{hp}q_{hp},\varphi) \\ &=(\lambda q-\lambda_{hp}q+\lambda_{hp}q-\lambda_{hp}q_{hp},\varphi) \\ &=(\lambda-\lambda_{hp})(q,\varphi)+\lambda_{hp}(q-q_{hp},\varphi). \end{aligned} \tag{3.3}$$

Choosing φ such that $(q,\varphi)=0$, let

$$\varphi=\gamma^2(q-q_{hp})-\widehat{C}q:=\gamma^2(q-q_{hp})-(q-q_{hp},q)q,$$

we have

$$\begin{aligned} (q,\varphi) &=\big(q,\gamma^2(q-q_{hp})-(q-q_{hp},q)q\big) \\ &=\gamma^2(q,q-q_{hp})-(q-q_{hp},q)(q,q)=0, \end{aligned}$$

where we used $\|q\|_{L^2(\Omega)} = \gamma$. Therefore,

$$\widehat{C} = (q - q_{hp}, q) = \frac{1}{\lambda}(q - q_{hp}, z) = \frac{1}{\lambda}\big(y - y_d, y - y(q_{hp})\big),$$

and $\widehat{C}$ can be estimated as

$$|\widehat{C}| \le C\|y - y(q_{hp})\|_{L^2(\Omega)} \le \|q - q_{hp}\|_{L^2(\Omega)}.$$

Using the above equations, we write

$$\begin{aligned}\lambda_{hp}(q - q_{hp}, \varphi) &= \lambda_{hp}\big(q - q_{hp}, \gamma^2(q - q_{hp}) - \widehat{C}q\big) \\ &= \lambda_{hp}\big(q - q_{hp}, \gamma^2(q - q_{hp})\big) - \lambda_{hp}(q - q_{hp}, \widehat{C}q). \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\lambda_{hp}\gamma^2(q - q_{hp}, q - q_{hp}) = \big(z - z_{hp}, \gamma^2(q - q_{hp}) - \widehat{C}q\big) + \lambda_{hp}(q - q_{hp}, \widehat{C}q).$$

The above expression can be rewritten as

$$\|q - q_{hp}\|_{L^2(\Omega)} \le c(\varepsilon)\big\{\|z - z_{hp}\|_{L^2(\Omega)} + |\widehat{C}|\big\} + \varepsilon\|q - q_{hp}\|_{L^2(\Omega)},$$

and using Lemma 3.3, we get

$$\|q - q_{hp}\|_{L^2(\Omega)} \le C\|z(q_{hp}) - z_{hp}\|_{L^2(\Omega)}.$$

The proof is complete. □

It follows from the discrete optimality conditions (2.5) and the auxiliary system that

$$\mathscr{A}\big(y_{hp} - y(q_{hp}), w_{hp}\big) = 0 \quad \text{for all} \quad w_{hp} \in Y_0^p(\mathscr{T}, \Omega), \tag{3.5a}$$

$$\mathscr{A}\big(\varphi_{hp}, z_{hp} - z(q_{hp})\big) = \big(y_{hp} - y(q_{hp}), \varphi_{hp}\big) \quad \text{for all} \quad \varphi_{hp} \in Y_0^p(\mathscr{T}, \Omega). \tag{3.5b}$$

Lemma 3.5. *If* $(y(q_{hp}), z(q_{hp}))$ *and* (q_{hp}, y_{hp}, z_{hp}) *are respectively the solutions of the auxiliary system* (3.1) *and the discrete optimality conditions* (2.11)*, then*

$$\|y(q_{hp}) - y_{hp}\|^2_{H^1(\Omega)} + \|z(q_{hp}) - z_{hp}\|^2_{H^1(\Omega)} \le C(\eta_1 + \eta_2 + \eta_3 + \eta_4),$$

where

$$\begin{aligned}\eta_1 &= \sum_{\tau\in\mathscr{T}} \frac{h_\tau^2}{p_\tau^2}\|\Delta y_{hp} + f + q_{hp}\|^2_{L^2(\tau)}, \\ \eta_2 &= \sum_{e\in\varepsilon_0(\mathscr{T})} \frac{h_e}{p_e}\left\|\left[\frac{\partial y_{hp}}{\partial n_e}\right]\right\|^2_{L^2(e)}, \\ \eta_3 &= \sum_{\tau\in\mathscr{T}} \frac{h_\tau^2}{p_\tau^2}\|\Delta z_{hp} + y_{hp} - y_d\|^2_{L^2(\tau)}, \\ \eta_4 &= \sum_{e\in\varepsilon_0(\mathscr{T})} \frac{h_e}{p_e}\left\|\left[\frac{\partial z_{hp}}{\partial n_e}\right]\right\|^2_{L^2(e)}\end{aligned}$$

with n_e *and* $[v]$ *respectively referring to the outward unit normal to* e *and the jump of a function* v *across an edge.*

