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Identifying the Order and a Space Source Term in a Time Fractional Diffusion-Wave Equation

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Abstract. This paper is devoted to identifying the order of time fractional derivative and a space-dependent source term in a time fractional diffusion-wave equation from some additional measured data in a subdomain or on a subboundary with a small time period. The Lipschitz continuity of forward operators mapping the unknown order and source term into the given data are established based on the stability estimates of solution for the direct problem. We prove the uniqueness of the considered inverse problems by using the asymptotic behavior of the solution at $t = 0$, the Titchmarsh convolution theorem and the Duhamel principle. Moreover, a Tikhonov-type regularization method is proposed with H^1-norm as a penalty term. The existence of the regularized solution and its convergence to the exact solution under a suitable regularization parameter choice are obtained. Then we employ a linearized iteration algorithm combined with the piecewise linear finite element approximation to find simultaneously the approximate order and space source term. Three numerical examples for one- and two-dimensional cases are tested and the numerical results demonstrate the effectiveness of the proposed method.

AMS subject classifications: 65M10, 78A48

Key words: Time fractional diffusion-wave equation, order and source, uniqueness, numerical method.

1. Introduction

In the last few decades, fractional partial differential equations have attracted wide attentions due to their non-local properties which enable more accurate descriptions of complex physical and mechanical processes with historical memory and spatial correlation. The time-fractional diffusion equations generalize the classical diffusion equation by replacing the first-order time derivative with a fractional-order derivative of order $\alpha \in (0,1)$. This modification captures anomalous subdiffusion phenomena observed in complex media, such as porous materials, biological tissues, and turbulent fluids [3,29,36]. The time-fractional diffusion-wave equations are deduced by replacing the standard second-order

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time derivative with a time-fractional derivative of order $\alpha \in (1,2)$ which can be used to describes anomalous superdiffusion process with a faster rate such as the propagation of stress waves in viscoelastic solids and wave propagation in viscoelastic material bio-engineering [31]. There have a lots of literatures for the initial-boundary value problems of time-fractional equations on the well-posedness of solution [2, 21, 34] and numerical methods [5, 10, 12, 16, 38]. Only a very few references are listed here.

Inverse problems for time-fractional partial differential equations aim to recover unknown parameters (e.g., fractional order, diffusion coefficient, potential coefficient) or initial/boundary conditions and source function from additional observation data. In recent decades, some linear inverse problems for time-fractional diffusion or diffusion-wave equations such as inverse initial value problem and inverse source problem have been investigated widely [11, 35, 43, 45, 51]. Furthermore, some nonlinear inverse problems for fractional partial differential equations were studied in [37, 41, 42, 49, 50].

In time-fractional partial differential equations, the order of time-fractional derivative is a power of time about mean square displacement of particles, which is unknown exactly and need to be determined before further applications. Sometimes, other parameters such as initial value or source function or diffusion coefficient are also known, it is required to identify them simultaneously. Such inverse problems are more difficult to be solved since the mixed influence of multi-parameters. For subdiffusion equations, the inverse problems of recovering simultaneously the fractional order and diffusion coefficient or potential coefficient have been investigated in [4, 6, 17, 23]. For superdiffusion equations, one can see [19, 20, 40, 44, 47] for recent researches on the simultaneous identifications of fractional order and diffusion coefficient or time potential coefficient. However, there are only a few studies determining the source $f(x)$ and the order α simultaneously. Ruan *et al.* [33] use a single point measured data for one-dimensional case and use two sets of data including the integral data as well as the final time data for two-dimensional case to recover the source $f(x)$ and the order only for a time-fractional diffusion equation in which the used observation data seem to be too much. In [24], Li and Zhang consider a special piecewise source function and by using the boundary additional data to obtain a uniqueness for recovering the order and special source, but no numerical method is proposed. Other kinds of papers to determine the source $f(x)$ and the order by using the final time data, one can see the references [25, 48]. In this paper, we investigate two simultaneous determination problem of the order and space-source for a time-fractional diffusion-wave equation by using two-kinds of measurement data which are different from [24, 33]. All the uniqueness proof and numerical method can be easily applied to time-fractional diffusion equations with a slight modification of the solution regularities. Usually, inverse problems for time-fractional diffusion-wave equations are more complicated than the corresponding ones of time-fractional diffusion equations since the Mittag-Leffler functions for $1 < \alpha < 2$ do not keep a positive or a negative property and there is no a maximum principle for the former.

The mathematical modelling of inverse problem investigated in this paper is as follows:

$$\partial_t^\alpha u(x,t) + Au(x,t) = f(x)p(t), \quad x \in \Omega, \quad t \in (0,T], \tag{1.1a}$$

$$\partial_\nu u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T], \tag{1.1b}$$

$$u(x,0)=\phi(x), \qquad x\in\bar{\Omega}, \tag{1.1c}$$

$$u_t(x,0)=\psi(x), \qquad x\in\bar{\Omega}, \tag{1.1d}$$

where $\Omega\subset\mathbb{R}^d$, $1\le d\le 3$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, and ∂_t^α denotes the Caputo fractional left-sided derivative of order α defined by

$$\partial_t^\alpha u(x,t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t(t-\tau)^{1-\alpha}u_{\tau\tau}(x,\tau)\,\mathrm{d}\tau, \quad 0<t\le T, \quad 1<\alpha<2, \tag{1.2}$$

the second order elliptic operator A is defined by

$$Au(x,t)=-\sum_{i,j=1}^d \partial_{x_i}\big(a_{ij}(x)\partial_{x_j}u(x,t)\big)+c(x)u(x,t).$$

The boundary condition is

$$\partial_\nu u(x,t)=\sum_{i,j=1}^d a_{ij}(x)\partial_{x_j}u(x,t)\nu_i(x), \quad x\in\partial\Omega$$

with the unit outward normal vector $\vec{\nu}=(\nu_1(x),\ldots,\nu_d(x))^\top$ at $x\in\partial\Omega$, and the coefficients satisfying the conditions

$$\begin{aligned}
&a_{ij}(x)=a_{ji}(x)\in C^\infty(\bar{\Omega}), && 1\le i,j\le d,\\
&\sum_{i,j=1}^d a_{i,j}(x)\xi_i\xi_j\ge a_0\sum_{j=1}^d\xi_j^2, && x\in\bar{\Omega}, \quad \xi=(\xi_1,\ldots,\xi_d)\in\mathbb{R}^d, \quad a_0>0,\\
&c(x)\in C^\infty(\bar{\Omega}), \quad c(x)\ge c_0>0, \quad x\in\bar{\Omega}.
\end{aligned}$$

If the functions ϕ,ψ,f,p and the order α are known, the problem (1.1) is a well-posed direct problem. It can be used to describe the propagation of mechanical waves in viscoelastic media, see [27, 28] for the background of model. However, in real applications the order of time-fractional derivative α is unknown and sometimes the space-dependent source term $f(x)$ in the Eq. (1.1) is also difficult to be measured, while the time-dependent component $p(t)$ can be given in advance. For instance, in the diffusion of harmful radioactive contaminants in soil, we can suppose $p(t)=e^{-\lambda t}$ with $\lambda>0$ as a half-life period [45] and in the underground pipeline leakage monitoring, the time-dependent term $p(t)$ is controlled by external devices and can be directly recorded using pressure sensors or voltage, meanwhile the pipeline leakage location $f(x)$ is unknown and cannot be directly observed, it must be estimated through indirect measurements. The proposed methodology employs real-time pressure signal monitoring $p(t)$, computes time delays via cross-correlation techniques, and subsequently determines the leakage location by incorporating known wave propagation velocities [32]. Therefore, it is required to identify simultaneously the order α and the source term $f(x)$.

This paper is devoted to two inverse problems for identifying order α and source term $f(x)$ in problem (1.1) by using two kinds of additional conditions:

(P_1) By an additional condition $u(x,t) = g(x,t),\ x \in \omega,\ t \in [0,T_0]$, $0 < T_0 < T$, determine α and $f(x)$, where $\omega \subset \Omega$ is a non-empty subdomain if no other specified.

(P_2) By an additional condition $u(x,t) = g(x,t),\ x \in \Gamma \subset \partial\Omega,\ t \in [0,T_0]$, $0 < T_0 < T$, determine α and $f(x)$ where $\Gamma \subset \partial\Omega$ is a non-empty subboundary.

There are works on the identifications of space-source components while the order is known. Jiang *et al.* [15] obtained the uniqueness of space-source by using additional data on a subboundary for subdiffusion case based on a new unique continuation result. For supperdiffusion case, Yan *et al.* [46] proved the uniqueness of inverse space-source component by using the data on a part of boundary and propose a numerical method to find an approximate solution. Yamamoto [45] showed the uniqueness of space-source or time-source by using additional data over a subdomain or on a subboundary. Cheng *et al.* [7] got the uniqueness result of space-source for a general second-order elliptic operator with a convection term. We note that the order of fractional derivative can be determined by the data over a subdomain or on a subboundary with a small time period simultaneously. Therefore, we carry out this paper. Compared to a single source inversion, the simultaneous inversion for the order and source is more difficult since the problem becomes a nonlinear ill-posed problem. The main contributions in this paper include: we prove the uniqueness for recovering the order and space-source term by additional data over a subdomain or on a subboundary with a small time period based on the unique continuation theorems for elliptic problems as well as propose a regularization method and give a simple iterative algorithm. The numerical results are new issued.

In this paper, the Lipschitz continuity of the forward operators are obtained based on the solution properties for the direct problem. We also obtain the uniqueness for the considered inverse problems. Moreover, we propose a Tikhonov-type regularization method for solving the inverse problems and prove the existence of the regularized solution and its convergence to the exact solution. Then we use a linearized iterative method combined with the piecewise linear finite element approximation for solving the inverse problems (P_1) and (P_2). Numerical examples for one- and two-dimensional cases are provided to show the effectiveness of the method used.

This paper is organized as follows. In Section 2, we present the regularities of a weak solution for the direct problem. In Section 3, we prove the Lipschitz continuity of the forward operators. In Section 4, we prove the uniqueness of the inverse fractional order and the space-dependent source term. In Section 5, a Tikhonov-type regularization method is proposed. The existence and convergence of the regularized solution are obtained. In Section 6, we use a linearized iterative algorithm to solve the regularization problem in which the finite element method is used to solve the well-posed direct problem and the sensitive problem. The numerical results for two examples in one-dimensional case and one example in two-dimensional case are provided to illustrate the efficiency of our method in Section 7. Finally, we give a brief conclusion in Section 8.

2. Regularities of Weak Solution for Direct Problem

In this paper, denote the inner product and norm in $L^2(\Omega)$ by $(\cdot,\cdot)$ and $\|\cdot\|$. Let $H^s(\Omega)$, $s\in\mathbb{R}$ be the standard Sobolev space [1]. Denote $D(A) := \{u\in H^2(\Omega) \mid \partial_\nu u = 0 \text{ on } \partial\Omega\}$. Let $\{\lambda_n,\varphi_n\}_{n=1}^\infty$ be the eigensystem of A in $D(A)$ such that $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\cdots$. It is known from the spectral theory of the self-adjoint elliptic operators, that $\lambda_n=\mathscr{O}(n^{2/d})$, and $\{\varphi_n\}_{n=1}^\infty\subset D(A)$ can be assumed as an orthonormal basis of $L^2(\Omega)$.

Define Hilbert scale $D(A^\gamma)$ for $\gamma\ge 0$ by

$$D(A^\gamma)=\left\{\Psi\in L^2(\Omega),\ \sum_{n=1}^\infty \lambda_n^{2\gamma}|(\Psi,\varphi_n)|^2<\infty\right\},$$

the fractional power of operator A as

$$A^\gamma\Psi=\sum_{n=1}^\infty \lambda_n^\gamma(\Psi,\varphi_n)\varphi_n,\quad \Psi\in D(A^\gamma),$$

and the norm $\|\Psi\|_{D(A^\gamma)}=\|A^\gamma\Psi\|_{L^2(\Omega)}$. It is known [39] that $D(A^\gamma)\subset H^{2\gamma}(\Omega)$, and

$$C_1\|\Psi\|_{H^{2\gamma}(\Omega)}\le\|\Psi\|_{D(A^\gamma)}\le C_2\|\Psi\|_{H^{2\gamma}(\Omega)},\quad \Psi\in D(A^\gamma),\quad 0\le\gamma\le 1,\quad \gamma\ne 1/4,3/4.$$

Throughout this paper, $AC[0,T]$ refers to the space of absolutely continuous functions on $[0,T]$, and

$$AC^n[0,T]:=\left\{z(t)\mid z\in C^{n-1}[0,T],\ z^{(n-1)}(t)\in AC[0,T]\right\},\quad n\ge 2.$$

Besides, C means a positive constant which can be different at different appearances.

According to the Fourier method [34], the weak solution of the direct problem (1.1) has the form

$$u(x,t)=\mathscr{S}_1(t)\phi+\mathscr{S}_2(t)\psi+\mathscr{S}_3(t)f \tag{2.1}$$

with the operators $\mathscr{S}_i$, $i=1,2,3$ defined by

$$\begin{aligned}
\mathscr{S}_1(t)\phi &:= \sum_{n=1}^\infty(\phi,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x),\\
\mathscr{S}_2(t)\psi &:= \sum_{n=1}^\infty(\psi,\varphi_n)tE_{\alpha,2}(-\lambda_n t^\alpha)\varphi_n(x),\\
\mathscr{S}_3(t)f &:= \sum_{n=1}^\infty(f,\varphi_n)p_n(t)\varphi_n(x),
\end{aligned}$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^\infty z^k/\Gamma(\alpha k+\beta),\quad z\in\mathbb{C},\quad \alpha>0,\quad \beta\in\mathbb{R},$$

$$p_n(t)=\int_0^t p(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n(t-\tau)^{\alpha-1}\big)d\tau.$$

If the initial values ϕ, ψ, the source function f and the source term p hold a certain regularity, the existence, uniqueness and regularity of the weak solution u of the direct problem (1.1) can be found in [34]. In order to get the continuity of forward operators and the uniqueness of inverse problems considered in this paper, we need some more strong regularities to the weak solution of the direct problem (1.1). We review them in the following lemmas, the proofs are not difficult and can be found in [34, 40].

Lemma 2.1. *If $\phi(x) \in L^2(\Omega)$, $\psi(x) \in D(A^{-1/\alpha})$ and $f(x) \in L^2(\Omega), p(t) \in L^1(0,T)$, then the weak solution $u \in C([0,T]; L^2(\Omega))$ for problem* (1.1) *satisfies the following stability estimate:*

$$\|u\|_{C([0,T];L^2(\Omega))} \leq C\left(\|\phi\|_{L^2(\Omega)} + \|\psi\|_{D(A^{-1/\alpha})} + \|f\|_{L^2(\Omega)}\|p\|_{L^1(0,T)}\right).$$

Moreover, if $0 \leq \gamma \leq 1$, $\phi(x) \in D(A^\gamma)$, $\psi(x) \in D(A^{\gamma-1/\alpha})$ and $f(x) \in L^2(\Omega)$, $p(t) \in AC[0,T]$, then the weak solution $u \in C([0,T]; D(A^\gamma))$ of the problem (1.1) *satisfies the following stability estimate:*

$$\|u\|_{C([0,T];D(A^\gamma))} \leq C\left(\|\phi\|_{D(A^\gamma)} + \|\psi\|_{D(A^{\gamma-1/\alpha})} + \|f\|_{L^2(\Omega)}(\|p\|_{C[0,T]} + \|p'\|_{L^1(0,T)})\right).$$

If $\gamma \geq 0$, $\phi \in D(A^{\gamma+1/\alpha+\epsilon})$, $\psi \in D(A^{\gamma+\epsilon})$ for sufficiently small $\epsilon > 0$, $p(t) \in AC[0,T]$, and $f(x) \in D(A^\gamma)$, then the weak solution of the direct problem (1.1) *has the regularity $u \in AC^2([0,T]; D(A^\gamma))$ and there are constants C_1, C_2, C_3 such that*

$$\begin{aligned}
&\|u\|_{C([0,T];D(A^\gamma))} \leq C_1\left(\|\phi\|_{D(A^\gamma)} + \|\psi\|_{D(A^{\gamma-1/\alpha})} + \|f\|_{D(A^{\gamma-1})}(\|p\|_{C[0,T]} + \|p'\|_{L^1(0,T)})\right),\\
&\|\partial_t u\|_{C([0,T];D(A^\gamma))} \leq C_2\left(\|\phi\|_{D(A^{\gamma+1/\alpha})} + \|\psi\|_{D(A^\gamma)} + \|f\|_{D(A^{\gamma-1+1/\alpha})}(\|p\|_{C[0,T]} + \|p'\|_{L^1(0,T)})\right),\\
&\|\partial_{tt} u\|_{L^1((0,T);D(A^\gamma))} \leq C_3\left(\|\phi\|_{D(A^{\gamma+1/\alpha+\epsilon})} + \|\psi\|_{D(A^{\gamma+\epsilon})} + \|f\|_{D(A^\gamma)}(\|p\|_{C[0,T]} + \|p'\|_{L^1(0,T)})\right).
\end{aligned}$$

For general source functions $s(x,t)$ in (1.1), the following lemma hold true.

