East Asian Journal on Applied Mathematics Vol. **16**, No. 1, pp. 152-167
doi: 10.4208/eajam.2024-193.170325 February 2026

Asymptotic Behavior of Solutions for a Free Boundary Problem with a Monostable Nonlinearity

Jingjing Cai*, Siqi Liu and Li Xu

School of Mathematics and Physics, Shanghai University of Electric Power, Shanghai 200090, China.

Received 20 November 2024; Accepted (in revised version) 17 March 2025.

Abstract. A free boundary problem with Dirichlet boundary conditions is studied. Such problems can be used for describing the spread of chemical substances or biological species, which live in a moving region $[0,h(t)]$. In this case, the free boundary $h(t)$ represents the spreading front. If the density of the substance or the population at the boundary exceeds a threshold value, they will be going to spread outwards. On the other hand, the outside environment may be not very beneficial for spreading and this generates a decay rate. We mainly analyze how the decay rate and threshold value affect the solutions. There is a trichotomy result: the solution is either shrinking, or the transition case, or spreading. Besides, if the decay rate or threshold value is large, only the shrinking happens.

AMS subject classifications: 35R35, 35B40, 35K57
Key words: Free boundary problem, asymptotic behavior, monostable, reaction-diffusion equation.

1. Introduction

Consider the following free boundary problem:

$$
\begin{aligned}
&u_t = u_{xx} + f(u), && t>0, && 0<x<h(t), && (1.1a)\\
&u(t,0)=0, \quad u(t,h(t))=\sigma, && t>0, && && (1.1b)\\
&h'(t) = -u_x(t,h(t)) - \alpha, && t>0, && && (1.1c)\\
&h(0)=h_0, \quad u(0,x)=u_0(x), && && 0\le x\le h_0, && (1.1d)
\end{aligned}
$$

where $x=h(t)$ is a moving boundary, $f:[0,+\infty)\to\mathbb{R}$ a monostable nonlinearity, and $\alpha>0$, $\sigma\in(0,1)$ are given constants. In this paper, we always assume that the function $f\in C^1$ and satisfies the following conditions:

$$
\begin{aligned}
&f(u)>0, \quad u\in(0,1),\\
&f(u)<0, \quad u\in(1,+\infty),\\
&f(0)=f(1)=0, \quad f'(0)>0, \quad f'(1)<0.
\end{aligned} \tag{1.2}
$$

*Corresponding author. *Email address:* `cjjing1983@163.com` (J. Cai)

The initial function u_0 belongs to $\mathscr{X}(h_0)$ for an $h_0 > 0$, where

$$\mathscr{X}(h_0) := \Big\{ \phi \in C^2([0,h_0]) : \phi(0) = 0,\ \phi(h_0) = \sigma,\ \phi(x) \geq (\not\equiv) 0 \ \text{ in } \ (0,h_0) \Big\}.$$

Free boundary problems can describe the spreading of species or chemical substances invading into a new area. The solution $u(t,x)$ represents their density. The moving interval $[0,h(t)]$ is the area the species occupied at time $t > 0$, the free boundary $h(t)$ is moving depending on t, its speed is determined by the gradient of the species at the boundary. There is a resistance force with strength $\alpha > 0$ at the boundary caused by bad environment — cf. [4, 14, 24]. If species or chemical substances exceed a threshold value $\sigma > 0$, they will move outside. We assume $\sigma \in (0,1)$. As noted in [8], species are not too crowded for available resources in the given environment. However, it seems from $u(t,h(t)) = \sigma$ that the species may expand or shrink — i.e. spreading and shrinking of free boundary $h(t)$ is more complicated if $\sigma = 0$. We will analyze such cases and study the long time behavior of solutions.

Note that there are many free boundary problems with $\alpha = \sigma = 0$. If $u(t,0) = 0$ is replaced by $u_x(t,0)$, Du and Lin [9] studied the long time behaviour of solutions for a free boundary problem for $f(u)$ of logistic type. Besides spreading, they proved that vanishing — i.e. when $h(t)$ tends to a positive finite number and u tends to 0, may also happen. Apparently, in describing the spreading of species, free boundary problems are more reasonable than Cauchy problems, which have only hair-trigger result (any nonnegative solutions will spread) for monostable nonlinearity. For monostable nonlinearity, there are many works devoted to the long time behavior of solutions for free boundaries — cf. Refs. [2–4, 10, 12, 16, 18, 21]. Gu and Lou [17] added advection term βu_x and obtained different results, while Gu *et al.* [16] estimated the spreading speeds of the free boundaries. When nonlinearity $f(u)$ is bistable, Du and Lou [10] studied the Eq. (1.1a) with free boundaries

$$u\big(t,g(t)\big) = u\big(t,h(t)\big) = 0, \quad g'(t) = -\mu u\big(t,g(t)\big), \quad h'(t) = -\mu u\big(t,h(t)\big).$$

They obtained a trichotomy result — i.e. besides spreading and vanishing, there is also a transition case when $h(t) \to +\infty$ and u converges to a stationary solution defined on $\mathbb{R}$ as $t \to +\infty$. In [11], the authors answered several interesting questions left in [10], and completed the general theory for one-dimensional nonlinear free boundary problems. There are also some other free boundary problems with bistable nonlinearity, such as [17, 19], they obtained transition cases. Sharp estimates for the spreading speeds when spreading happens are obtained in [10, 12].

For $\sigma = 0$ and $\alpha > 0$, a free boundary problem for Fisher-KPP equation was studied in [2–4, 7]. Using the parameter α, the authors determined decay rate at the boundary, where species have growth resistance caused by bad environment outside of the existence interval (or say, the boundary). From their results, it is more difficult to spread for the solution when $\alpha > 0$ than $\alpha = 0$, and $h'(t) > 0$ if and only if $-u_x(t,h(t)) > \alpha$. They obtained different asymptotic behavior of solutions, especially the transition and vanishing cases.