Proof. Set $E^y = y(q_{hp}) - y_{hp}$ and $E_I^y = \widehat{\Pi} E^y$, where the interpolator $\widehat{\Pi}$ is defined in Lemma 3.1. Then

$$\begin{aligned}
& c\|y(q_{hp}) - y_{hp}\|^2_{H^1(\Omega)} \\
&= c\|E^y\|^2_{H^1(\Omega)} \leq \mathscr{A}(E^y, E^y) \\
&= \mathscr{A}\left(E^y, E^y - E_I^y\right) + \mathscr{A}\left(E^y, E_I^y\right) \\
&= \mathscr{A}\left(y(q_{hp}) - y_{hp}, E^y - E_I^y\right) \\
&= \mathscr{A}\left(y(q_{hp}), E^y - E_I^y\right) - \mathscr{A}\left(y_{hp}, E^y - E_I^y\right) \\
&= \sum_{\tau\in\mathscr{T}} \left(f + q_{hp} + \Delta y_{hp}, E^y - E_I^y\right)_\tau - \sum_{e\in\varepsilon_0(\mathscr{T})} \int_e \left[\frac{\partial y_{hp}}{\partial n_e}\right]\left(E^y - E_I^y\right) \\
&= \sum_{\tau\in\mathscr{T}} \int_\tau (f + q_{hp} + \Delta y_{hp})\left(E^y - E_I^y\right) - \sum_{e\in\varepsilon_0(\mathscr{T})} \int_e \left[\frac{\partial y_{hp}}{\partial n_e}\right]\left(E^y - E_I^y\right).
\end{aligned}$$

Using (3.1a), (3.5a) and (2.11) gives

$$\begin{aligned}
& c\|y(q_{hp}) - y_{hp}\|^2_{H^1(\Omega)} \\
\leq\ & C \sum_{\tau\in\mathscr{T}} \|f + q_{hp} + \Delta y_{hp}\|_{L^2(\tau)} \left\|E^y - E_I^y\right\|_{L^2(\tau)} \\
& + C \sum_{e\in\varepsilon_0(\mathscr{T})} \left\|\left[\frac{\partial y_{hp}}{\partial n_e}\right]\right\|_{L^2(e)} \left\|E^y - E_I^y\right\|_{L^2(e)} \\
\leq\ & C \sum_{\tau\in\mathscr{T}} \frac{h_\tau}{p_\tau} \|f + q_{hp} + \Delta y_{hp}\|_{L^2(\tau)} \|\nabla E^y\|_{L^2(\omega_\tau)} \\
& + C \sum_{e\in\varepsilon_0(\mathscr{T})} \sqrt{\frac{h_e}{p_e}} \left\|\left[\frac{\partial y_{hp}}{\partial n_e}\right]\right\|_{L^2(e)} \|\nabla E^y\|_{L^2(\omega_e)} \\
\leq\ & C(\delta) \left\{ \sum_{\tau\in\mathscr{T}} \frac{h_\tau^2}{p_\tau^2} \|f + q_{hp} + \Delta y_{hp}\|^2_{L^2(\tau)} + \sum_{e\in\varepsilon_0(\mathscr{T})} \frac{h_e}{p_e} \left\|\left[\frac{\partial y_{hp}}{\partial n_e}\right]\right\|^2_{L^2(e)} \right\} + \delta\|E^y\|^2_{H^1(\Omega)}.
\end{aligned}$$

Choosing $\delta = c/2$ yields

$$\begin{aligned}
& \|y_{hp} - y(q_{hp})\|^2_{H^1(\Omega)} \\
\leq\ & C \left\{ \sum_{\tau\in\mathscr{T}} \frac{h_\tau^2}{p_\tau^2} \|f + q_{hp} + \Delta y_{hp}\|^2_{L^2(\tau)} + \sum_{e\in\varepsilon_0(\mathscr{T})} \frac{h_e}{p_e} \left\|\left[\frac{\partial y_{hp}}{\partial n_e}\right]\right\|^2_{L^2(e)} \right\}.
\end{aligned} \tag{3.6}$$