Lemma 2.2. *If $1 < \alpha < 2$, $\gamma \geq 0$, $\phi(x) \in D(A^\gamma)$, $\psi(x) \in D(A^{\gamma-1/\alpha})$ and $s(x,t) \in L^1(0,T); D(A^\gamma))$, then the weak solution $u \in L^\infty((0,T); D(A^\gamma))$ satisfies the following stability estimate:*

$$\|u\|_{L^\infty((0,T);D(A^\gamma))} \leq C\left(\|\phi\|_{D(A^\gamma)} + \|\psi\|_{D(A^{\gamma-1/\alpha})} + \|s\|_{L^1((0,T);D(A^\gamma))}\right).$$

Lemma 2.3 (cf. Liao *et al.* [26]). *For any $\underline{\beta}, \overline{\beta}$ such that $1 < \underline{\beta} \leq \beta \leq \overline{\beta} < 2$, $\beta - 1 \leq \eta \leq 2$, we have*

$$|E_{\beta,\eta}(-x)| \leq \frac{C_0}{1+x}, \quad x > 0,$$

where the constant C_0 depends on $\underline{\beta}, \overline{\beta}$ only.

Remark 2.1. Based on the uniform upper bound estimate for the Mittag-Leffler function in Lemma 2.3, if $\alpha \in [\underline{\alpha}, \overline{\alpha}] \subset (1,2)$, then the constants C in Lemmas 2.1-2.2 depend only on $\underline{\alpha}$, $\overline{\alpha}$. This point is important for proving the local Lipschitz continuity of forward operators.

3. Lipschitz Continuity of Forward Operators

In this paper, we consider two kinds of inverse problems (P_1) and (P_2) for recovering the fractional order α and the source term $f(x)$ simultaneously. If the initial values $\phi(x)$, $\psi(x)$ and source $f(x)$, $p(t)$ satisfy certain conditions, the Lipschitz continuity of the forward operators mapping the order and source into the given data can be obtained. At first, we give the definitions of the forward operators respectively.

For the problem (P_1), let $\phi \in L^2(\Omega)$, $\psi \in D(A^{-1/\alpha})$, $p \in L^1(0,T)$. We define the forward operator by

$$\mathscr{F} : (\alpha, f) \in D(\mathscr{F}) \longmapsto u(x,t)|_{\omega\times[0,T_0]} \in L^2\big(0,T_0; L^2(\omega)\big), \tag{3.1}$$

where

$$D(\mathscr{F}) = \left\{(\alpha, f) \mid \alpha \in (1,2),\ f \in L^2(\Omega)\right\},$$

and $u(x,t)$ is the weak solution (2.1) of problem (1.1) for given α and f. By Lemma 2.1, it can be seen the operator $\mathscr{F}$ is well-defined. Thus, the inverse problem is to solve $\mathscr{F}(\alpha, f) = g$ where $g = u(x,t)$, $x \in \omega$, $t \in [0, T_0]$.

For the problem (P_2), suppose $\phi \in D(A^{1/4})$, $\psi \in D(A^{1/4-1/\alpha})$, $p \in AC[0,T]$. Define the forward operator

$$\mathscr{F} : (\alpha, f) \in D(\mathscr{F}) \longmapsto u(x,t)|_{\Gamma\times[0,T_0]} \in L^2\big(0,T_0; L^2(\Gamma)\big), \tag{3.2}$$

where

$$D(\mathscr{F}) = \left\{(\alpha, f) \mid \alpha \in (1,2),\ f \in L^2(\Omega)\right\},$$

and $u(x,t)$ is the solution of problem (1.1) for given α and f. By Lemma 2.1, the solution $u(x,t)$ of problem (1.1) satisfies $u \in C([0,T_0]; D(A^{1/4}))$. Since $D(A^{1/4}) \subset H^{1/2}(\Omega)$, its trace is in $L^2(\partial\Omega)$. Therefore, the forward operator $\mathscr{F}$ defined in (3.2) is well-defined. In this case the inverse problem consists in finding $\mathscr{F}(\alpha, f) = g$ where $g = u(x,t)$, $x \in \Gamma$, $t \in [0, T_0]$.

In the following theorem we present the Lipschitz continuity of the forward operator $\mathscr{F}$ for the problem (P_1).

Theorem 3.1. *Let $\phi(x) \in D(A^{1/\alpha+\epsilon})$, $\psi(x) \in D(A^{\epsilon})$ for sufficiently small $\epsilon > 0$, $p \in AC[0,T]$. Suppose $\alpha_i \in [\underline{\alpha}, \overline{\alpha}] \subset (1,2)$ and $f_i(x) \in L^2(\Omega)$ such that $\|f_i\|_{L^2(\Omega)} \leq M$, $i = 1,2$. Then for the problem (P_1), the following Lipschitz continuity estimate holds:*

$$\|\mathscr{F}(\alpha_1, f_1) - \mathscr{F}(\alpha_2, f_2)\|_{L^2(0,T_0;L^2(\omega))} \leq C\left(|\alpha_1 - \alpha_2| + \|f_1 - f_2\|_{L^2(\Omega)}\right), \tag{3.3}$$

where $C > 0$ is a constant depending only on ϕ, ψ, p and $\underline{\alpha}, \overline{\alpha}$, M.

Proof. Let u_1 and u_2 be the solution of the direct problem (1.1) corresponding to (α_1, f_1) and (α_2, f_2), respectively. Denote $u(x,t) = u_1(x,t) - u_2(x,t)$, $\alpha = \alpha_1 - \alpha_2$, $f = f_1 - f_2$.

Then in terms of the problems satisfied by u_1 and u_2 respectively, it is easy to see that

$$\begin{aligned}
&\partial_{0+}^{\alpha_1}u(x,t)+Au(x,t)\\
&=\partial_{0+}^{\alpha_2}u_2(x,t)-\partial_{0+}^{\alpha_1}u_2(x,t)+f(x)p(t), && x\in\Omega, && t\in(0,T],\\
&\partial_\nu u(x,t)=0, && x\in\partial\Omega, && t\in(0,T],\\
&u(x,0)=0, && x\in\bar{\Omega},\\
&u_t(x,0)=0, && x\in\bar{\Omega}.
\end{aligned} \tag{3.4}$$

According to Lemma 2.2 with $\gamma=0$, the following estimate holds:

$$\begin{aligned}
\|u\|_{L^2(0,T_0;L^2(\omega))} &\le C\|u\|_{L^\infty(0,T;L^2(\Omega))}\\
&\le C\left(\|\partial_{0+}^{\alpha_2}u_2-\partial_{0+}^{\alpha_1}u_2\|_{L^1(0,T;L^2(\Omega))}+\|f\|_{L^2(\Omega)}\|p\|_{L^1(0,T)}\right).
\end{aligned}$$

By (3.1) and the estimate above, we have

$$\begin{aligned}
&\|\mathscr{F}(\alpha_1,f_1)-\mathscr{F}(\alpha_2,f_2)\|_{L^2(0,T_0;L^2(\omega))}\\
&\le C\left(\|\partial_{0+}^{\alpha_2}u_2-\partial_{0+}^{\alpha_1}u_2\|_{L^1(0,T;L^2(\Omega))}+\|p\|_{L^1(0,T)}\|f\|_{L^2(\Omega)}\right).
\end{aligned} \tag{3.5}$$

The first term at the right-hand side of inequality (3.5) is estimated below. Using the Caputo derivative defined in (1.2), we have

$$\partial_{0+}^{\alpha_2}u_2-\partial_{0+}^{\alpha_1}u_2=\int_0^t\left[\frac{1}{\Gamma(2-\alpha_2)}(t-\tau)^{1-\alpha_2}-\frac{1}{\Gamma(2-\alpha_1)}(t-\tau)^{1-\alpha_1}\right]\partial_{\tau\tau}u_2(\cdot,\tau)\,\mathrm{d}\tau.$$

By the result in [14], the following inequality holds:

$$\begin{aligned}
&\left|\frac{1}{\Gamma(2-\alpha_2)}(t-\tau)^{1-\alpha_2}-\frac{1}{\Gamma(2-\alpha_1)}(t-\tau)^{1-\alpha_1}\right|\\
&\le C_0\big(1+|\ln(t-\tau)|\big)\max\big[(t-\tau)^{-(\alpha_1-1)},(t-\tau)^{-(\alpha_2-1)}\big]|\alpha_1-\alpha_2|, \quad t>\tau.
\end{aligned} \tag{3.6}$$

Denote $C_{\sigma,T}:=\max_{t\in[0,T]}(1+|\ln t|)t^\sigma$ for a sufficiently small number $\sigma>0$, then we have $(1+|\ln t|)\le C_{\sigma,T}t^{-\sigma}$, $t\in(0,T]$. Thus, the generalized Minkowski inequality yields

$$\begin{aligned}
&\left\|\partial_{0+}^{\alpha_2}u_2(\cdot,t)-\partial_{0+}^{\alpha_1}u_2(\cdot,t)\right\|_{L^2(\Omega)}\\
&\le C\int_0^t\big[(t-\tau)^{-(\alpha_1-1)-\sigma}+(t-\tau)^{-(\alpha_2-1)-\sigma}\big]\|\partial_{\tau\tau}u_2(\cdot,\tau)\|_{L^2(\Omega)}\,\mathrm{d}\tau|\alpha_1-\alpha_2|.
\end{aligned}$$

Taking $\sigma=1-\overline{\alpha}/2>0$, it is not hard to prove that

$$\|t^{-(\alpha_1-1)-\sigma}\|_{L^1(0,T)},\|t^{-(\alpha_2-1)-\sigma}\|_{L^1(0,T)}\le C(\underline{\alpha},\overline{\alpha},T),$$

further by the Fubini theorem and Lemma 2.1 in which take $\gamma=0$, we have

$$\begin{aligned}
&\left\|\partial_{0+}^{\alpha_2}u_2-\partial_{0+}^{\alpha_1}u_2\right\|_{L^1(0,T;L^2(\Omega))}\\
&\le C(\underline{\alpha},\overline{\alpha},T)\|\partial_{\tau\tau}u_2\|_{L^1(0,T;L^2(\Omega))}|\alpha_1-\alpha_2|\\
&\le C|\alpha_1-\alpha_2|.
\end{aligned} \tag{3.7}$$

Therefore, substituting (3.7) into (3.5), the estimate (3.3) is satisfied. □

In what follows, we show the Lipschitz continuity of the forward operator for the problem (P_2).

Theorem 3.2. *Let $\phi(x) \in D(A^{1/4+1/\alpha+\epsilon})$, $\psi(x) \in D(A^{1/4+\epsilon})$ for sufficiently small $\epsilon > 0$, $p \in AC[0,T]$. Suppose for $i = 1,2$, $\alpha_i \in [\underline{\alpha},\overline{\alpha}] \subset (1,2)$ and $f_i(x) \in D(A^{1/4})$ such that $\|f_i\|_{D(A^{1/4})} \le M$. Then for the problem ($P_2$), the following Lipschitz continuity estimate holds:*

$$\|\mathscr{F}(\alpha_1,f_1)-\mathscr{F}(\alpha_2,f_2)\|_{L^2(0,T_0;L^2(\Gamma))} \le C\big(|\alpha_1-\alpha_2|+\|f_1-f_2\|_{L^2(\Omega)}\big),$$

where $C > 0$ is a constant depending only on ϕ, ψ, p and $\underline{\alpha},\overline{\alpha}$, M.

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be the solution of the direct problem (1.1) corresponding to (α_1,f_1) and (α_2,f_2) respectively. Denote $u(x,t) = u_1(x,t)-u_2(x,t)$, $\alpha = \alpha_1-\alpha_2$ and $f = f_1(x)-f_2(x)$. The u satisfies problem (3.4), by (3.2), the trace theorem and Lemmas 2.1-2.2 in which taking $\gamma = 1/4$, we have

$$\begin{aligned}
&\|\mathscr{F}(\alpha_1,f_1)-\mathscr{F}(\alpha_2,f_2)\|_{L^2(0,T_0;L^2(\Gamma))}\\
&\le C\|u\|_{L^\infty(0,T;D(A^{1/4}))}\\
&\le C\left(\|\partial_{0+}^{\alpha_2}u_2-\partial_{0+}^{\alpha_1}u_2\|_{L^1(0,T;D(A^{1/4}))}+\|f\|_{L^2(\Omega)}(\|p\|_{C[0,T]}+\|p'\|_{L^1(0,T)})\right).
\end{aligned}$$

We estimate the first term in the right-hand side. Using the Caputo derivative defined in (1.2) and (3.6), we have

$$\begin{aligned}
&\left\|\partial_{0+}^{\alpha_2}u_2(\cdot,t)-\partial_{0+}^{\alpha_1}u_2(\cdot,t)\right\|_{D(A^{1/4})}\\
&\le C\int_0^t\left[(t-\tau)^{-(\alpha_1-1)-\sigma}+(t-\tau)^{-(\alpha_2-1)-\sigma}\right]\|\partial_{\tau\tau}u_2(\cdot,\tau)\|_{D(A^{1/4})}\,\mathrm{d}\tau|\alpha_1-\alpha_2|.
\end{aligned}$$

Moreover, by Lemma 2.1 with $\gamma = 1/4$ and the same proof of (3.7), we know

$$\begin{aligned}
&\|\mathscr{F}(\alpha_1,f_1)-\mathscr{F}(\alpha_2,f_2)\|_{L^2(0,T_0;L^2(\Gamma))}\\
&\le C\left(\|\partial_{\tau\tau}u_2\|_{L^1(0,T;D(A^{1/4}))}|\alpha_1-\alpha_2|+\|f\|_{L^2(\Omega)}(\|p\|_{C[0,T]}+\|p'\|_{L^1(0,T)})\right)\\
&\le C\left(|\alpha_1-\alpha_2|+\|f_1-f_2\|_{L^2(\Omega)}\right),
\end{aligned}$$

where $C > 0$ is a constant depending only to ϕ, ψ, p and $\underline{\alpha}$, $\overline{\alpha}$, M. □

4. Uniqueness for Inverse Problems

In this section, we prove the uniqueness of the fractional order α and the source term $f(x)$ in problems (P_1) and (P_2) through the following lemma and theorem.

Lemma 4.1 (Titchmarsh Convolution Theorem, cf. Doss [9]). *Let $f(t)$, $g(t)$ be two integrable functions on $(0,T)$, and vanish on $(-\infty,0)$. Suppose that*

$$\int_0^t f(t-\tau)g(\tau)d\tau = 0, \quad t \in (0,T).$$

Then there are nonnegative numbers α,β with $\alpha+\beta \ge T$ such that $f(t) = 0$ for almost all $t \in (0,\alpha)$ and $g(t) = 0$ for almost $t \in (0,\beta)$.

