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A Fast Discontinuous Galerkin Finite Element Method for a Bond-Based Linear Peridynamic Model

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Abstract. A fast discontinuous Galerkin finite element algorithm on a non-uniform mesh for solution of a one-dimensional bond-based linear peridynamic model with fractional noise is developed. It is based on the approximation of the stiffness matrix corresponding to the discontinuous Galerkin finite element method by its hierarchical representation. The fast algorithm reduces the storage requirement for the stiffness matrix from $\mathscr{O}(N^2)$ to $\mathscr{O}(kN)$, where k is a parameter controlling the accuracy of hierarchical matrices. The computational complexities of assembling the stiffness matrix and the Krylov subspace method for solving linear systems are also reduced from $\mathscr{O}(N^2)$ to $\mathscr{O}(kN)$. Numerical results show the utility of the numerical method.

AMS subject classifications: 65C30, 65B99

Key words: Peridynamic model, discontinuous Galerkin finite element, hierarchical matrix, fast algorithm, fractional noise.

1. Introduction

The peridynamic (PD) theory aimed to model problems with evolving discontinuities has been successfully used in various applications, including polycrystal fracture [11], quasi-static crack propagation [14], brittle fracture [15], damage in concrete [10], geomaterial fragmentation by impulse loads [19], failure and damage in composite laminates [23], failure in a stiffened composite curved panel [23]. To solve PD models, a number of numerical methods have been proposed. In particular, Chen and Gunzburger [4] developed a Galerkin finite element method for the one-dimensional steady-state PD model

$$
\begin{aligned}
&\frac{1}{\delta^2}\int_{x-\delta}^{x+\delta}\frac{u(x)-u(x')}{|x-x'|}dx' = b(x), && x\in\Omega,\\
&u(x)=g(x), && x\in\Gamma.
\end{aligned}
\tag{1.1}
$$

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Discontinuous Galerkin finite element method has been applied to various models and extended to various novel discretization schemes [22,26,29]. In PD models (1.1), continuous Galerkin finite element is not suitable if the solutions contains jump discontinuities. Therefore, in this paper a discontinuous Galerkin (DG) finite element method is used. However, numerical methods for peridynamic models usually yield dense stiffness matrices due to their non-local property, thus the direct solvers require $\mathcal{O}(N^2)$ memory and $\mathcal{O}(N^3)$ computational complexity, where N is the number of spatial unknowns. Meanwhile Krylov subspace iterative methods require $\mathcal{O}(N^2)$ computational complexity per iteration, which can be also very expensive. Extensive efforts have been made to improve the computational efficiency [8,21,25]. Wang and Tian [25] developed a fast method for a linear steady-state bond-based peridynamic model by utilizing the positive-definite tridiagonal-plus-Toeplitz structure of the stiffness matrix. Du *et al.* [8] developed a fast collocation method for a state-based linear PD model equation in one space dimension. Both approaches reduce the computational cost from $\mathcal{O}(N^3)$ to $\mathcal{O}(N\log^2 N)$ and the memory requirements from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$. So far, main progress has been made on reducing the computational cost and memory requirement based on a uniform mesh. However, the existing fast methods are not applicable to PD models if a non-uniform mesh is used.

In this work, we present a fast discontinuous Galerkin (FDG) finite element method for a steady-state one-dimensional PD model based on a non-uniform mesh using Hierarchical matrices ($\mathscr{H}$-matrices) [2]. $\mathscr{H}$-matrices are data-sparse approximations of dense matrices. They are successfully used for solving integral equations [1,2], fractional differential equations [20,28] and elliptic partial differential equations [18]. Matrices stored in $\mathscr{H}$-matrix format provide the reduction of storage requirement from $\mathcal{O}(N^2)$ to $\mathcal{O}(kN)$ and the computational complexity from $\mathcal{O}(N^2)$ to $\mathcal{O}(kN)$ per matrix-vector multiplication, where k is a parameter controlling the accuracy. The key aim of the $\mathscr{H}$-matrix method is to approximate the stiffness matrix $\mathbb{A}$ by $\tilde{\mathbb{A}}$ which can be stored in a data-sparse (not necessary sparse) format. To this end, truncated Taylor expansion is adopted to approximate the kernel of the PD model. Let k be the number of terms in the truncated Taylor expansion, we will prove theoretically that the approximation error of the $\mathscr{H}$-matrices $\tilde{\mathbb{A}}$ decays as $\mathcal{O}(3^{-k})$.

Meanwhile, most of the problems are affected by ubiquitous noise perturbations. This motivates the study of mathematical models driven by stochastic noise and numerical methods for solving them [6,27]. For finding correlated random fluctuations occurred in more general systems see [5,7,12]. An approximate model for correlated noise is the fractional Brownian motion (fBm) $W_B(x)$ [3,17,24], which satisfies

$$Cov(x,y) := E\big(W_B(x)W_B(y)\big) = \frac{1}{2}\left(|x|^{2B} + |y|^{2B} - |x-y|^{2B}\right),$$

where $x,y\in[0,\infty)$ and $B\in(0,1)$ is the Hurst index. Here, fBm denotes the Brownian motion if $B=1/2$. Otherwise the increments of fBm are dependent.

The rest of the paper is organized as follows. In Section 2, we introduce a bond-based PD model with fractional noise and its discontinuous Galerkin finite element discretization based on a non-uniform mesh. In Section 3, we present the $\mathscr{H}$-matrix representation of stiffness matrices corresponding to the discretization scheme and present the error analysis. In Section 4, numerical experiments are presented to demonstrate the accuracy and