Analogously, setting $E^z = z(q_{hp}) - z_{hp}$ and $E_I^z = \widehat{\Pi} E^z$ with the interpolator $\widehat{\Pi}$ defined in Lemma 3.1, we have

$$c\|z(q_{hp}) - z_{hp}\|^2_{H^1(\Omega)} = c\|E^z\|^2_{H^1(\Omega)} \leq \mathscr{A}(E^z, E^z)$$

$$
\begin{aligned}
&= \mathscr{A}(E^z, E^z - E_I^z) + \mathscr{A}(E^z, E_I^z) \\
&= \mathscr{A}\left(z(q_{hp}) - z_{hp}, E^z - E_I^z\right) + \left(y(q_{hp}) - y_{hp}, E_I^z\right) \\
&= \mathscr{A}\left(z(q_{hp}), E^z - E_I^z\right) - \mathscr{A}\left(z_{hp}, E^z - E_I^z\right) + \left(y(q_{hp}) - y_{hp}, E_I^z\right) \\
&= \left(y_{hp} - y_d, E^z - E_I^z\right) + \left(y(q_{hp}) - y_{hp}, E^z\right) - \mathscr{A}\left(z_{hp}, E^z - E_I^z\right).
\end{aligned}
$$

Taking into account (3.1b), (3.5b) and (2.11), we write

$$
\begin{aligned}
& c\|z(q_{hp}) - z_{hp}\|_{H^1(\Omega)}^2 \\
&= \sum_{\tau\in\mathscr{T}} \int_\tau (y_{hp} - y_d + \Delta z_{hp})\left(E^z - E_I^z\right) + \left(y(q_{hp}) - y_{hp}, E^z\right) \\
&\quad - \sum_{e\in\varepsilon_0(\mathscr{T})} \int_e \left[\frac{\partial z_{hp}}{\partial n_e}\right]\left(E^z - E_I^z\right) \\
&\le \sum_{\tau\in\mathscr{T}} \|y_{hp} - y_d + \Delta z_{hp}\|_{L^2(\tau)} \left\|E^z - E_I^z\right\|_{L^2(\tau)} \\
&\quad + \|y(q_{hp}) - y_{hp}\|_{L^2(\Omega)} \|E^z\|_{L^2(\Omega)} \\
&\quad + C \sum_{e\in\varepsilon_0(\mathscr{T})} \left\|\left[\frac{\partial z_{hp}}{\partial n_e}\right]\right\|_{L^2(e)} \left\|E^z - E_I^z\right\|_{L^2(e)} \\
&\le C \sum_{\tau\in\mathscr{T}} \frac{h_\tau}{p_\tau} \|y_{hp} - y_d + \Delta z_{hp}\|_{L^2(\tau)} \|\nabla E^z\|_{L^2(\omega_\tau)} \\
&\quad + C \sum_{e\in\varepsilon_0(\mathscr{T})} \sqrt{\frac{h_e}{p_e}} \left\|\left[\frac{\partial z_{hp}}{\partial n_e}\right]\right\|_{L^2(e)} \|\nabla E^z\|_{L^2(\omega_e)} \\
&\quad + \|y(q_{hp}) - y_{hp}\|_{L^2(\Omega)} \|E^z\|_{L^2(\Omega)} \\
&\le C(\delta) \left\{ \sum_{\tau\in\mathscr{T}} \frac{h_\tau^2}{p_\tau^2} \|y_{hp} - y_d + \Delta z_{hp}\|_{L^2(\tau)}^2 \right. \\
&\qquad \left. + \sum_{e\in\varepsilon_0(\mathscr{T})} \frac{h_e}{p_e} \left\|\left[\frac{\partial z_{hp}}{\partial n_e}\right]\right\|_{L^2(e)}^2 + \|y(q_{hp}) - y_{hp}\|_{L^2(\Omega)}^2 \right\} + \delta \|E^z\|_{H^1(\Omega)}^2.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\|z(q_{hp}) - z_{hp}\|_{H^1(\Omega)}^2 \le C &\left\{ \sum_{\tau\in\mathscr{T}} \frac{h_\tau^2}{p_\tau^2} \|y_{hp} - y_d + \Delta z_{hp}\|_{L^2(\tau)}^2 + \sum_{e\in\varepsilon_0(\mathscr{T})} \frac{h_e}{p_e} \left\|\left[\frac{\partial z_{hp}}{\partial n_e}\right]\right\|_{L^2(e)}^2 \right\} \\
&+ C\|y(q_{hp}) - y_{hp}\|_{L^2(\Omega)}^2. \qquad (3.7)
\end{aligned}
$$

Summing up the relations (3.6)-(3.7) finishes the proof . □

Finally, we arrive at the following theorem.