In what follows, we consider firstly the uniqueness of the source term $f(x)$ if the order is given. Let u and v be the solutions of the following direct problems:

$$\begin{array}{lll} \partial_{0+}^{\alpha}u(x,t)+Au(x,t)=f(x)p(t), & x\in\Omega, & 0<t\le T,\\ u(x,0)=0, & x\in\bar{\Omega}, & \\ \partial_t u(x,0)=0, & x\in\bar{\Omega}, & \\ \partial_\nu u(x,t)=0, & x\in\partial\Omega, & 0<t\le T, \end{array} \tag{4.1}$$

$$\begin{array}{lll} \partial_{0+}^{\alpha}v(x,t)+Av(x,t)=0, & x\in\Omega, & 0<t\le T,\\ v(x,0)=f(x), & x\in\bar{\Omega}, & \\ \partial_t v(x,0)=0, & x\in\bar{\Omega}, & \\ \partial_\nu v(x,t)=0, & x\in\partial\Omega, & 0<t\le T. \end{array} \tag{4.2}$$

Lemma 4.2 (Duhamel Principle). *Let $u(x,t)$ be the solution of* (4.1) *in which $p(t)\in AC[0,T]$ and $f(x)\in D(A^{1/\alpha+\epsilon})$. Then the solution $u(x,t)$ has the representation*

$$u(\cdot,t)=\int_0^t \mu_p(t-s)v(\cdot,s)ds,\quad 0<t\le T,$$

where v is the solution of problem (4.2) *with $f(x)$ as the initial data and $\mu_p(t)$*

$$\mu_p(t)=\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t p(s)(t-s)^{\alpha-1}ds=\frac{1}{\Gamma(\alpha-1)}\int_0^t p(s)(t-s)^{\alpha-2}ds.$$

Proof. The proof can be completed by a small modification in [46] such that the condition to p is weaken to $p\in AC[0,T]$ from $p\in C^1[0,T]$ and $f\in D(A^{1/\alpha+\epsilon})$ instead of $f\in D(A)$. □

Lemma 4.3. *Assume that $f\in D(A^{d/2-1/\alpha+\epsilon})$ for sufficiently small $\epsilon>0$ and v is the solution of* (4.2). *If $v(x,t)=0$ for $0<t\le T_0<\infty$, $x\in\omega\subset\Omega$, or $x\in\Gamma\subset\partial\Omega$, then $f(x)=0$ in $L^2(\Omega)$.*

Proof. According to (2.1), the solution $v(x,t)$ is has the form

$$v(x,t)=\sum_{n=1}^{\infty}(f,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x),\quad x\in\Omega,\quad t\in(0,T). \tag{4.3}$$

Since the Mittag-Leffler function $E_{\alpha,\beta}(-\lambda z)$ is an entire function, we know that $E_{\alpha,1}(-\lambda_n z^\alpha)$ is analytic over the domain

$$S_1=\left\{z\in\mathbb{C}|\ |z|\ge t_0,-\pi<\arg(z)\le\pi\right\},$$

$t_0>0$ is any fixed number. Restrict $E_{\alpha,1}(-\lambda_n z^\alpha)$ to the sector domain

$$S_2=\left\{z\in\mathbb{C}|\ |\arg(z)|<\frac{(\pi-\mu)}{\alpha}\right\}$$

with $\alpha\pi/2 < \mu < \pi$ such that

$$|E_{\alpha,1}(-\lambda_n z^\alpha)| \le \frac{C}{1+|\lambda_n z^\alpha|}$$

holds [22].

Next we prove the uniform convergence of series (4.3). In terms of the regularity of φ_n, we know that $\varphi_n \in D(A) \subset H^2(\Omega)$. For $d \le 3$ we have $\varphi_n \in C(\bar{\Omega})$ and

$$\|\varphi_n\|_{C(\bar{\Omega})} \le C\|\varphi_n\|_{H^{d/2+\epsilon}(\Omega)} \le C\|\varphi_n\|_{D(A^{d/4+\epsilon/2})} \le C\lambda_n^{d/4+\epsilon/2}$$

for sufficiently small $\epsilon > 0$. By the Cauchy-Schwarz inequality, noting that $\lambda_n \sim n^{2/d}$, $n \in \mathbb{N}$ [8], for $z \in S_1 \cap S_2$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} |E_{\alpha,1}(-\lambda_n z^\alpha)(f,\varphi_n)\varphi_n(x)| \\
\le& \frac{C}{t_0^\alpha}\sum_{n=1}^{\infty}|(f,\varphi_n)|\lambda_n^{d/4+\epsilon/2-1} \\
\le& \frac{C}{t_0^\alpha}\left(\sum_{n=1}^{\infty}\left(\lambda_n^{d/2-1+\epsilon}(f,\varphi_n)\right)^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{2(d/4+\epsilon/2)}}\right)^{1/2}. \qquad (4.4)
\end{aligned}$$

Note that

$$\frac{1}{\lambda_n^{2(d/4+\epsilon/2)}} \le C\frac{1}{n^{4/d(d/4+\epsilon/2)}}$$

for $n \ge 1$. Then the series above converges if $f \in D(A^{d/2-1+\epsilon})$. Thus,

$$A = \sum_{n=1}^{\infty}(f,\varphi_n)E_{\alpha,1}(-\lambda_n z^\alpha)\varphi_n(x)$$

converges uniformly over $S_1 \cap S_2$ for $x \in \bar{\Omega}$ and is analytic on $S_1 \cap S_2$ for $x \in \bar{\Omega}$. Therefore, the solution $v(x,t)$ can be extended to $\bar{\Omega}\times\{S_1\cap S_2\}$, in particular $v(x,t)$ can be extended to $\bar{\Omega}\times(0,\infty)$ analytically. By the uniqueness of analytical extension, the conditions $v(x,t) = 0$ for $0 < t \le T_0$, $x \in \omega$ or $x \in \Gamma$ imply

$$\sum_{n=1}^{\infty}(f,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x) = 0, \quad t > 0, \quad x \in \omega \quad \text{or} \quad x \in \Gamma. \qquad (4.5)$$

Note that

$$|E_{\alpha,1}(-\lambda_n t^\alpha)| \le \frac{C}{1+|\lambda_n t^\alpha|} \le C(\lambda_n t^\alpha)^{-\varrho}$$

for $0 \le \varrho \le 1$. In what follows, we take $\varrho = 1/\alpha - \epsilon/4$ with sufficiently small $\epsilon > 0$. Then

$$\left|\sum_{n=1}^{\infty}(f,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x)\right| \le C\sum_{n=1}^{\infty}|(f,\varphi_n)|(\lambda_n t^\alpha)^{-(1/\alpha-\epsilon/4)}\lambda_n^{d/4+\epsilon/2}$$

$$\leq Ct^{-1+\alpha\epsilon/4}\left(\sum_{n=1}^{\infty}\left(\lambda_n^{d/2-1/\alpha+\epsilon}(f,\varphi_n)\right)^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{2(d/4+\epsilon/4)}}\right)^{1/2}.$$

Note that

$$\frac{1}{\lambda_n^{2(d/4+\epsilon/4)}}\leq C\frac{1}{n^{4/d(d/4+\epsilon/4)}}$$

for $n\geq 1$. Then the series above converges if $f\in D(A^{d/2-1/\alpha+\epsilon})$.

Since $e^{-tRe(z)}t^{-1+\alpha\epsilon/4}$ is integrable in $t\in(0,\infty)$ for fixed z satisfying $Re(z)>0$, by the Lebesgue convergence theorem, we can take the Laplace transform for (4.5). It is known [22] that

$$\int_0^{\infty}e^{-zt}E_{\alpha,1}(-\lambda_n t^{\alpha})dt=\frac{z^{\alpha-1}}{z^{\alpha}+\lambda_n},\quad Re(z)>|\lambda_n|^{1/\alpha}. \tag{4.6}$$

Due to the boundedness of the Mittag-Leffler function, we know $\int_0^{\infty}e^{-zt}E_{\alpha,1}(-\lambda_n t^{\alpha})dt$ is analytic on $Re(z)>0$. And for $1<\alpha<2$, we can easily obtain that $z^{\alpha}+\lambda_n\neq 0$ on $Re(z)>0$, then both sides of (4.6) can be extended to $Re(z)>0$. Then the Laplace transform of (4.5) yields

$$\sum_{n=1}^{\infty}(f,\varphi_n)\varphi_n(x)\frac{z^{\alpha-1}}{z^{\alpha}+\lambda_n}=0,\quad Re(z)>0,\quad x\in\omega\quad\text{or}\quad x\in\Gamma.$$

Let $\eta=z^{\alpha}$, the equality above deduces

$$\sum_{n=1}^{\infty}(f,\varphi_n)\varphi_n(x)\frac{1}{\eta+\lambda_n}=0,\quad Re(\eta)>0,\quad x\in\omega\quad\text{or}\quad x\in\Gamma. \tag{4.7}$$

Since

$$\sum_{n=1}^{\infty}\left|(f,\varphi_n)\varphi_n(x)\frac{1}{\eta+\lambda_n}\right|\leq C\sum_{n=1}^{\infty}\lambda_n^{d/4+\epsilon/2-1}|(f,\varphi_n)|\left|\frac{\lambda_n}{\eta+\lambda_n}\right|,$$

the above series is internally closed uniformly convergent in $\eta\in\mathbb{C}\backslash\{-\lambda_n\}_{n=1}^{\infty}$ if $f\in D(A^{d/2-1+\epsilon})$ according to the proof of (4.4) and the bounded property of $|\lambda_n/(\eta+\lambda_n)|$ for large n while η is in a bounded domain. Using the Weierstrass theorem, the left side of (4.7) is analytic in $\eta\in\mathbb{C}\backslash\{-\lambda_n\}_{n=1}^{\infty}$. Therefore, the equality (4.7) holds for $\eta\in\mathbb{C}\backslash\{-\lambda_n\}_{n=1}^{\infty}$.

We denote the different λ_n as $0<\mu_1<\mu_2<\cdots$ and $\{\varphi_{n_k}\}_{1\leq k\leq m_n}$ as an orthonormal basis in $\ker(\mu_n I-A)$. Then we can rewrite (4.7) as

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{m_n}(f,\varphi_{n_k})\varphi_{n_k}(x)\right)\frac{1}{\eta+\mu_n}=0,\quad \eta\in\mathbb{C}\backslash\{-\mu_n\}_{n=1}^{\infty},\quad x\in\omega\quad\text{or}\quad x\in\Gamma. \tag{4.8}$$

We can choose a suitable disk which only includes μ_s and does not include $\{\mu_n\}_{n\neq s}$. By the Cauchy integral theorem, integrating (4.8) along the boundary of disk, we have

$$v_s:=\sum_{k=1}^{m_s}(f,\varphi_{s_k})\varphi_{s_k}(x)=0,\quad x\in\omega\quad\text{or}\quad x\in\Gamma.$$

For the case $x \in \Gamma$, since $(A-\mu_s I)v_s = 0$ in Ω, and $v_s = \partial_\nu v_s = 0$ on Γ, the uniqueness of the Cauchy problem for elliptic equations (e.g., see Isakov [13, p. 58, Theorem 3.3.1]) implies $v_s = 0$ in Ω for all $s \in \mathbb{N}$. Combining the linearly independence of $\{\varphi_{s_k}\}_{1\le k\le m_s}$ in Ω, we obtain that $(f, \varphi_{s_k}) = 0$ for $1 \le k \le m_s$, $s \in \mathbb{N}$, then we have $f = 0$ in $L^2(\Omega)$. For the case $x \in \omega$, by the uniqueness continuation of elliptic operators, it is similar to deduce that $f = 0$. The proof is complete. □

In what follows, we prove that the uniqueness of the space-dependent source term for the fractional diffusion wave equation if the order is given.

Lemma 4.4. *Let $f \in D(A^{\tilde{\gamma}+\epsilon})$ with $\tilde{\gamma} = \max\{1/\alpha, d/2-1/\alpha\}$, $p \in AC[0,T]$ and*

$$\frac{1}{\Gamma(\alpha-1)}\int_0^t p(s)(t-s)^{\alpha-2}ds \neq 0, \quad t \in (0,T_0).$$

Let u be the solution of problem (4.1)*, then $u(x,t) = 0$, $0 < t \le T_0$, $x \in \omega$ or $x \in \Gamma$ implies $f(x) = 0$, $x \in \Omega$.*

Proof. Lemma 4.2 yields

$$u(x,t) = \int_0^t \mu_p(t-s)v(x,s)ds = 0, \quad 0 < t \le T_0, \quad x \in \omega \quad \text{or} \quad x \in \Gamma,$$

where v is the solution of problem (4.2) and μ_p is given in Lemma 4.2. Since $\mu_p(t) \neq 0$ on $(0,T_0)$, then using Lemma 4.1 we have $v(x,t) = 0$, $0 < t \le T_0$, $x \in \omega$ or $x \in \Gamma$, thus by Lemma 4.3 we know $f(x) = 0$. □

In the following theorem, we give the uniqueness of recovering the order and source term simultaneously.

Theorem 4.1. *Let $\phi(x) \in D(A^{1+d/2+\epsilon_0})$, $\psi(x) \in D(A^{1+d/2+\epsilon_0-1/\alpha})$ for sufficiently small ϵ_0, $p(t) \in AC[0,T]$. Suppose α, $\tilde{\alpha} \in (1,2)$, $f, \tilde{f} \in D(A^{\gamma+\epsilon})$ with $\gamma = \max\{1/\alpha, d/2\}$ for sufficiently small ϵ and there is a point $x_0 \in \omega$ or $x_0 \in \Gamma$ such that*

$$p(0)f(x_0) - (A\phi)(x_0) \neq 0, \quad p(0)\tilde{f}(x_0) - (A\phi)(x_0) \neq 0.$$

Assume that

$$\int_0^t p(\tau)(t-\tau)^{\alpha-2}d\tau \neq 0 \quad \text{for all} \quad t \in (0,T_0).$$

Let $u(x,t)$, $\tilde{u}(x,t)$ be the solutions of the direct problem (1.1) *corresponding to (α, f) and $(\tilde{\alpha}, \tilde{f})$. If*

$$u(x,t) = \tilde{u}(x,t), \quad t \in (0,T_0), \quad x \in \omega \quad \text{or} \quad x \in \Gamma,$$

then $\alpha = \tilde{\alpha}$, $f(x) = \tilde{f}(x)$, $x \in \Omega$.

