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Improved Random Feature Method for Continuous Solution Reconstruction from Sparse Observation

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Abstract. The reconstruction of continuous solutions using all available mechanisms and data is essential for high-precision simulations and forecasts. This paper presents an improved random feature method (IRFM) that combines observational data with models for solution reconstruction. For the multi-region neuron approximation, we employ a global activation function with robust local approximation capabilities instead of the traditional piecewise method. This enhances smoothness across regions and accelerates convergence. We also derived the equivalence condition for the optimal solution of multi-constrained optimization problems, established criteria for determining the weights in the cost function and the distribution of randomly generated collocation points, reducing biases from subjective choices. Additionally, we introduce a weighted scheme for computing the cost function related to sparse observations, reducing interpolation errors and improving stability against noise. Numerical examples demonstrate that the IRFM is more accurate and converges faster than the original RFM. Its efficiency and accuracy are validated through comparisons with physics-informed neural networks, and its flexibility is shown by successful continuous solution reconstruction in complex domains.

AMS subject classifications: 65M32, 68T09

Key words: Improved random feature method, continuous solution reconstruction, data assimilation, sparse observation.

1. Introduction

Partial differential equation (PDE) models are widely used to describe the physical behavior of systems in various scientific and engineering applications. However, these models often suffer from inaccuracies due to limitations in the model itself, missing initial and boundary conditions, and parameter uncertainties, all of which affect the simulation's accuracy. By combining observational data with model predictions, it is possible to correct the model state effectively, significantly improving simulation precision. This is particularly

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crucial in numerical forecasting, as it provides enhanced predictive accuracy when dealing with imperfect models and incomplete observational data.

To address uncertainties and sparse observational data, the problem can be formulated as the following optimization problem:

$$u^* = \arg\min_u \frac{\lambda}{2}\|u-u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mathscr{H}u-h\|^2_{L^2(\mathscr{D})}$$

subject to the PDE constraints

$$\mathscr{L}u(x,t) = f(x,t), \quad x\in\Omega, \quad t\in\mathscr{T},$$

where u is the continuous function to be reconstructed, $\mathscr{H}$ is the observation operator, h is the observation data, λ is the regularization parameter, and u_0 is the prior solution. The space domain is denoted as Ω, the time domain as $\mathscr{T}$, and the space-time domain as $\mathscr{D}=\Omega\times\mathscr{T}$.

When observational data are available at only a limited number of spatiotemporal points, the problem becomes one with sparse observations

$$u^* = \arg\min_u \frac{\lambda}{2}\|u-u_0\|^2_{L^2(\Omega)} + \frac{\Delta_x\Delta_t}{2}\sum_{j=1}^{N_t}\sum_{k=1}^{N_s}\left(\mathscr{H}u_{j,k}-h_{j,k}\right)^2,$$

where Δ_x and Δ_t are constants related to the density of spatial and temporal observation points, and $u_{j,k}$ and $h_{j,k}$ represent the solution and observations at discrete time and space points, respectively.

This type of problems are known as data assimilation in numerical forecasting [1]. The concept of data assimilation originated from meteorology [7] to integrate simulation results from weather models with real observations, improving the accuracy of weather forecasts. By the late 20th century, data assimilation had expanded to various fields, including Earth sciences [13], oceanography [17], and hydrology [15]. In these fields, we frequently encounter challenges related to imperfect models, incomplete observations, and data errors. Data assimilation offers an effective method to overcome these challenges by merging observational data with numerical models.

The reconstruction of continuous solutions from sparse observational data is one of the key challenges in data assimilation. In many applications, observational data is often limited and discontinuous, while PDE models require complete solutions across all spatial and temporal domains. For example, in numerical weather prediction (NWP), data may come from only a few weather stations or satellite paths, yet a comprehensive simulation of the atmospheric state demands a continuous solution in three-dimensional space and time [12]. This highlights the necessity of reconstructing high-precision continuous solutions from sparse observational data.

The primary methods of data assimilation include variational data assimilation (3D/4DVar) [20] and ensemble Kalman filter (EnKF) [10, 11], each with its own advantages and disadvantages. 3D/4DVar optimizes a cost function to find the best estimate of the system state, making it effective for integrating continuous observational data. However,