Theorem 3.1. *If* (q,y,z) *and* (q_{hp},y_{hp},z_{hp}) *are respectively the solutions of the optimality conditions* (2.5) *and* (2.11)*, then the following a posteriori error estimates hold:*

$$\|q-q_{hp}\|^2_{L^2(\Omega)}+\|y-y_{hp}\|^2_{H^1(\Omega)}+\|z-z_{hp}\|^2_{H^1(\Omega)}\leq C\{\eta_1+\eta_2+\eta_3+\eta_4\},$$

where $\eta_1,\eta_2,\eta_3,\eta_4$ *are defined in Lemma* 3.5.

Proof. Using the triangle inequality gives

$$\begin{aligned}&\|y-y_{hp}\|^2_{H^1(\Omega)}+\|z-z_{hp}\|^2_{H^1(\Omega)}\\ \leq&\|y-y(q_{hp})\|^2_{H^1(\Omega)}+\|y(q_{hp})-y_{hp}\|^2_{H^1(\Omega)}+\|z-z(q_{hp})\|^2_{H^1(\Omega)}+\|z(q_{hp})-z_{hp}\|^2_{H^1(\Omega)}\\ \leq& C\left\{\|q-q_{hp}\|^2_{L^2(\Omega)}+\|y(q_{hp})-y_{hp}\|^2_{H^1(\Omega)}+\|z-z(q_{hp})\|^2_{H^1(\Omega)}+\|z(q_{hp})-z_{hp}\|^2_{H^1(\Omega)}\right\}.\end{aligned}$$

This and Lemmas 3.3-3.5 complete the proof. □

As far as the applications are concerned, we have to use the L^2-norm for estimating state and costate approximation errors — viz. we require L^2-L^2-(a posteriori) error estimates of hp spectral element discretization. In order to evaluate $\|z(q_{hp})-z_{hp}\|_{L^2(\Omega)}$, we consider an auxiliary problem — viz.

$$\begin{aligned}-\Delta\xi&=f\quad\text{in }\ \Omega,\\ \xi&=0\qquad\text{on }\ \partial\Omega\end{aligned}\tag{3.8}$$

such that

$$\|\xi\|_{H^2(\Omega)}\leq C\|f\|_{L^2(\Omega)}.\tag{3.9}$$

Lemma 3.6. *If* $(y(q_{hp}),z(q_{hp}))$ *and* (q_{hp},y_{hp},z_{hp}) *are respectively the solutions of the auxiliary system* (3.8) *and the discrete optimality conditions* (2.11)*, then*

$$\|y(q_{hp})-y_{hp}\|^2_{L^2(\Omega)}+\|z(q_{hp})-z_{hp}\|^2_{L^2(\Omega)}\leq C(\eta_5+\eta_6+\eta_7+\eta_8),$$

where

$$\begin{aligned}\eta_5&=\sum_{\tau\in\mathcal{T}}\frac{h_\tau^4}{p_\tau^4}\|\Delta y_{hp}+f+q_{hp}\|^2_{L^2(\tau)},\\ \eta_6&=\sum_{e\in\varepsilon(\mathcal{T})\backslash\partial\Omega}\frac{h_e^3}{p_e^3}\left\|\left[\frac{\partial y_{hp}}{\partial n_e}\right]\right\|^2_{L^2(e)},\\ \eta_7&=\sum_{\tau\in\mathcal{T}}\frac{h_\tau^4}{p_\tau^4}\|\Delta z_{hp}+y_{hp}-y_d\|^2_{L^2(\tau)},\\ \eta_8&=\sum_{e\in\varepsilon(\mathcal{T})\backslash\partial\Omega}\frac{h_e^3}{p_e^3}\left\|\left[\frac{\partial z_{hp}}{\partial n_e}\right]\right\|^2_{L^2(e)}.\end{aligned}$$