Proof. Noting that the conditions $\phi(x)\in D(A)$, $\psi(x)\in D(A^{1-1/\alpha})$, $p(t)\in AC[0,T]$, $f\in L^2(\Omega)$ hold, by Lemma 2.1, we know $u\in C([0,T];D(A))$, then by the embedding theorem for $d\le 3$, we have $u\in C([0,T];C(\bar{\Omega}))$ and thus there is

$$\begin{aligned}u(x_0,t)=&\sum_{n=1}^{\infty}(\phi,\varphi_n)E_{\alpha,1}(-\lambda_n t^{\alpha})\varphi_n(x_0)+\sum_{n=1}^{\infty}(\psi,\varphi_n)tE_{\alpha,2}(-\lambda_n t^{\alpha})\varphi_n(x_0)\\&+\sum_{n=1}^{\infty}(f,\varphi_n)\int_0^t p(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})\,\mathrm{d}\tau\,\varphi_n(x_0)\\=:&\ I_1(t)+I_2(t)+I_3(t).\end{aligned}\tag{4.9}$$

Instead of u,α,f in the above formula by $\tilde{u}$, $\tilde{\alpha}$, $\tilde{f}$, one has

$$\tilde{u}(x_0,t):=\tilde{I}_1(t)+\tilde{I}_2(t)+\tilde{I}_3(t).$$

Next, we consider each term of (4.9) separately. Recalling the following identities:

$$\begin{aligned}&E_{\alpha,1}(-\lambda_n t^{\alpha})=1-\lambda_n t^{\alpha}E_{\alpha,\alpha+1}(-\lambda_n t^{\alpha}),\\&E_{\alpha,2}(-\lambda_n t^{\alpha})=1-\lambda_n t^{\alpha}E_{\alpha,\alpha+2}(-\lambda_n t^{\alpha}),\\&E_{\alpha,\alpha}(-\lambda_n t^{\alpha})=\frac{1}{\Gamma(\alpha)}-\lambda_n t^{\alpha}E_{\alpha,2\alpha}(-\lambda_n t^{\alpha}),\end{aligned}$$

from [22], we write

$$\begin{aligned}I_1(t)=&\sum_{n=1}^{\infty}(\phi,\varphi_n)\varphi_n(x_0)-t^{\alpha}\sum_{n=1}^{\infty}(\phi,\varphi_n)\lambda_n E_{\alpha,\alpha+1}(-\lambda_n t^{\alpha})\varphi_n(x_0)\\=:&\ I_{11}-t^{\alpha}I_{12}(t),\end{aligned}\tag{4.10}$$

$$\begin{aligned}I_2(t)=&\ t\sum_{n=1}^{\infty}(\psi,\varphi_n)\varphi_n(x_0)-t^{\alpha}\sum_{n=1}^{\infty}(\psi,\varphi_n)\lambda_n tE_{\alpha,\alpha+1}(-\lambda_n t^{\alpha})\varphi_n(x_0)\\=:&\ tI_{21}-t^{\alpha}I_{22}(t),\end{aligned}\tag{4.11}$$

$$\begin{aligned}I_3(t)=&\sum_{n=1}^{\infty}(f,\varphi_n)\int_0^t\left(p(\tau)(t-\tau)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\right)\mathrm{d}\tau\varphi_n(x_0)\\&-\sum_{n=1}^{\infty}(f,\varphi_n)\int_0^t p(\tau)(t-\tau)^{\alpha-1}\lambda_n(t-\tau)^{\alpha}E_{\alpha,2\alpha}\big(-\lambda_n(t-\tau)^{\alpha}\big)\,d\tau\varphi_n(x_0)\\=:&\ t^{\alpha}I_{31}(t)-I_{32}(t).\end{aligned}\tag{4.12}$$

Since $|\varphi_n(x_0)|\le C\lambda_n^{d/4+\varepsilon_0/2}$ for sufficiently small $\varepsilon_0>0$, the Eq. (4.10) and the Cauchy inequality lead to the estimate

$$\begin{aligned}|I_{11}|\le&\ C\sum_{n=1}^{\infty}|(\phi,\varphi_n)|\lambda_n^{d/4+\varepsilon_0/2}\lambda_n^{d/4+\varepsilon_0/2}\lambda_n^{-(d/4+\varepsilon_0/2)}\\\le&\ C\left(\sum_{n=1}^{\infty}(\phi,\varphi_n)^2\lambda_n^{2(d/2+\varepsilon_0)}\right)^{1/2}\left(\sum_{n=1}^{\infty}\lambda_n^{-(d/2+\varepsilon_0)}\right)^{1/2}\le C\|\phi\|_{D(A^{d/2+\varepsilon_0})}.\end{aligned}$$

Therefore,

$$I_{11} = \sum_{n=1}^{\infty}(\phi, \varphi_n)\varphi_n(x_0) = \phi(x_0).$$

It is noted that

$$I_{12}(t) = \sum_{n=1}^{\infty}(\phi, \varphi_n)\lambda_n E_{\alpha,\alpha+1}(-\lambda_n t^\alpha)\varphi_n(x_0). \tag{4.13}$$

The Cauchy inequality yields

$$\begin{aligned}|I_{12}(t)| &\le C\sum_{n=1}^{\infty}|(\phi, \varphi_n)|\lambda_n\lambda_n^{d/4+\varepsilon_0/2}\lambda_n^{d/4+\varepsilon_0/2}\lambda_n^{-(d/4+\varepsilon_0/2)} \\ &\le C\left(\sum_{n=1}^{\infty}(\phi, \varphi_n)^2\lambda_n^{2(1+d/2+\varepsilon_0)}\right)^{1/2}\left(\sum_{n=1}^{\infty}\lambda_n^{-(d/2+\varepsilon_0)}\right)^{1/2} \le C\|\phi\|_{D(A^{1+d/2+\varepsilon_0})},\end{aligned}$$

so that $I_{12}(t)$ converges uniformly on $t \in [0, T]$ and

$$\lim_{t\to 0} I_{12}(t) = \sum_{n=1}^{\infty}(\phi, \varphi_n)\lambda_n\varphi_n(x_0)\frac{1}{\Gamma(\alpha+1)} = (A\phi)(x_0)\frac{1}{\Gamma(\alpha+1)}. \tag{4.14}$$

For Eq. (4.11) similar considerations show that

$$I_{21} = \sum_{n=1}^{\infty}(\psi, \varphi_n)\varphi_n(x_0) = \psi(x_0), \tag{4.15}$$

and since

$$I_{22}(t) = \sum_{n=1}^{\infty}(\psi, \varphi_n)\lambda_n t E_{\alpha,\alpha+2}(-\lambda_n t^\alpha)\varphi_n(x_0),$$

the Cauchy inequality and the estimate $|x^\nu E_{\alpha,\beta}(-x)| \le C, x \ge 0, 0 \le \nu \le 1$ yield

$$\begin{aligned}|I_{22}(t)| &\le C\sum_{n=1}^{\infty}|(\psi, \varphi_n)|\lambda_n\lambda_n^{-1/\alpha}(\lambda_n t^\alpha)^{1/\alpha}E_{\alpha,\alpha+2}(-\lambda_n t^\alpha)\lambda_n^{d/4+\varepsilon_0/2}\lambda_n^{d/4+\varepsilon_0/2}\lambda_n^{-(d/4+\varepsilon_0/2)} \\ &\le C\left(\sum_{n=1}^{\infty}(\psi, \varphi_n)^2\lambda_n^{2(1+d/2+\varepsilon_0-1/\alpha)}\right)^{1/2}\left(\sum_{n=1}^{\infty}\lambda_n^{-(d/2+\varepsilon_0)}\right)^{1/2} \le C\|\psi\|_{D(A^{1+d/2+\varepsilon_0-1/\alpha})}.\end{aligned}$$

Therefore, $I_{22}(t)$ converges uniformly on $t \in [0, T]$ and

$$\lim_{t\to 0} I_{22}(t) = 0. \tag{4.16}$$

For Eq. (4.12), using the second mean value theorem for integrals, we write

$$\begin{aligned}I_{31}(t) &= t^{-\alpha}\sum_{n=1}^{\infty}(f, \varphi_n)\int_0^t p(\tau)(t-\tau)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\,\mathrm{d}\tau\,\varphi_n(x_0) \\ &= \sum_{n=1}^{\infty}(f, \varphi_n)p(\xi(t))\frac{1}{\Gamma(\alpha+1)}\varphi_n(x_0),\end{aligned}$$

where $\xi(t)\in(0,t)$. Further we have

$$|I_{31}(t)|\le C\sum_{n=1}^{\infty}|(f,\varphi_n)|\frac{1}{\Gamma(\alpha+1)}\lambda_n^{d/4+\varepsilon_0/2}\le C\|f\|_{D(A^{d/2+\varepsilon_0})},$$

so that $I_{31}(t)$ converges uniformly on $t\in[0,T]$. Hence,

$$\lim_{t\to 0}I_{31}(t)=\sum_{n=1}^{\infty}(f,\varphi_n)p(0)\varphi_n(x_0)\frac{1}{\Gamma(\alpha+1)}=f(x_0)p(0)\frac{1}{\Gamma(\alpha+1)}. \tag{4.17}$$

Since

$$I_{32}(t)=\sum_{n=1}^{\infty}(f,\varphi_n)\int_0^t p(\tau)(t-\tau)^{\alpha-1}\lambda_n(t-\tau)^{\alpha}E_{\alpha,2\alpha}(-\lambda_n(t-\tau)^{\alpha})\,\mathrm{d}\tau\,\varphi_n(x_0),$$

the Cauchy inequality and the estimate $|x^{\nu}E_{\alpha,\beta}(-x)|\le C, x\ge 0, 0\le\nu\le 1$ give

$$\begin{aligned}
|I_{32}(t)|\le&\int_0^t(t-\tau)^{\alpha-1+\varepsilon}\sum_{n=1}^{\infty}|(f,\varphi_n)||\lambda_n p(\tau)(t-\tau)^{\alpha-\varepsilon}E_{\alpha,2\alpha}(-\lambda_n(t-\tau)^{\alpha})||\varphi_n(x_0)|\mathrm{d}\tau\\
\le& C\int_0^t(t-\tau)^{\alpha-1+\varepsilon}\sum_{n=1}^{\infty}|(f,\varphi_n)||\lambda_n^{\varepsilon/\alpha}(\lambda_n(t-\tau)^{\alpha})^{(\alpha-\varepsilon)/\alpha}E_{\alpha,2\alpha}(-\lambda_n(t-\tau)^{\alpha})|\\
&\qquad\times\lambda_n^{d/4+\varepsilon_0/2}\lambda_n^{d/4+\varepsilon_0/2}\lambda_n^{-(d/4+\varepsilon_0/2)}\,\mathrm{d}\tau\\
\le& C\int_0^t(t-\tau)^{\alpha-1+\varepsilon}\left(\sum_{n=1}^{\infty}(f,\varphi_n)^2\lambda_n^{2(\varepsilon/\alpha+d/2+\varepsilon_0)}\right)^{1/2}\left(\sum_{n=1}^{\infty}\lambda_n^{-(d/2+\varepsilon_0)}\right)^{1/2}\mathrm{d}\tau\\
\le& C\int_0^t(t-\tau)^{\alpha-1+\varepsilon}\|f\|_{D(A^{d/2+\varepsilon_0+\varepsilon/\alpha})}\,\mathrm{d}\tau\\
\le& C\|f\|_{D(A^{d/2+\varepsilon_0+\varepsilon/\alpha})}t^{\alpha+\varepsilon}=\mathscr{O}(t^{\alpha+\varepsilon}),
\end{aligned}\tag{4.18}$$

where $\varepsilon>0$ is sufficiently small. It follows from (4.9), (4.13) and (4.15) and $u(x_0,t)=\tilde{u}(x_0,t)$ that

$$\begin{aligned}
&\phi(x_0)-t^{\alpha}I_{12}(t)+t\psi(x_0)-t^{\alpha}I_{22}(t)+t^{\alpha}I_{31}(t)-I_{32}(t)\\
=&\phi(x_0)-t^{\tilde{\alpha}}\tilde{I}_{12}(t)+t\psi(x_0)-t^{\tilde{\alpha}}\tilde{I}_{22}(t)+t^{\tilde{\alpha}}\tilde{I}_{31}(t)-\tilde{I}_{32}(t).
\end{aligned}\tag{4.19}$$

Next we show that $\alpha=\tilde{\alpha}$. Assume that $\alpha<\tilde{\alpha}$. Removing $\phi(x_0)$ and $t\psi(x_0)$ and dividing both sides of (4.19) by t^{α} yield

$$\begin{aligned}
&-I_{12}(t)-I_{22}(t)+I_{31}(t)-t^{-\alpha}I_{32}(t)\\
=&t^{\tilde{\alpha}-\alpha}\left(-\tilde{I}_{12}(t)-\tilde{I}_{22}(t)+\tilde{I}_{31}(t)-t^{-\tilde{\alpha}}\tilde{I}_{32}(t)\right).
\end{aligned}$$

Taking $t\to 0$, by (4.14), (4.16)-(4.18), we get

$$f(x_0)p(0)-(A\phi)(x_0)=0.$$

Since $f(x_0)p(0)-(A\phi)(x_0)\neq 0$, we have a contradiction. Analogously, one shows that $\alpha>\tilde{\alpha}$ is impossible. Thus, $\alpha=\tilde{\alpha}$. The uniqueness of $f(x)$ can be obtained from Lemma 4.4. ☐

5. Tikhonov-Type Regularization Method

In this section, the inverse problems of recovering α and $f(x)$ are solved by a variational method based on the Tikhonov type regularization with H^1 penalty term for f.

Define the functional $J(\alpha,f)$ by

$$J(\alpha,f)=\frac{1}{2}\|\mathscr{F}(\alpha,f)-g^\delta\|^2_{L^2(0,T_0;L^2(\omega))}+\frac{\mu}{2}\|f\|^2_{H^1(\Omega)}+\frac{\mu}{2}(\alpha-\alpha^*)^2,$$

or

$$J(\alpha,f)=\frac{1}{2}\|\mathscr{F}(\alpha,f)-g^\delta\|^2_{L^2(0,T_0;L^2(\Gamma))}+\frac{\mu}{2}\|f\|^2_{H^1(\Omega)}+\frac{\mu}{2}(\alpha-\alpha^*)^2,$$

and consider the minimization problem

$$\min_{(\alpha,f)\in D} J(\alpha,f), \tag{5.1}$$

where $D:=\{(\alpha,f)\in[\underline{\alpha},\overline{\alpha}]\times H^1(\Omega)\}$, $\mu>0$ is a regularization parameter, α^* a suitable guess for α, and $g^\delta(x,t)$ the noisy measurement of $g(x,t)=u(x,t), x\in\omega, t\in(0,T_0)$ in the problem (P_1) or $u(x,t), x\in\Gamma, t\in(0,T_0)$ in the problem (P_2).

The minimizer of (5.1) is called the regularized solution of the inverse problem (P_1) or (P_2). The existence and convergence of the regularized solution can be obtained by using the continuity of the forward operator $\mathscr{F}$. In what follows, we denote $Y=L^2(0,T_0;L^2(\omega))$ or $L^2(0,T_0;L^2(\Gamma))$.

Theorem 5.1. *For $g^\delta\in L^2(0,T_0;L^2(\omega))$ or $g^\delta\in L^2(0,T_0;L^2(\Gamma))$ and any $\mu>0$, there exists a pair of minimizers $(\alpha^\delta_\mu,f^\delta_\mu)\in D$ for the variational problem* (5.1).

