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Identifying the Order and a Space Source Term in a Time Fractional Diffusion-Wave Equation

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Abstract. This paper is devoted to identifying the order of time fractional derivative and a space-dependent source term in a time fractional diffusion-wave equation from some additional measured data in a subdomain or on a subboundary with a small time period. The Lipschitz continuity of forward operators mapping the unknown order and source term into the given data are established based on the stability estimates of solution for the direct problem. We prove the uniqueness of the considered inverse problems by using the asymptotic behavior of the solution at $t = 0$, the Titchmarsh convolution theorem and the Duhamel principle. Moreover, a Tikhonov-type regularization method is proposed with H^1-norm as a penalty term. The existence of the regularized solution and its convergence to the exact solution under a suitable regularization parameter choice are obtained. Then we employ a linearized iteration algorithm combined with the piecewise linear finite element approximation to find simultaneously the approximate order and space source term. Three numerical examples for one- and two-dimensional cases are tested and the numerical results demonstrate the effectiveness of the proposed method.

AMS subject classifications: 65M10, 78A48

Key words: Time fractional diffusion-wave equation, order and source, uniqueness, numerical method.

1. Introduction

In the last few decades, fractional partial differential equations have attracted wide attentions due to their non-local properties which enable more accurate descriptions of complex physical and mechanical processes with historical memory and spatial correlation. The time-fractional diffusion equations generalize the classical diffusion equation by replacing the first-order time derivative with a fractional-order derivative of order $\alpha \in (0,1)$. This modification captures anomalous subdiffusion phenomena observed in complex media, such as porous materials, biological tissues, and turbulent fluids [3, 29, 36]. The time-fractional diffusion-wave equations are deduced by replacing the standard second-order

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time derivative with a time-fractional derivative of order $\alpha \in (1,2)$ which can be used to describes anomalous superdiffusion process with a faster rate such as the propagation of stress waves in viscoelastic solids and wave propagation in viscoelastic material bio-engineering [31]. There have a lots of literatures for the initial-boundary value problems of time-fractional equations on the well-posedness of solution [2, 21, 34] and numerical methods [5, 10, 12, 16, 38]. Only a very few references are listed here.

Inverse problems for time-fractional partial differential equations aim to recover unknown parameters (e.g., fractional order, diffusion coefficient, potential coefficient) or initial/boundary conditions and source function from additional observation data. In recent decades, some linear inverse problems for time-fractional diffusion or diffusion-wave equations such as inverse initial value problem and inverse source problem have been investigated widely [11, 35, 43, 45, 51]. Furthermore, some nonlinear inverse problems for fractional partial differential equations were studied in [37, 41, 42, 49, 50].

In time-fractional partial differential equations, the order of time-fractional derivative is a power of time about mean square displacement of particles, which is unknown exactly and need to be determined before further applications. Sometimes, other parameters such as initial value or source function or diffusion coefficient are also known, it is required to identify them simultaneously. Such inverse problems are more difficult to be solved since the mixed influence of multi-parameters. For subdiffusion equations, the inverse problems of recovering simultaneously the fractional order and diffusion coefficient or potential coefficient have been investigated in [4, 6, 17, 23]. For superdiffusion equations, one can see [19, 20, 40, 44, 47] for recent researches on the simultaneous identifications of fractional order and diffusion coefficient or time potential coefficient. However, there are only a few studies determining the source $f(x)$ and the order α simultaneously. Ruan *et al.* [33] use a single point measured data for one-dimensional case and use two sets of data including the integral data as well as the final time data for two-dimensional case to recover the source $f(x)$ and the order only for a time-fractional diffusion equation in which the used observation data seem to be too much. In [24], Li and Zhang consider a special piecewise source function and by using the boundary additional data to obtain a uniqueness for recovering the order and special source, but no numerical method is proposed. Other kinds of papers to determine the source $f(x)$ and the order by using the final time data, one can see the references [25, 48]. In this paper, we investigate two simultaneous determination problem of the order and space-source for a time-fractional diffusion-wave equation by using two-kinds of measurement data which are different from [24, 33]. All the uniqueness proof and numerical method can be easily applied to time-fractional diffusion equations with a slight modification of the solution regularities. Usually, inverse problems for time-fractional diffusion-wave equations are more complicated than the corresponding ones of time-fractional diffusion equations since the Mittag-Leffler functions for $1 < \alpha < 2$ do not keep a positive or a negative property and there is no a maximum principle for the former.

The mathematical modelling of inverse problem investigated in this paper is as follows:

$$\partial_t^\alpha u(x,t) + Au(x,t) = f(x)p(t), \quad x \in \Omega, \quad t \in (0,T], \tag{1.1a}$$

$$\partial_\nu u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T], \tag{1.1b}$$