Proof. If ξ is the solution of (3.8) with $f = z(q_{hp}) - z_{hp}$, and $\xi_{hp} = \Lambda\xi$ with the interpolator Λ defined in Lemma 3.2, then

$$
\begin{aligned}
\|z(q_{hp}) - z_{hp}\|^2_{L^2(\Omega)} &= \big(f, z(q_{hp}) - z_{hp}\big) \\
&= \mathscr{A}\big(\xi, z(q_{hp}) - z_{hp}\big) \\
&= \mathscr{A}\big(\xi - \xi_{hp}, z(q_{hp}) - z_{hp}\big) + \mathscr{A}\big(\xi_{hp}, z(q_{hp}) - z_{hp}\big) \\
&= \big(y(q_{hp}) - y_d, \xi - \xi_{hp}\big) - \mathscr{A}\big(\xi - \xi_{hp}, z_{hp}\big) + \big(y(q_{hp}) - y_{hp}, \xi_{hp}\big) \\
&= \big(y_{hp} - y_d, \xi - \xi_{hp}\big) - \mathscr{A}\big(\xi - \xi_{hp}, z_{hp}\big) + \big(y(q_{hp}) - y_{hp}, \xi\big).
\end{aligned}
$$

Using (3.1b), (3.5b), (3.9) yields

$$
\begin{aligned}
&\|z(q_{hp}) - z_{hp}\|^2_{L^2(\Omega)} \\
=& \sum_{\tau\in\mathscr{T}} \int_\tau (\Delta z_{hp} + y_{hp} - y_d)(\xi - \xi_{hp}) - \sum_{e\in\varepsilon_0(\mathscr{T})} \int_e \left[\frac{\partial z_{hp}}{\partial n_e}\right](\xi - \xi_{hp}) \\
&+ \big(y(q_{hp}) - y_{hp}, \xi\big) \\
\le& \sum_{\tau\in\mathscr{T}} \|\Delta z_{hp} + y_{hp} - y_d\|_{L^2(\tau)} \|\xi - \xi_{hp}\|_{L^2(\tau)} \\
&+ \|y_{hp} - y(q_{hp})\|_{L^2(\Omega)} \|\xi\|_{L^2(\Omega)} \\
&+ \sum_{e\in\varepsilon_0(\mathscr{T})} \left\|\left[\frac{\partial z_{hp}}{\partial n_e}\right]\right\|_{L^2(e)} \|\xi - \xi_{hp}\|_{L^2(e)} \\
\le& C(\varepsilon) \left\{ \sum_{\tau\in\mathscr{T}} \frac{h_\tau^4}{p_\tau^4} \|\Delta z_{hp} + y_{hp} - y_d\|^2_{L^2(\tau)} + \sum_{e\in\varepsilon_0(\mathscr{T})} \frac{h_e^3}{p_e^3} \left\|\left[\frac{\partial z_{hp}}{\partial n_e}\right]\right\|^2_{L^2(e)} \right. \\
&\left. + \|y_{hp} - y(q_{hp})\|^2_{L^2(\Omega)} \right\} + \varepsilon \|f\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|z(q_{hp}) - z_{hp}\|^2_{L^2(\Omega)} \le C(\varepsilon) &\left\{ \sum_{\tau\in\mathscr{T}} \frac{h_\tau^4}{p_\tau^4} \|\Delta z_{hp} + y_{hp} - y_d\|^2_{L^2(\tau)} + \sum_{e\in\varepsilon_0(\mathscr{T})} \frac{h_e^3}{p_e^3} \left\|\left[\frac{\partial z_{hp}}{\partial n_e}\right]\right\|^2_{L^2(e)} \right. \\
&\left. + \|y_{hp} - y(q_{hp})\|^2_{L^2(\Omega)} \right\},
\end{aligned}
$$

and

$$
\|z_{hp} - z(q_{hp})\|^2_{L^2(\Omega)} \le C(\eta_5 + \eta_6) + C\|y_{hp} - y(q_{hp})\|^2_{L^2(\Omega)}. \tag{3.10}
$$

Now let us estimate $\|y(q_{hp}) - y_{hp}\|^2_{L^2(\Omega)}$. Similar to the previous considerations, we assume that ξ is the solution of (3.8) with $f = y(q_{hp}) - y_{hp}$. Setting $\xi_{hp} = \Lambda\xi$ with the interpolator