Proof. It is clear that $\inf_{(\alpha,f)\in D}J(\alpha,f)$ is finite. Then there exists a sequence $(\alpha_n,f_n)\in D$ such that $J(\alpha_n,f_n)\to d:=\inf_{(\alpha,f)\in D}J(\alpha,f)$, which implies that $\{f_n\}$ is bounded in $H^1(\Omega)$. Since $\{\alpha_n\}$ is also bounded, there exists a subsequence (α_{n_k},f_{n_k}) such that $\alpha_{n_k}\to\alpha^\delta_\mu$ and $(f_{n_k}\rightharpoonup f^\delta_\mu)$ in $H^1(\Omega)$ and $f_{n_k}\to f^\delta_\mu$ in $L^2(\Omega)$ as $k\to\infty$. It is easily seen that $(\alpha^\delta_\mu,f^\delta_\mu)\in D$. The Lipschitz continuity of $\mathscr{F}$, cf. Theorems 3.1 and 3.2, yields

$$\left\|\mathscr{F}(\alpha_{n_k},f_{n_k})-\mathscr{F}\big(\alpha^\delta_\mu,f^\delta_\mu\big)\right\|_Y\le C\left|\alpha_{n_k}-\alpha^\delta_\mu\right|+C\left\|f_{n_k}-f^\delta_\mu\right\|_{L^2(\Omega)}\to 0,\quad k\to\infty.$$

Since $\|f^\delta_\mu\|_{H^1(\Omega)}\le\liminf_{k\to\infty}\|f_{n_k}\|_{H^1(\Omega)}$, we obtain

$$J\big(\alpha^\delta_\mu,f^\delta_\mu\big)\le\liminf_{k\to\infty}J(\alpha_{n_k},f_{n_k})=d.$$

Therefore $(\alpha^\delta_\mu,f^\delta_\mu)$ is a pair of minimizer of $J(\alpha,f)$. □

Theorem 5.2. *Suppose the conditions in Theorems* 3.1, 3.2, *and* 4.1 *hold. Let $(\hat\alpha,\hat f)\in D$ be the unique solution of $\mathscr{F}(\hat\alpha,\hat f)=g$, where g is the exact data and g^δ satisfy*

$$\|g-g^\delta\|_Y\le\delta.$$

Take $\mu=\mu(\delta)$ such that $\mu(\delta)\to 0$, $\delta^2/\mu(\delta)\to 0$ as $\delta\to 0$. If $(\alpha^\delta_{\mu(\delta)}, f^\delta_{\mu(\delta)})$ is the minimizer of $J(\alpha,f)$ with $\mu=\mu(\delta)$, then

$$\begin{aligned} &\alpha^\delta_{\mu(\delta)} \to \hat{\alpha}, && \delta\to 0,\\ &f^\delta_{\mu(\delta)} \to \hat{f} \quad \text{in } H^1(\Omega), && \delta\to 0. \end{aligned}$$

Proof. Since $(\alpha^\delta_{\mu(\delta)}, f^\delta_{\mu(\delta)})$ is the minimizer of $J(\alpha,f)$ with $\mu=\mu(\delta)$, we have

$$\begin{aligned} J\left(\alpha^\delta_{\mu(\delta)}, f^\delta_{\mu(\delta)}\right) &\le \frac{1}{2}\left\|\mathscr{F}(\hat{\alpha},\hat{f})-g^\delta\right\|_Y^2+\frac{\mu(\delta)}{2}\|\hat{f}\|^2_{H^1(\Omega)}+\frac{\mu(\delta)}{2}(\hat{\alpha}-\alpha^*)^2\\ &\le \frac{1}{2}\delta^2+\frac{\mu(\delta)}{2}\|\hat{f}\|^2_{H^1(\Omega)}+\frac{\mu(\delta)}{2}(\hat{\alpha}-\alpha^*)^2. \end{aligned}$$

Then

$$\frac{1}{2}\left\|f^\delta_{\mu(\delta)}\right\|^2_{H^1(\Omega)} \le \frac{1}{2}\frac{\delta^2}{\mu(\delta)}+\frac{1}{2}\|\hat{f}\|^2_{H^1(\Omega)}+\frac{1}{2}(\hat{\alpha}-\alpha^*)^2,$$

which implies that for each sequence $\delta_j\to 0$, $\|f^{\delta_j}_{\mu(\delta_j)}\|_{H^1(\Omega)}$ is bounded. Thus, there exists a subsequence, denote also as $\{\alpha^{\delta_j}_{\mu(\delta_j)}, f^{\delta_j}_{\mu(\delta_j)}\}$ and $\tilde{\alpha}\in\mathbb{R}, \tilde{f}\in H^1(\Omega)$ such that $\alpha^{\delta_j}_{\mu(\delta_j)}\to\tilde{\alpha}$ and $f^\delta_{\mu(\delta_j)}\rightharpoonup\tilde{f}$ in $H^1(\Omega)$ as well as $f^{\delta_j}_{\mu(\delta_j)}\to\tilde{f}$ in $L^2(\Omega)$ as $\delta_j\to 0$. By the fact that $(\alpha^{\delta_j}_{\mu(\delta_j)}, f^{\delta_j}_{\mu(\delta_j)})\in D$, it can be easily obtained that $(\tilde{\alpha},\tilde{f})\in D$. The Lipschitz continuity of $\mathscr{F}$ in Theorems 3.1 or 3.2 gives

$$\mathscr{F}\left(\alpha^{\delta_j}_{\mu(\delta_j)}, f^{\delta_j}_{\mu(\delta_j)}\right)\to\mathscr{F}(\tilde{\alpha},\tilde{f}) \quad \text{in } L^2\big(0,T_0;L^2(\omega)\big) \quad \text{or} \quad L^2\big(0,T_0;L^2(\Gamma)\big).$$

Consequently, it follows that

$$\begin{aligned} &\frac{1}{2}\|\mathscr{F}(\tilde{\alpha},\tilde{f})-g\|_Y^2\\ &=\frac{1}{2}\lim_{j\to\infty}\left\|\mathscr{F}\left(\alpha^{\delta_j}_{\mu(\delta_j)}, f^{\delta_j}_{\mu(\delta_j)}\right)-g^{\delta_j}\right\|_Y^2\\ &\le\liminf_{j\to\infty}\left[\frac{1}{2}\left\|\mathscr{F}\left(\alpha^{\delta_j}_{\mu(\delta_j)}, f^{\delta_j}_{\mu(\delta_j)}\right)-g^{\delta_j}\right\|_Y^2+\frac{\mu(\delta_j)}{2}\left\|f^{\delta_j}_{\mu(\delta_j)}\right\|^2_{H^1(\Omega)}+\frac{\mu(\delta_j)}{2}\left(\alpha^{\delta_j}_{\mu(\delta_j)}-\alpha^*\right)^2\right]\\ &\le\liminf_{j\to\infty}\left[\frac{1}{2}\left\|\mathscr{F}(\hat{\alpha},\hat{f})-g^{\delta_j}\right\|_Y^2+\frac{\mu(\delta_j)}{2}\|\hat{f}\|^2_{H^1(\Omega)}+\frac{\mu(\delta_j)}{2}(\hat{\alpha}-\alpha^*)^2\right]\\ &=\frac{1}{2}\|\mathscr{F}(\hat{\alpha},\hat{f})-g\|_Y^2=0, \end{aligned}$$

which implies $\mathscr{F}(\tilde{\alpha},\tilde{f})=g$. Thus, we obtained that

$$\alpha^{\delta_j}_{\mu(\delta_j)}\to\tilde{\alpha}, \quad f^{\delta_j}_{\mu(\delta_j)}\to\tilde{f} \quad \text{in } L^2(\Omega) \quad \text{as} \quad j\to\infty.$$

Moreover, we can prove that $f^{\delta_j}_{\mu(\delta_j)} \to \tilde{f}$ in $H^1(\Omega)$ as $j \to \infty$. Indeed, using the weak low semi-continuity of the norm in $H^1(\Omega)$, we have

$$
\begin{aligned}
&\frac{1}{2}\|\tilde{f}\|^2_{H^1(\Omega)} + \frac{1}{2}(\tilde{\alpha}-\alpha^*)^2 \\
&\le \liminf_{j\to\infty}\left(\frac{1}{2}\left\|f^{\delta_j}_{\mu(\delta_j)}\right\|^2_{H^1(\Omega)} + \frac{1}{2}\left(\alpha^{\delta_j}_{\mu(\delta_j)} - \alpha^*\right)^2\right) \\
&\le \limsup_{j\to\infty}\left(\frac{1}{2}\left\|f^{\delta_j}_{\mu(\delta_j)}\right\|^2_{H^1(\Omega)} + \frac{1}{2}\left(\alpha^{\delta_j}_{\mu(\delta_j)} - \alpha^*\right)^2\right) \\
&\le \limsup_{j\to\infty}\left\{\frac{1}{\mu(\delta_j)}\left[\frac{\mu(\delta_j)}{2}\left\|f^{\delta_j}_{\mu(\delta_j)}\right\|^2_{H^1(\Omega)} + \frac{\mu(\delta_j)}{2}\left(\alpha^{\delta_j}_{\mu(\delta_j)} - \alpha^*\right)^2\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. + \frac{1}{2}\left\|\mathscr{F}\left(\alpha^{\delta_j}_{\mu(\delta_j)}, f^{\delta_j}_{\mu(\delta_j)}\right) - g^{\delta_j}\right\|^2_Y\right]\right\} \\
&\le \limsup_{j\to\infty}\left\{\frac{1}{\mu(\delta_j)}\left[\frac{1}{2}\left\|\mathscr{F}(\tilde{\alpha},\tilde{f}) - g^{\delta_j}\right\|^2_Y + \frac{\mu(\delta_j)}{2}\|\tilde{f}\|^2_{H^1(\Omega)} + \frac{\mu(\delta_j)}{2}(\tilde{\alpha}-\alpha^*)^2\right]\right\} \\
&\le \limsup_{j\to\infty}\left[\frac{\delta_j^2}{2\mu(\delta_j)} + \frac{1}{2}\|\tilde{f}\|^2_{H^1(\Omega)} + \frac{1}{2}(\tilde{\alpha}-\alpha^*)^2\right] \\
&= \frac{1}{2}\|\tilde{f}\|^2_{H^1(\Omega)} + \frac{1}{2}(\tilde{\alpha}-\alpha^*)^2,
\end{aligned}
$$

which implies that

$$\lim_{j\to\infty}\frac{1}{2}\left\|f^{\delta_j}_{\mu(\delta_j)}\right\|_{H^1(\Omega)} = \frac{1}{2}\|\tilde{f}\|_{H^1(\Omega)}.$$

Hence, $f^{\delta_j}_{\mu(\delta_j)} \to \tilde{f}$ in $H^1(\Omega)$ as $j \to \infty$.

It is known that if $(\hat{\alpha}, \hat{f})$ is the unique solution of $\mathscr{F}(\alpha, f) = g$, then

$$
\begin{aligned}
&\alpha^{\delta}_{\mu(\delta)} \to \hat{\alpha}, && \delta \to 0, \\
&f^{\delta}_{\mu(\delta)} \to \hat{f} \quad \text{in } H^1(\Omega), && \delta \to 0.
\end{aligned}
$$

The proof is complete. □

6. Numerical Algorithms

In order to solve the minimization problem (5.1), we employ a linearized iterative method for recovering the order α and the source component $f(x)$ numerically. This is similar to the well-known Levenberg-Marquardt method [18].

Given an initial value (α_0, f^0) and the found approximate solution (α_k, f^k) in the k-th step, we determine $(k+1)$-th step approximate solution by solving the following linearized variational problem:

$$(\alpha_{k+1}, f^{k+1}) = \arg\min_{(\alpha,f)\in D} J_k(\alpha, f), \tag{6.1}$$

where

$$J_k(\alpha,f)=\frac{1}{2}\left\|\mathscr{F}(\alpha_k,f^k)-g^\delta+\mathscr{F}'_{f^k}(\alpha_k,f^k)(f-f^k)+\mathscr{F}'_{\alpha_k}(\alpha_k,f^k)(\alpha-\alpha_k)\right\|_Y^2 \\ +\frac{\mu}{2}\|f\|^2_{H^1(\Omega)}+\frac{\mu}{2}(\alpha-\alpha^*)^2+\frac{\nu_k}{2}\parallel f-f^k\parallel^2_{H^1(\Omega)}+\frac{\nu_k}{2}(\alpha-\alpha_k)^2,$$

where $\nu_k>0$ is a constant approaching to zero as k tends to ∞.

To find an approximate minimizer of (6.1), we use a finite-dimensional approximation method. Take nodes $x_i\in\bar{\Omega}$ for $i=1,2,\dots,m$. Assuming that $\{\theta_i(x),i=1,2,\dots,m\}$ are piecewise linear finite element basis functions in $H^1(\Omega)$ corresponding to node x_i, we let

$$f^k(x)=\sum_{i=1}^m f_i^k\theta_i(x),$$

where f_i^k are the expansion coefficients. In what follows, we denote $\mathbf{f}^k=(f_1^k,f_2^k,\cdots,f_m^k)^T$ as the coefficient vector.

The problem (6.1) is transformed into following minimization problem:

$$\min_{\alpha_{k+1},\mathbf{f}^{k+1}\in\mathbb{R}^m}\Bigg\{\frac{1}{2}\Bigg\|\mathscr{F}(\alpha_k,f^k)-g^\delta+\sum_{i=1}^m\big(f_i^{k+1}-f_i^k\big)\big(\mathscr{F}'_{f^k}(\alpha_k,f^k)\theta_i(x)\big) \\ +\mathscr{F}'_{\alpha_k}(\alpha_k,f^k)(\alpha_{k+1}-\alpha_k)\Bigg\|_Y^2 \\ +\frac{\mu}{2}\|f^{k+1}\|^2_{H^1(\Omega)}+\frac{\mu}{2}(\alpha_{k+1}-\alpha^*)^2 \\ +\frac{\nu_k}{2}\|f^{k+1}-f^k\|^2_{H^1(\Omega)}+\frac{\nu_k}{2}(\alpha_{k+1}-\alpha_k)^2\Bigg\}, \tag{6.2}$$

where $\mathscr{F}'_{f^k}(\alpha_k,f^k)\theta_i(x)=w_i(x,t)$ for $x\in\omega,t\in[0,T_0]$ in the problem (P_1) or $w_i(x,t)$ for $x\in\Gamma$, $t\in[0,T_0]$ in the problem (P_2) for $i=1,2,\dots,m$ in which $w_i(x,t)$ is the solution to the following sensitive problem with $\alpha=\alpha_k$, $h(x)=\theta_i(x)$:

$$\begin{aligned} &\partial^\alpha_{0+}w(x,t)+Aw(x,t)=h(x)p(t), && x\in\Omega, \quad 0<t\le T,\\ &w(x,0)=0, && x\in\bar{\Omega},\\ &w_t(x,0)=0, && \bar{x}\in\Omega,\\ &\partial_\nu w(x,t)=0, && x\in\partial\Omega,\quad 0\le t\le T. \end{aligned} \tag{6.3}$$

The derivative $\mathscr{F}'_{\alpha_k}(\alpha_k,f^k)$ is approximated as

$$\mathscr{F}'_{\alpha_k}(\alpha_k,f^k)\approx\frac{\mathscr{F}(\alpha_k+\varsigma,f^k)-\mathscr{F}(\alpha_k,f^k)}{\varsigma}, \tag{6.4}$$

where ς is a small step size. The minimizer for problem (6.2) can be then obtained by solving the system of linear equations

$$\big[Q^k+(\mu+\nu_k)(K_1+K_2)\big]\begin{bmatrix}\mathbf{f}^{k+1}\\ \alpha_{k+1}\end{bmatrix}=b^k+\big(Q^k+\nu_k(K_1+K_2)\big)\begin{bmatrix}\mathbf{f}^k\\ \alpha_k\end{bmatrix}+\mu(K_1+K_2)\begin{bmatrix}0\\ \alpha^*\end{bmatrix}, \tag{6.5}$$

where

$$
\begin{aligned}
Q^k &= \begin{bmatrix}
(\mathscr{F}'_{f^k}\theta_1, \mathscr{F}'_{f^k}\theta_1)_Y & \dots & (\mathscr{F}'_{f^k}\theta_1, \mathscr{F}'_{f^k}\theta_m)_Y & (\mathscr{F}'_{f^k}\theta_1, \mathscr{F}'_{\alpha_k})_Y \\
\vdots & \ddots & \vdots & \vdots \\
(\mathscr{F}'_{f^k}\theta_m, \mathscr{F}'_{f^k}\theta_1)_Y & \dots & (\mathscr{F}'_{f^k}\theta_m, \mathscr{F}'_{f^k}\theta_m)_Y & (\mathscr{F}'_{f^k}\theta_m, \mathscr{F}'_{\alpha_k})_Y \\
(\mathscr{F}'_{\alpha_k}, \mathscr{F}'_{f^k}\theta_1)_Y & \dots & (\mathscr{F}'_{\alpha_k}, \mathscr{F}'_{f^k}\theta_m)_Y & (\mathscr{F}'_{\alpha_k}, \mathscr{F}'_{\alpha_k})_Y
\end{bmatrix}, \\
K_1 &= \begin{bmatrix}
(\theta_1(x), \theta_1(x)) & \dots & (\theta_1(x), \theta_m(x)) & 0 \\
\vdots & \ddots & \vdots & \vdots \\
(\theta_m(x), \theta_1(x)) & \dots & (\theta_m(x), \theta_m(x)) & 0 \\
0 & \dots & 0 & 1
\end{bmatrix}, \\
K_2 &= \begin{bmatrix}
(\nabla\theta_1(x), \nabla\theta_1(x)) & \dots & (\nabla\theta_1(x), \nabla\theta_m(x)) & 0 \\
\vdots & \ddots & \vdots & \vdots \\
(\nabla\theta_m(x), \nabla\theta_1(x)) & \dots & (\nabla\theta_m(x), \nabla\theta_m(x)) & 0 \\
0 & \dots & 0 & 0
\end{bmatrix}, \\
b^k &= \begin{bmatrix}
(g^\delta - \mathscr{F}(\alpha_k, f^k), \mathscr{F}'_{f^k}\theta_1)_Y \\
\vdots \\
(g^\delta - \mathscr{F}(\alpha_k, f^k), \mathscr{F}'_{f^k}\theta_m)_Y \\
(g^\delta - \mathscr{F}(\alpha_k, f^k), \mathscr{F}'_{\alpha_k})_Y
\end{bmatrix}.
\end{aligned}
\tag{6.6}
$$

Denote the residual R_k at the k-th step as

$$R_k = \left\| \mathscr{F}(\alpha_k, f^k) - g^\delta \right\|_Y.$$

We use the Morozov discrepancy principle [30] to find a good stopping step i.e., choosing k_* satisfying the following inequality:

$$R_{k_*} \le \tau\delta < R_{k-1}, \quad k \le k_*,$$

where $\tau > 1$ is a constant. Then the linearized iterative method for solving the variational problem (6.1) is reviewed in Algorithm 6.1.

Algorithm 6.1 Linearized Iterative Algorithm

1: Given the values ς, τ, μ, ν_k, α^* and initialized value α_0, f_0. Set $k = 0$.
2: Solve the direct problem (1.1) with $(\alpha, f) = (\alpha_k, f^k)$ to get $\mathscr{F}(\alpha_k, f^k)$.
3: Solve the sensitive problem (6.3) with $\alpha = \alpha_k$ and $h(x) = \theta_i(x)$ for $i = 1, 2, \dots, m$ to get $\mathscr{F}'_{f^k}(\alpha_k, f^k)\theta_i(x)$ for $i = 1, 2, \dots, m$.
4: Compute $\mathscr{F}'_{\alpha_k}(\alpha_k, f^k)$ by (6.4).
5: Compute Q^k, K_1, K_2 and b^k by (6.6).
6: Solve the linear system of equations (6.5) to update α_{k+1} and $\mathbf{f}^{k+1}$.
7: Project α_{k+1} onto $[\underline{\alpha}, \overline{\alpha}]$ by $\alpha_{k+1} = \min(\bar{\alpha}, \max(\underline{\alpha}, \alpha_{k+1}))$.
8: Set $k = k + 1$ and return to step 2 until the given stopping criterion is satisfied.

7. Numerical Experiments

In this section, we test two examples in one-dimensional case and one example in two-dimensional case for problems (P_1) and (P_2) to illustrate the effectiveness of the proposed method.

The noisy data are generated by adding random perturbations — i.e.

$$g^{\delta}(x_i,t_j)=g(x_i,t_j)+\varepsilon g(x_i,t_j)(2r_{ij}-1),$$

where r_{ij} are random numbers distributed uniformly on $[0,1]$ and ε is a relative noise level of the data. The corresponding absolute noise level is calculated by $\delta=\|g^{\delta}-g\|_Y$ numerically.

To show the accuracy of numerical solutions, we compute approximate L^2 relative errors defined by

$$\mathrm{err}_k^{(\alpha)}=\frac{|\alpha^k-\alpha|}{\alpha},\quad \mathrm{err}_k^{f}=\frac{\|f^k-f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}},$$

where α, f are the exact fractional order and space-independent source, and α^k, f^k are the approximate numerical solution at the k-th step.

7.1. One-dimensional case

Setting $\Omega=(0,1)$, $T=1$, we divide the space-time region $[0,1]\times[0,1]$ into 100×100 equidistant meshes. In the iterative algorithm, the direct problem (1.1) and the sensitive problem (6.3) are solved by the piecewise linear finite element in space and $L1$-type discrete scheme in time. The additional data are obtained via solving the direct problem (1.1) by a discrete scheme with a finite difference method in space and the $L1$-type finite difference scheme in time. We take $\underline{\alpha}=1.01$, $\bar{\alpha}=1.99$. The maximum number of iteration steps is 20.

Example 7.1. Let

$$\begin{aligned}
&a(x)=2-\sin(\pi x)-\frac{1}{2}x, \quad c(x)=e^{3x}+3x+\sin(2\pi x)+1,\\
&\phi(x)=2-2\cos(2\pi x), \qquad \psi(x)=2-2\cos(2\pi x)-\frac{x^2}{2}+\frac{x^3}{3}.
\end{aligned}$$

Take

$$\begin{aligned}
&f(x)=\big(\sin(\pi x)-\pi x(1-x)\big)-\cos(2\pi x),\\
&p(t)=5e^{3-\alpha}(0.25t+2.75)+\cos(2\pi t)+1.
\end{aligned}$$

We recover a numerical solution for α and $f(x)$ by the noisy data with relative noise levels $\varepsilon=0.001, 0.01, 0.05$. Take $\mu=0.0001\times\delta^{3/2}$, $\nu_k=0.005\times 0.5^k$ and $\tau=1.01$, $\varsigma=0.0001$ if no other specified. The objective guess are $\alpha^*=1.5$ in the regularization functional, and the initial values are $f^0=0$ and $\alpha_0=1.5$ if no further specified.

Problem (P_1). To find the order and source function in this case, we use additional data $u(x,t)$, $x \in \omega = [0.15, 0.75]$, $t \in [0, T_0] = [0, 0.6]$. Numerical results for fractional orders $\alpha = 1.2, 1.5, 1.8$ are shown in Fig. 1 and the relative errors of numerical solutions are given in Table 1. It can be seen that the numerical results match the exact solution accurately and also stable to the noise levels. Meanwhile, the stopping steps are small which indicate the proposed iterative method is efficient and robust.

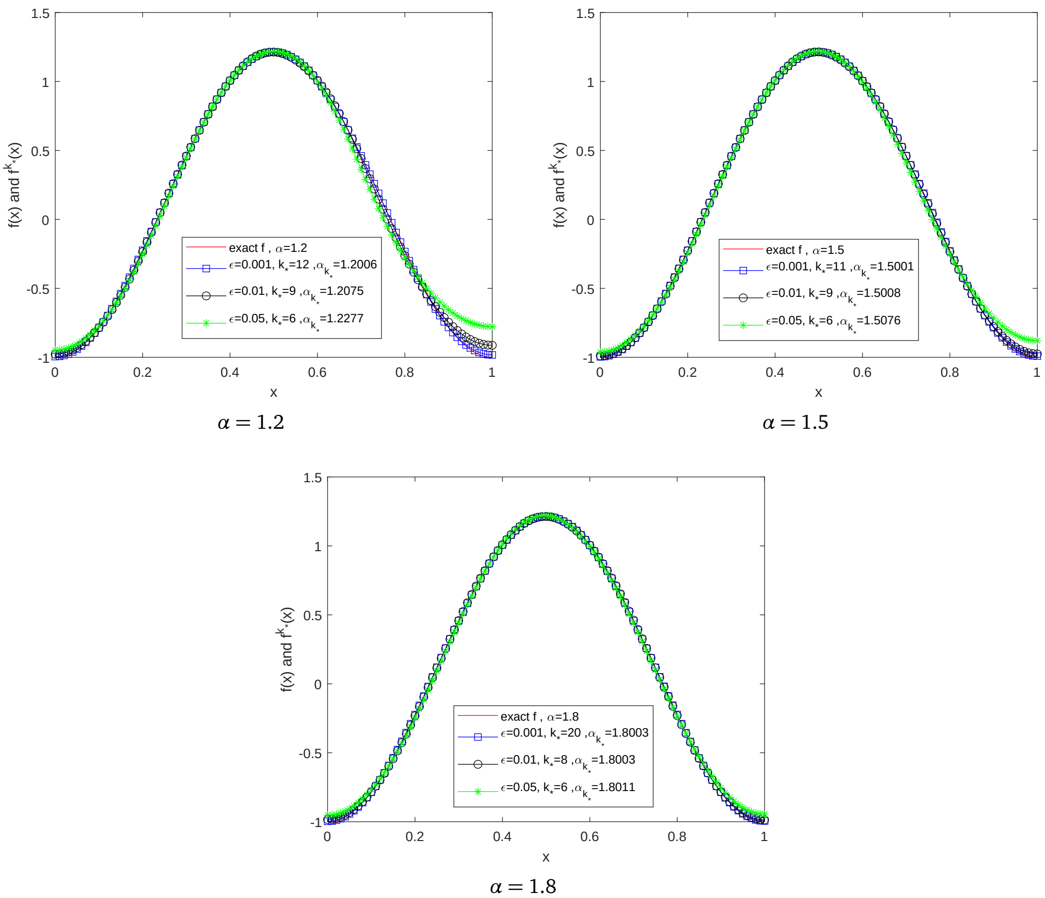


Figure 1: Example 7.1. Problem (P_1). Numerical solutions.

Table 1: Example 7.1. Problem (P_1). Relative errors of numerical solutions.

ε	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.8$	
	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$
0.001	1.2006/0.0005	0.0064	1.5001/0.0001	0.0037	1.8003/0.0001	0.0026
0.01	1.2075/0.0063	0.0347	1.5008/0.0005	0.0095	1.8003/0.0001	0.0076
0.05	1.2277/0.0231	0.0942	1.5076/0.0050	0.0513	1.8011/0.0006	0.0285

Problem (P_2). To solve the inverse problem in this case, we use additional data $u(x_0,t)$, $x_0=0$, $t\in[0,T_0]=[0,0.75]$. Numerical results for fractional order $\alpha=1.2,1.5,1.8$ are shown in Fig. 2, and the relative errors of numerical solutions are given in Table 2. It can be seen that for the problem (P_2), the algorithm recover effectively the fractional order and space-dependent source term by using a single point measurement. Meanwhile, the stopping steps are also small which indicate the proposed iterative method is also efficient and robust for the problem (P_2).

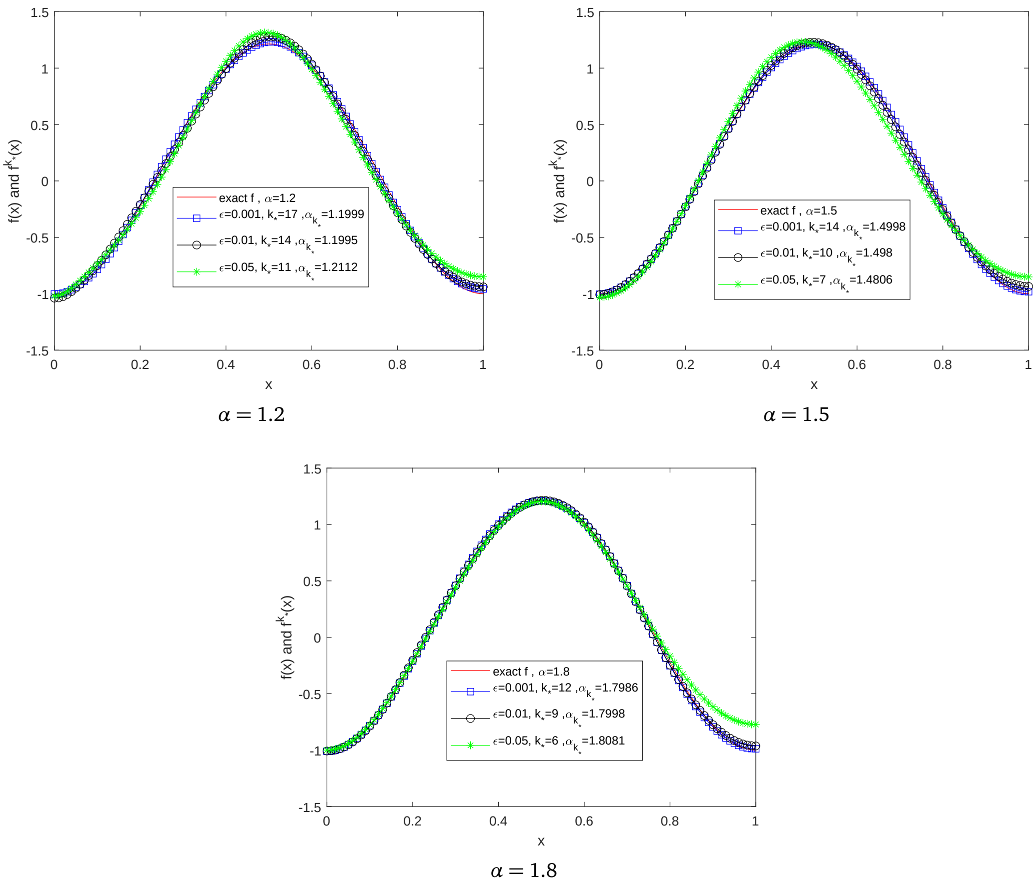


Figure 2: Example 7.1. Problem (P_2). Numerical solutions.

Table 2: Example 7.1. Problem (P_2). Relative errors of numerical solutions.

ε	$\alpha=1.2$		$\alpha=1.5$		$\alpha=1.8$	
	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$
0.001	1.1999/0.0000	0.0239	1.4998/0.0002	0.0090	1.7986/0.0008	0.0147
0.01	1.1995/0.0004	0.0538	1.498/0.0013	0.0329	1.7998/0.0001	0.0188
0.05	1.2112/0.0093	0.0948	1.4806/0.0129	0.1106	1.8081/0.0045	0.0967

Example 7.2. Let

$$a(x) = \sin(2\pi x) - x^2 + 2, \qquad c(x) = e^{x+2} + \sin(2\pi x) + (1+x)^2 - 6,$$
$$\phi(x) = 2\cos(2\pi x) + x^2(1-x)^2 + 1, \quad \psi(x) = 2\cos(2\pi x) + x^2(1-x)^2 + 1.$$

Take

$$f(x) = \frac{1}{2}x^2(1-x)^2 - \cos(4\pi x),$$
$$p(t) = 5\pi^2(t+2)^2 + 3e^{\alpha-t} + \cos(2\pi t) + 1.$$

We recover a numerical solution for α and $f(x)$ by the noisy data with relative noise levels $\varepsilon = 0.001, 0.01, 0.05$. Take $\mu = 0.0001 \times \delta^{3/2}$, $\nu_k = 0.005 \times 0.5^k$ and $\tau = 1.01$, $\varsigma = 0.0001$ if no other specified. The objective guess are $\alpha^* = 1.5$ in the regularization functional, and the initial values are $f^0 = 0$ and $\alpha_0 = 1.5$ if no further specified.

Problem (P$_1$). To find the order and source function in this case, we use additional data $u(x,t)$, $x \in \omega = [0.10, 0.65]$, $t \in [0, T_0] = [0, 0.55]$. Numerical results for fractional orders $\alpha = 1.2, 1.5, 1.8$ are shown in Fig. 3 and the relative errors of numerical solutions are given in Table 3. It can be seen that the numerical results match the exact solution accurately and also stable to the noise levels.