Λ defined in Lemma 3.2, we write

$$\begin{aligned}\|y(q_{hp})-y_{hp}\|^2_{L^2(\Omega)} &= \big(f, y(q_{hp})-y_{hp}\big)\\ &= \mathscr{A}\big(\xi, y(q_{hp})-y_{hp}\big)\\ &= \mathscr{A}\big(\xi-\xi_{hp}, y(q_{hp})-y_{hp}\big)+\mathscr{A}\big(\xi_{hp}, y(q_{hp})-y_{hp}\big)\\ &= (f+q_{hp}, \xi-\xi_{hp})-\mathscr{A}(\xi-\xi_{hp}, y_{hp}).\end{aligned}$$

Using (3.1a), (3.5a), (3.9) yields

$$\begin{aligned}&\|y(q_{hp})-y_{hp}\|^2_{L^2(\Omega)}\\ &=\sum_{\tau\in\mathscr{T}}\int_\tau(\Delta y_{hp}+f+q_{hp})(\xi-\xi_{hp})-\sum_{e\in\varepsilon_0(\mathscr{T})}\int_e\left[\frac{\partial y_{hp}}{\partial n_e}\right](\xi-\xi_{hp})\\ &\le\sum_{\tau\in\mathscr{T}}\|\Delta y_{hp}+f+q_{hp}\|_{L^2(\tau)}\|\xi-\xi_{hp}\|_{L^2(\tau)}+\sum_{e\in\varepsilon_0(\mathscr{T})}\left\|\left[\frac{\partial y_{hp}}{\partial n_e}\right]\right\|_{L^2(e)}\|\xi-\xi_{hp}\|_{L^2(e)}\\ &\le C(\delta)\left\{\sum_{\tau\in\mathscr{T}}\frac{h_\tau^4}{p_\tau^4}\|\Delta y_{hp}+f+q_{hp}\|^2_{L^2(\tau)}+\sum_{e\in\varepsilon_0(\mathscr{T})}\frac{h_e^3}{p_e^3}\left\|\left[\frac{\partial y_{hp}}{\partial n_e}\right]\right\|^2_{L^2(e)}\right\}+\delta\|\xi\|^2_{H^2(\Omega)}\\ &\le C(\delta)\left\{\sum_{\tau\in\mathscr{T}}\frac{h_\tau^4}{p_\tau^4}\|\Delta y_{hp}+f+q_{hp}\|^2_{L^2(\tau)}+\sum_{e\in\varepsilon_0(\mathscr{T})}\frac{h_e^3}{p_e^3}\left\|\left[\frac{\partial y_{hp}}{\partial n_e}\right]\right\|^2_{L^2(e)}\right\}+\delta\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

Choosing $\delta=1/2$ gives

$$\begin{aligned}\|y_{hp}-y(q_{hp})\|^2_{L^2(\Omega)}&\le C\left\{\sum_{\tau\in\mathscr{T}}\frac{h_\tau^4}{p_\tau^4}\|\Delta y_{hp}+f+q_{hp}\|^2_{L^2(\tau)}+\sum_{e\in\varepsilon_0(\mathscr{T})}\frac{h_e^3}{p_e^3}\left\|\left[\frac{\partial y_{hp}}{\partial n_e}\right]\right\|^2_{L^2(e)}\right\}\\ &\le C(\eta_7+\eta_8).\end{aligned}\tag{3.11}$$

Combining (3.10) and (3.11) finishes the proof. □

Theorem 3.2. *If* (q,y,z) *and* (q_{hp},y_{hp},z_{hp}) *are respectively the solutions of optimality conditions* (2.5) *and* (2.11)*, then the following a posteriori error estimates hold:*

$$\|q-q_{hp}\|^2_{L^2(\Omega)}+\|y-y_{hp}\|^2_{L^2(\Omega)}+\|z-z_{hp}\|^2_{L^2(\Omega)}\le C\{\eta_5+\eta_6+\eta_7+\eta_8\},$$

where $\eta_5,\eta_6,\eta_7,\eta_8$ *are defined in Lemma* 3.6.