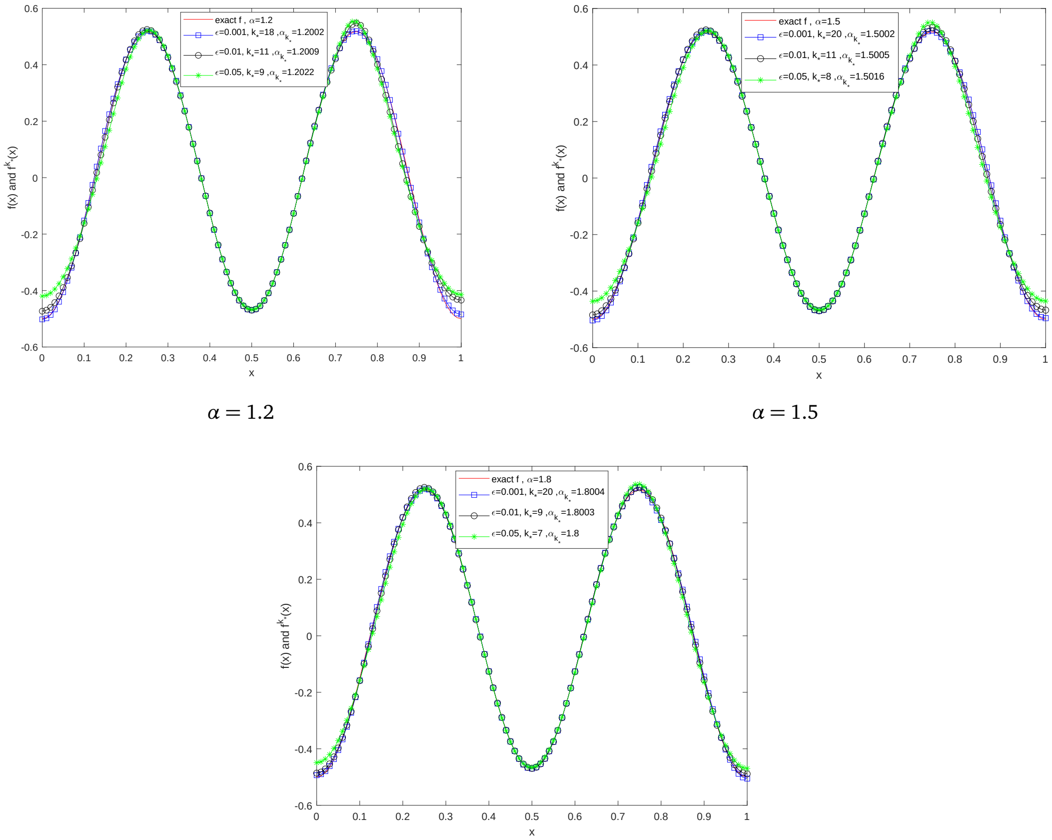
In order to see the effect of the initial guess α_0, we test Example 7.2 with the exact $\alpha = 1.5$ by taking $\alpha^* = \alpha_0 = 1.2, 1.4, 1.6, 1.8$, respectively. Under various relative noise levels $\varepsilon = 0.001, 0.01, 0.05$, we show the numerical results in Fig. 4 and Table 4 from which

Table 3: Example 7.2. Problem (P$_1$). Relative errors of numerical solutions.

ε	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.8$	
	$\alpha_{k_*}/\text{err}_{k_*}^{(\alpha)}$	$\text{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\text{err}_{k_*}^{(\alpha)}$	$\text{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\text{err}_{k_*}^{(\alpha)}$	$\text{err}_{k_*}^{(f)}$
0.001	1.2003/0.0003	0.0160	1.5003/0.0002	0.0081	1.8005/0.0003	0.0107
0.01	1.2014/0.0012	0.0794	1.501/0.0007	0.0399	1.8007/0.0004	0.0168
0.05	1.2043/0.0036	0.1961	1.5038/0.0025	0.1079	1.802/0.0011	0.0468

Table 4: Example 7.2. Problem (P$_1$). Relative errors of numerical solutions.

ε	$\alpha_0 = 1.2$		$\alpha_0 = 1.4$	
	$\alpha_{k_*}/\text{err}_{k_*}^{(\alpha)}$	$\text{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\text{err}_{k_*}^{(\alpha)}$	$\text{err}_{k_*}^{(f)}$
0.001	1.5002/0.0001	0.0052	1.5002/0.0001	0.0052
0.01	1.5005/0.0004	0.0293	1.5005/0.0004	0.0296
0.05	1.5016/0.0011	0.0766	1.5016/0.0011	0.0770
ε	$\alpha_0 = 1.6$		$\alpha_0 = 1.8$	
	$\alpha_{k_*}/\text{err}_{k_*}^{(\alpha)}$	$\text{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\text{err}_{k_*}^{(\alpha)}$	$\text{err}_{k_*}^{(f)}$
0.001	1.5002/0.0001	0.0052	1.5002/0.0001	0.0052
0.01	1.5005/0.0004	0.0297	1.5006/0.0004	0.0333
0.05	1.5016/0.0011	0.0771	1.5018/0.0012	0.0863



$\alpha = 1.2$ $\alpha = 1.5$

$\alpha = 1.8$

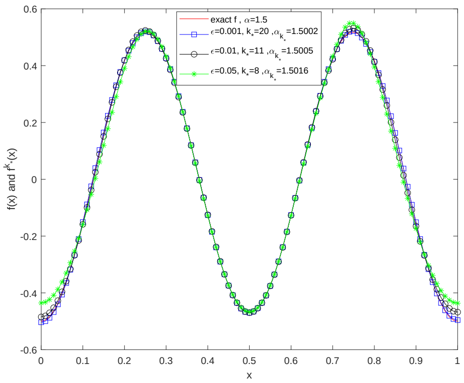
Figure 3: Example 7.2. Problem (P_1). Numerical solutions.

it can be observed that the different α_0 has a slightly impact on the reconstruction of source f and α, but can give quite good results for both of α and f.

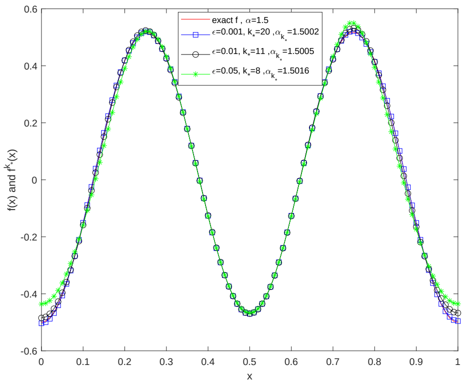
In what follows, we test the effect of the initial guess f^0 for Example 7.2 to $\alpha = 1.5$ with $\alpha^* = \alpha_0 = 1.8$ in which we take $\mu = 0.0001 \times \delta^{3/2}$, $\nu_k = 0.005 \times 0.5^k$. The initial guess f^0 are constant 0, linear function $f(0) + (f(1) - f(0))x$ through two endpoints and function $\cos(2\pi x)$. The numerical results are shown in Fig. 5 and Table 5. It can be seen that the numerical order α and source f are not sensitive to the initial value f^0.

Table 5: Example 7.2. Problem (P_1). Relative errors of numerical solutions.

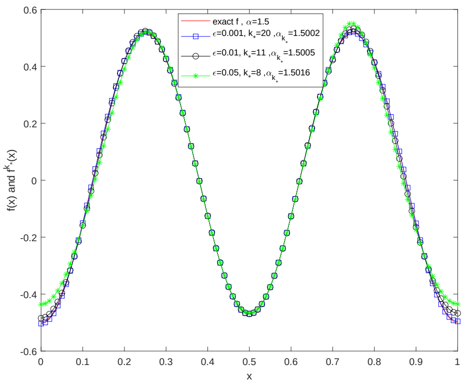
ε	$f^0 = 0$		$f^0 = f(0) + (f(1) - f(0))x$		$f^0 = \cos(2\pi x)$	
	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$
0.001	1.5002/0.0001	0.0052	1.5002/0.0001	0.0052	1.5002/0.0001	0.0052
0.01	1.5006/0.0004	0.0333	1.5005/0.0003	0.0278	1.5005/0.0003	0.0258
0.05	1.5018/0.0012	0.0863	1.5017/0.0011	0.0806	1.5016/0.0011	0.0798



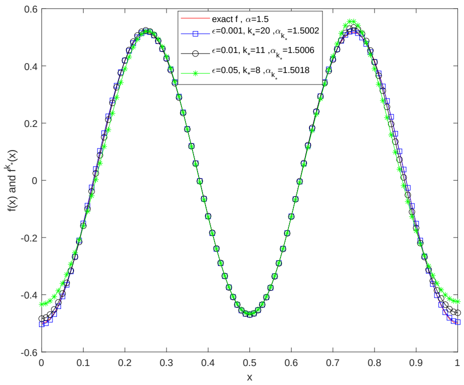
$\alpha_0 = 1.2$



$\alpha_0 = 1.4$



$\alpha_0 = 1.6$



$\alpha_0 = 1.8$

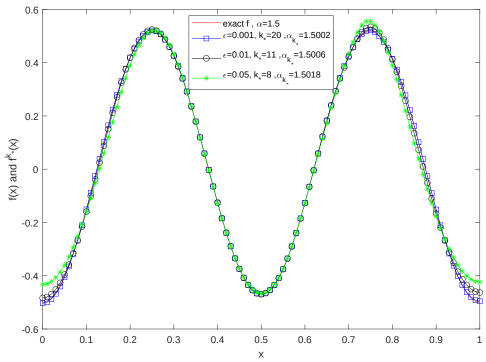
Figure 4: Example 7.2. Problem (P_1). Numerical solutions.

Problem (P_2). To solve the inverse problem in this case, we use additional data $u(x_0, t)$, $x_0 = 0$, $t \in [0, T_0] = [0, 0.75]$. Numerical results for fractional order $\alpha = 1.2, 1.5, 1.8$ are shown in Fig. 6, and the relative errors of numerical solutions are given in Table 6. Despite using fewer measurement data than the problem (P_1), the algorithm can still effectively recover the fractional order and space-independent source term. However, it is clear that the numerical results for (P_2) are not so good as ones to (P_1).

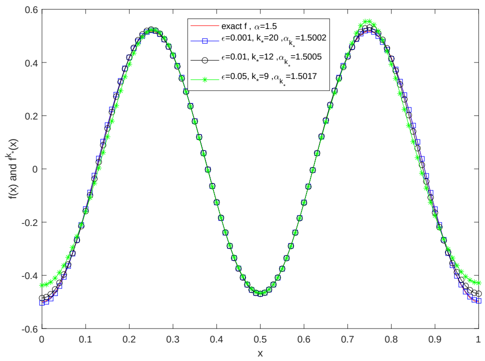
Table 6: Example 7.2. Problem (P_2). Relative errors of numerical solutions.

ε	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.8$	
	$\alpha_{k_*}/\text{err}_{k_*}^{(\alpha)}$	$\text{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\text{err}_{k_*}^{(\alpha)}$	$\text{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\text{err}_{k_*}^{(\alpha)}$	$\text{err}_{k_*}^{(f)}$
0.001	1.201/0.0009	0.0615	1.5001/0.0001	0.0451	1.7999/0.0000	0.0167
0.01	1.2034/0.0029	0.2071	1.5005/0.0003	0.1084	1.7948/0.0029	0.0800
0.05	1.1883/0.0097	0.4702	1.5019/0.0013	0.2407	1.7833/0.0093	0.2195

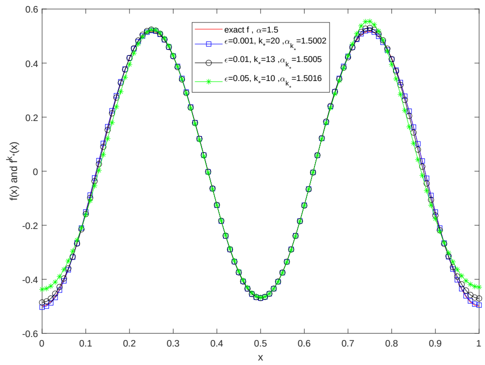
The influence of initial guess α_0 and initial guess f^0 for Example 7.2 to the problem (P_2) are also small, we just show the relative errors in Tables 7-8 for the case of $\alpha = 1.5$.



$f^0 = 0$



$f^0 = f(0) + (f(1) - f(0))x$

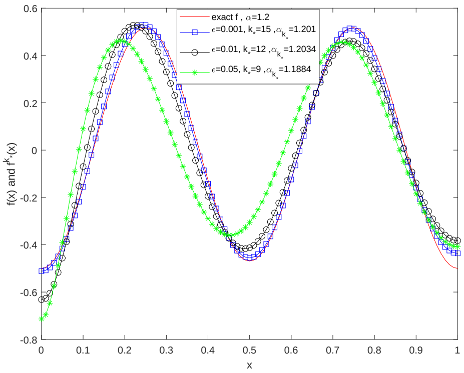


$f^0 = \cos(2\pi x)$

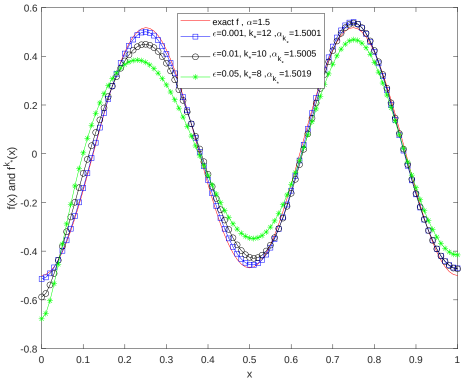
Figure 5: Example 7.2. Problem (P_1). Numerical solutions.

Table 7: Example 7.2. Problem (P_2). Relative errors of numerical solutions.

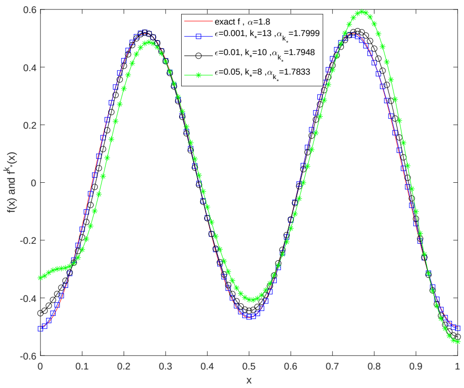
ε	$\alpha_0 = 1.2$		$\alpha_0 = 1.4$	
	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$
0.001	1.5001/0.0001	0.0451	1.5001/0.0001	0.0451
0.01	1.5004/0.0003	0.1109	1.5005/0.0003	0.1087
0.05	1.5018/0.0012	0.2439	1.5019/0.0013	0.2413
ε	$\alpha_0 = 1.6$		$\alpha_0 = 1.8$	
	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$
0.001	1.5002/0.0001	0.0451	1.5002/0.0001	0.0397
0.01	1.5005/0.0003	0.1078	1.5008/0.0005	0.1041
0.05	1.502/0.0013	0.2419	1.5028/0.0019	0.2633



$\alpha = 1.2$



$\alpha = 1.5$



$\alpha = 1.8$

Figure 6: Example 7.2. Problem (P_2). Numerical solutions.

Table 8: Example 7.2. Problem (P_2). Relative errors of numerical solutions.

ε	$f^0 = 0$		$f^0 = f(0) + (f(1) - f(0))x$		$f^0 = \cos(2\pi x)$	
	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$	$\alpha_{k_*}/\mathrm{err}_{k_*}^{(\alpha)}$	$\mathrm{err}_{k_*}^{(f)}$
0.001	1.5002/0.0001	0.0397	1.5002/0.0001	0.0426	1.5003/0.0002	0.0504
0.01	1.5008/0.0005	0.1041	1.5009/0.0006	0.1008	1.5009/0.0006	0.1104
0.05	1.5028/0.0019	0.2633	1.504/0.0027	0.2941	1.5038/0.0025	0.2810

7.2. Two-dimensional case

Write (x, y) for (x_1, x_2), $\underline{\alpha} = 1.01$, $\bar{\alpha} = 1.99$. Let Ω be a smooth domain shown in Fig. 7. The mesh nodes on Ω and grid point in $[0, T]$ are 437 and 101 for solving direct problem (1.1), sensitive problem (6.3) in the inverse iterative process by a piecewise linear finite element scheme in space and $L1$-type discrete scheme in time. The maximum number of iteration steps is 15.