Proof. Applying the triangle inequality yields

$$\begin{aligned}&\|y-y_{hp}\|^2_{L^2(\Omega)}+\|z-z_{hp}\|^2_{L^2(\Omega)}\\ &\le\|y-y(q_{hp})\|^2_{L^2(\Omega)}+\|y(q_{hp})-y_{hp}\|^2_{L^2(\Omega)}+\|z-z(q_{hp})\|^2_{L^2(\Omega)}+\|z(q_{hp})-z_{hp}\|^2_{L^2(\Omega)}\\ &\le C\left\{\|q-q_{hp}\|^2_{L^2(\Omega)}+\|y(q_{hp})-y_{hp}\|^2_{L^2(\Omega)}+\|z-z(q_{hp})\|^2_{L^2(\Omega)}+\|z(q_{hp})-z_{hp}\|^2_{L^2(\Omega)}\right\}.\end{aligned}$$

Recalling Lemmas 3.3, 3.4, 3.6 finishes the proof. □

4. Numerical Examples

In this section, we carry out a numerical experiment to demonstrate the error estimates. We present an iterative algorithm for solving optimal control problems efficiently, and summarize the algorithm as follows.

Algorithm 4.1

1: Fix $\rho > 0$ and select an initial approximation $q_{hp}^0 \in U^{\boldsymbol{p}}(\mathscr{T},\Omega)$, seek $y_{hp}^0 \in Y_0^{\boldsymbol{p}}(\mathscr{T},\Omega)$ such that

$$\mathscr{A}\left(y_{hp}^0, w_{hp}\right) = \left(f + q_{hp}^0, w_{hp}\right) \quad \text{for all} \quad w_{hp} \in Y_0^{\boldsymbol{p}}(\mathscr{T},\Omega).$$

2: Find $z_{hp}^n \in Y_0^{\boldsymbol{p}}(\mathscr{T},\Omega)$ such that

$$\mathscr{A}\left(z_{hp}^n, \varphi_{hp}\right) = \left(y_{hp}^n - y_d, \varphi_{hp}\right) \quad \text{for all} \quad \varphi_{hp} \in Y_0^{\boldsymbol{p}}(\mathscr{T},\Omega).$$

3: Set $q_{hp}^{n+1/2} = q_{hp}^n - \rho z_{hp}^n$, then

$$q_{hp}^{n+1} = \frac{q_{hp}^{n+1/2}}{\max\left\{\|q_{hp}^{n+1/2}\|_{L^2(\Omega)}, \gamma\right\}}.$$

4: Find $y_{hp}^{n+1} \in Y_0^{\boldsymbol{p}}(\mathscr{T},\Omega)$ such that

$$\mathscr{A}\left(y_{hp}^{n+1}, w_{hp}\right) = \left(f + q_{hp}^{n+1}, w_{hp}\right) \quad \text{for all} \quad w_{hp} \in Y_0^{\boldsymbol{p}}(\mathscr{T},\Omega).$$

5: **if** $\|q_{hp}^{n+1} - q_{hp}^n\|_{L^2(\Omega)} \leq$ tolerance **then**
6: Stop
7: **else**
8: Set $n = n+1$ and go to Step 2.
9: **end if**

It follows from the definition of q_{hp}^{n+1} that

$$\left\|q_{hp}^{n+1}\right\|_{L^2(\Omega)} = \frac{\|q_{hp}^{n+1/2}\|_{L^2(\Omega)}}{\max\left\{\|q_{hp}^{n+1/2}\|_{L^2(\Omega)}, \gamma\right\}} \leq 1.$$

Therefore, we can prove that $q_{hp}^{n+1} \in K$.

Next, we perform a numerical experiment to verify theoretical results. Assume that the iteration parameter $\rho < 1$ and let

$$K := \left\{\omega \in L^2(\Omega) : \|\omega\|_{L^2(\Omega)} \leq 1\right\}.$$

On the domain $\Omega = (-1,1)^2 \subset R^2$, we consider the model problem

$$\begin{aligned} \min_{q\in Q} J(q,y) &= \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} \\ \text{s.t.} -\Delta y &= f+q \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with the exact solution

$$\begin{aligned} y &= \frac{1}{2\pi^2}\sin\pi x_1 \sin\pi x_2, \\ q &= \sin\pi x_1 \sin\pi x_2, \\ z &= \lambda q, \quad \lambda = -0.5. \end{aligned}$$

Applying the hp spectral element method, we evaluate $\|q-q_{hp}\|_{L^2(\Omega)}$, $\|y-y_{hp}\|_{H^1(\Omega)}$, $\|z-z_{hp}\|_{H^1(\Omega)}$, $|\lambda-\lambda_{hp}|$, $\|y-y_{hp}\|_{L^2(\Omega)}$, $\|z-z_{hp}\|_{L^2(\Omega)}$, and demonstrate the results in Tables 1-2 and Fig. 1.

Tables 1-2 show that the errors decay exponentially fast. We can obtain the convergence of error estimate by increasing the degree of the polynomials or the number of nonoverlapping regular elements.

Table 1: Error estimates, $N_\tau = 4$.

p	2	4	8	16	32
$\|q-q_{hp}\|_{L^2(\Omega)}$	1.0398e-002	2.3911e-005	4.5766e-011	3.7298e-013	2.2714e-013
$\|y-y_{hp}\|_{H^1(\Omega)}$	4.3930e-003	1.4953e-005	5.7735e-011	8.1736e-013	5.3219e-013
$\|z-z_{hp}\|_{H^1(\Omega)}$	3.5603e-002	1.3692e-004	5.6060e-010	7.9023e-012	5.2260e-012
$\lvert\lambda-\lambda_{hp}\rvert$	2.3000e-003	1.7822e-007	1.6653e-016	3.7747e-015	1.3656e-014
$\|y-y_{hp}\|_{L^2(\Omega)}$	6.8736e-004	1.3515e-006	2.2364e-012	1.9406e-014	1.1772e-014
$\|z-z_{hp}\|_{L^2(\Omega)}$	5.6761e-003	1.1949e-005	2.2882e-011	1.8753e-013	1.1455e-013

Table 2: Error estimates, $p = 4$.

N_τ	2	4	8	16	32
$\|q-q_{hp}\|_{L^2(\Omega)}$	2.1563e-003	2.3911e-005	3.8927e-007	6.1433e-009	9.6227e-011
$\|y-y_{hp}\|_{H^1(\Omega)}$	6.9266e-004	1.4953e-005	4.5152e-007	1.3978e-008	4.3576e-010
$\|z-z_{hp}\|_{H^1(\Omega)}$	5.5261e-003	1.3692e-004	4.3697e-006	1.3727e-007	4.2954e-009
$\lvert\lambda-\lambda_{hp}\rvert$	1.025e-003	1.7822e-007	7.056e-010	2.7653e-012	1.3100e-014
$\|y-y_{hp}\|_{L^2(\Omega)}$	1.1239e-004	1.3515e-006	2.0294e-008	3.1348e-010	4.8837e-012
$\|z-z_{hp}\|_{L^2(\Omega)}$	9.2833e-004	1.1949e-005	1.9460e-007	3.0715e-008	4.8110e-011

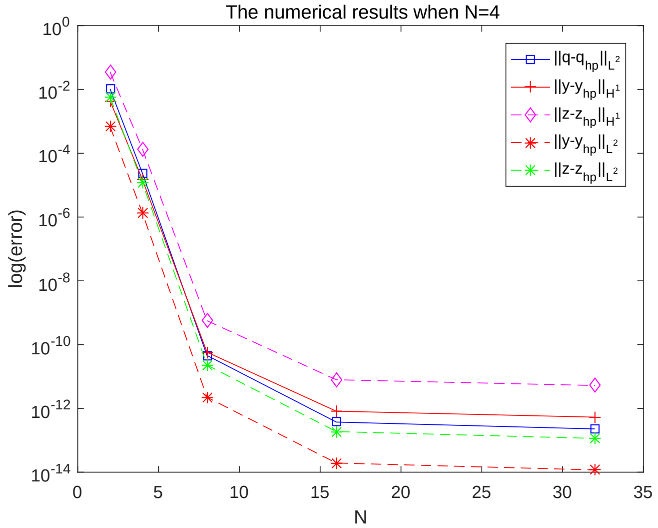

Figure 1: Error estimates, $N_\tau = 4$.

5. Conclusion

In this paper, we consider *hp* spectral element approximation for an elliptic optimal control problem with L^2-norm control constraint. We established optimality conditions and constructed discretization scheme. The method can be implemented efficiently. A posteriori error estimates are rigorously proved. Numerical tests confirm the efficiency of the *hp* spectral element approximation for optimal control problems. The application of the method to a more complex optimal control problems will be considered elsewhere.

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