Example 7.3. Let

$$\begin{aligned} &a_{11}(x,y)=x^2+3, && a_{12}(x,y)=x+y+1,\\ &a_{22}(x,y)=y^2+3, && c(x,y)=x^2+y^2+1,\\ &\phi(x)=\cos(\pi x)\cos(\pi y)-\frac{1}{2}, && \psi(x)=\frac{1}{2}\cos(\pi x)\cos(\pi y)-2. \end{aligned}$$

Take

$$\begin{aligned} &p(t)=p(t)=2(t+2)^2+6\cos(2\pi t)+10\pi e^{4-\alpha},\\ &f(x)=\cos(\pi x)\cos(\pi y)\big(0.4^2-(x^2+y^2)\big)\chi, \end{aligned}$$

where

$$\chi=\begin{cases}1, & \text{if} \quad x^2+y^2\le 0.4^2,\\ 0, & \text{otherwise.}\end{cases}$$

We recover a numerical solution for α and $f(x,y)$ by the noisy data with relative noise levels $\varepsilon=0.001,0.01,0.05$. Take $\mu=0.01\times\delta^{3/2}$, $\nu_k=0.01\times 0.5^k$ and $\tau=1.001$, $\varsigma=0.0001$ if no other specified. The objective guess are $\alpha^*=1.5$ in the regularization functional, and the initial values are $f^0=0$ and $\alpha_0=1.5$ if no further specified.

Problem (P$_1$). The measured domain ω can be seen in Fig. 7 and $T_0=0.7$. The additional data $u(x,y,t)$ over $\omega\times[0,T_0]$ is obtained by solving the direct problem (1.1). The exact solution is shown in Fig. 8. The numerical results for α and $f(x,y)$ corresponding to different fractional orders $\alpha=1.3,1.7$ are shown in Fig. 9 for $\varepsilon=0.001$ and Fig. 10 for $\varepsilon=0.01$. It can be seen that the numerical results match the exact solutions reasonably which indicate that the proposed iterative method is effective in two-dimensional case. The numerical solutions for $\varepsilon=0.001$ have much better accuracy than those for $\varepsilon=0.01$. The results for $\alpha=1.7$ are better than ones for $\alpha=1.3$.

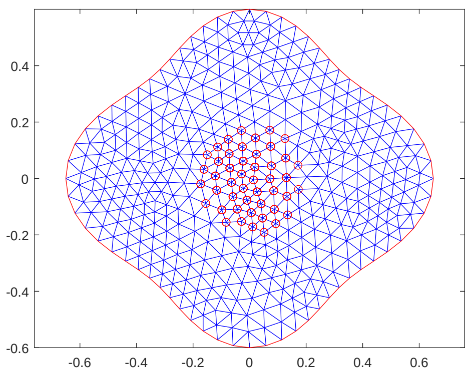


Figure 7: Domain and mesh.

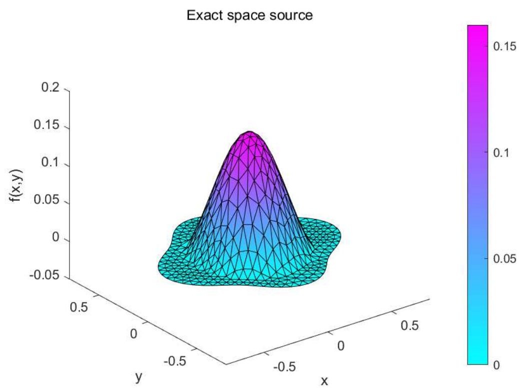
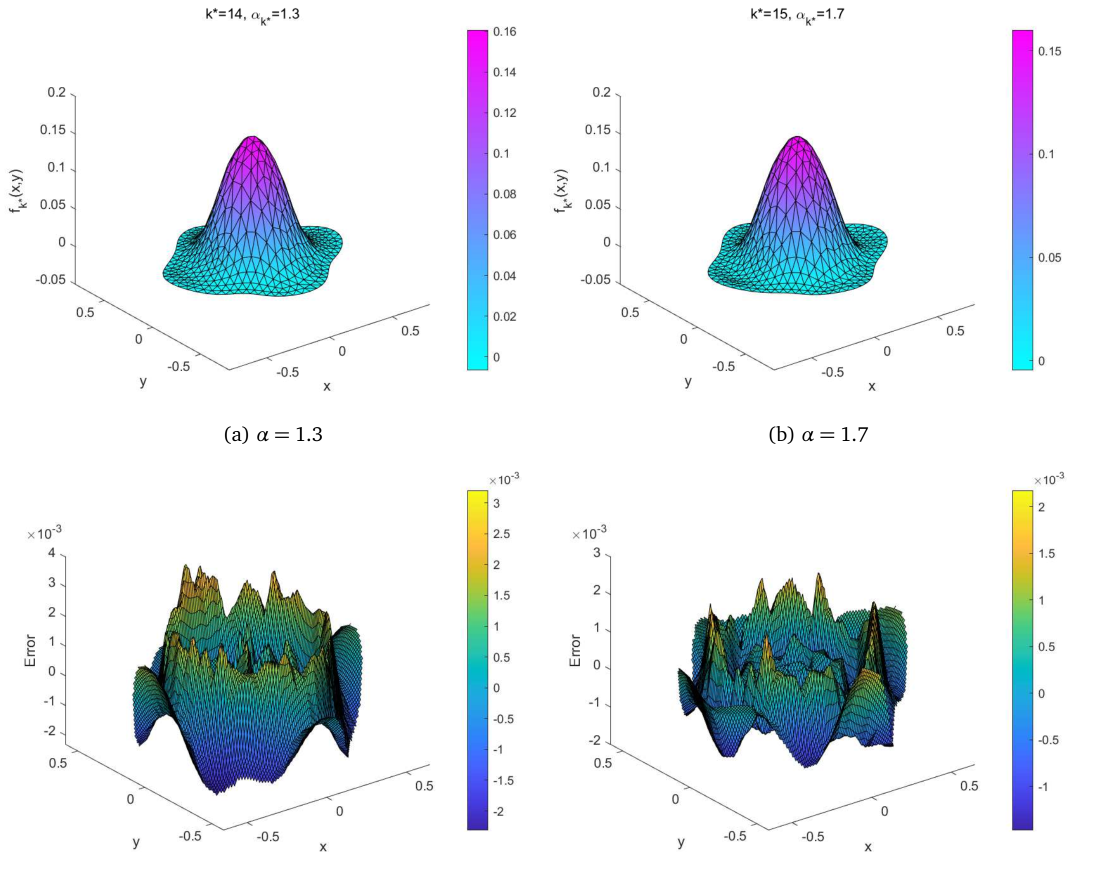


Figure 8: Exact solution.



(a) $\alpha = 1.3$ (b) $\alpha = 1.7$

(c) $\alpha = 1.3$ (d) $\alpha = 1.7$

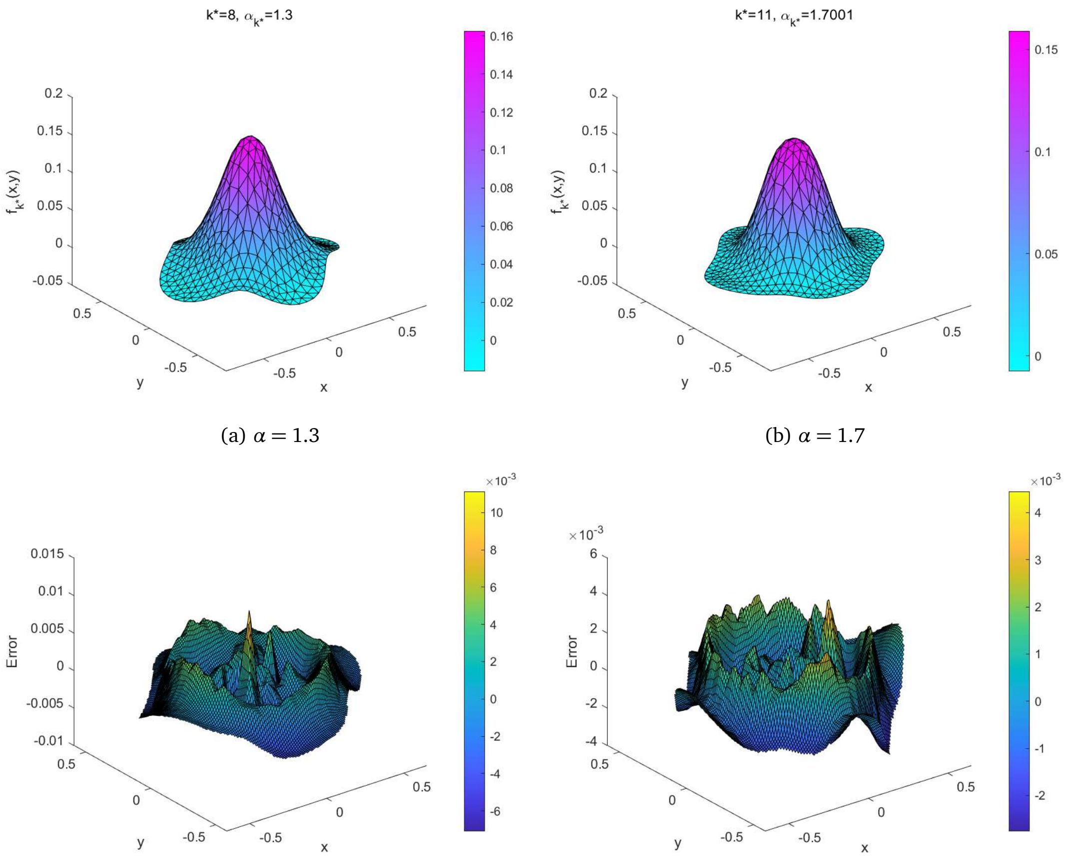
Figure 9: Example 7.3. Problem (P_1). Numerical solutions and absolute errors, $\varepsilon = 0.001$.

Problem (P_2). The measured boundary Γ can be seen in Fig. 11 and take $T_0 = 0.7$. The exact source function is shown in Fig. 12.

In Figs. 13-14, we show the numerical results and absolute errors for different fractional orders $\alpha = 1.3, 1.7$ with the noise levels $\varepsilon = 0.001, 0.01$, respectively. We can see the numerical results become worse for a little large noise level $\varepsilon = 0.01$ since for this case the measurement data are few. However, the proposed iterative method is also effect for solving the inverse problem (P_2) by using the boundary measured data.

8. Conclusion

This paper is devoted to identifying the order of fractional derivative and the space-dependent source term for a time-fractional diffusion-wave equation. The Lipschitz continuity of forward operators are obtained based on the solution of direct problem. Using the asymptotic properties of the solution, the Titchmarsh convolution theorem, and



(a) $\alpha = 1.3$ (b) $\alpha = 1.7$

(c) $\alpha = 1.3$ (d) $\alpha = 1.7$

Figure 10: Example 7.3. Problem (P_1). Numerical solutions and absolute errors, $\varepsilon = 0.01$.

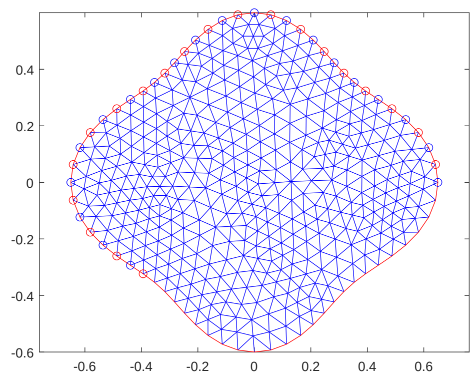


Figure 11: Domain and mesh.

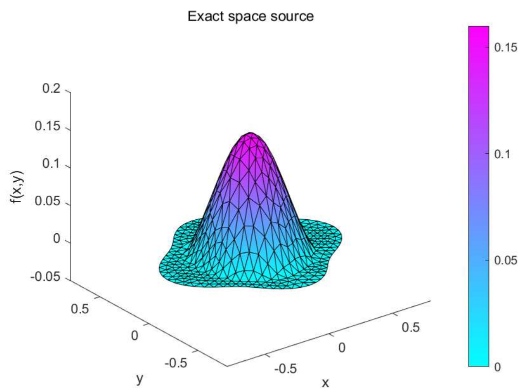
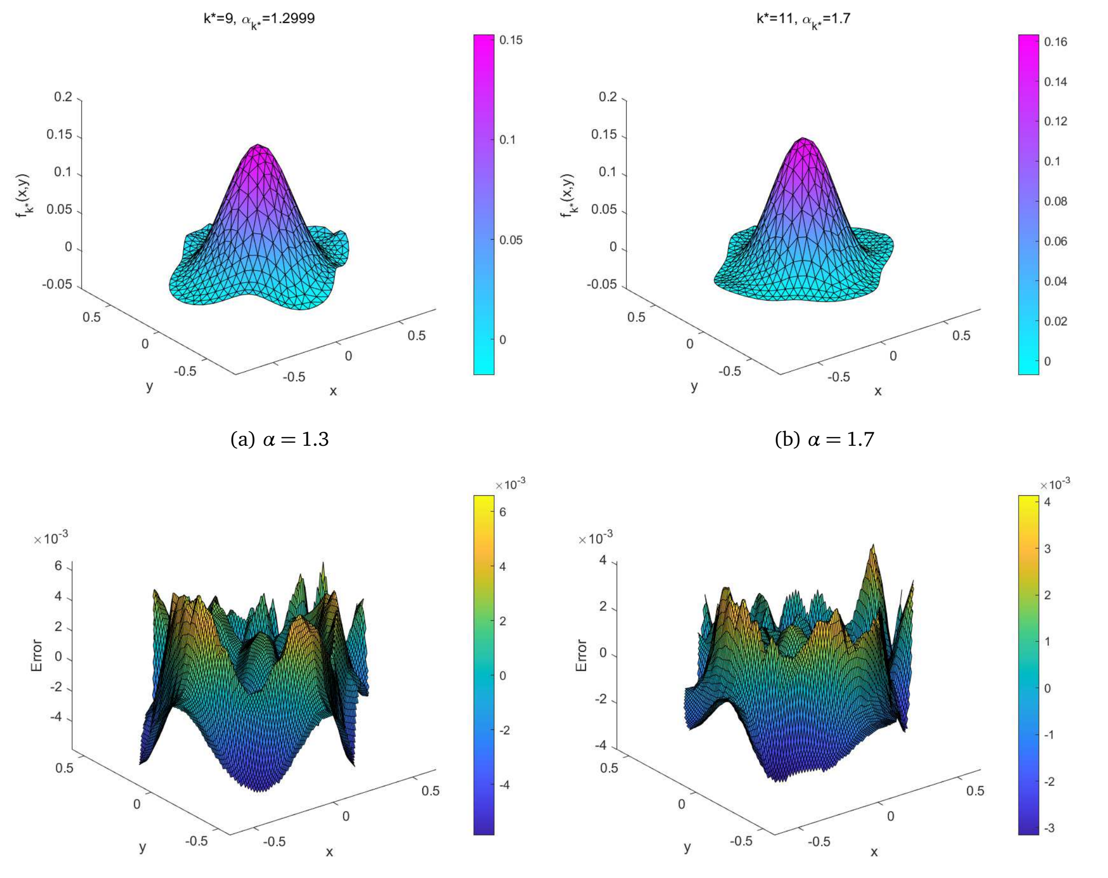


Figure 12: Exact solution.



(a) $\alpha = 1.3$ (b) $\alpha = 1.7$

(c) $\alpha = 1.3$ (d) $\alpha = 1.7$

Figure 13: Example 7.3. Problem (P_2). Numerical solutions and absolute errors, $\varepsilon = 0.001$.

the Duhamel principle, we prove the uniqueness of the inverse order and source term. We propose a Tikhonov-type regularization method to transform the inverse problem into a variational problem and prove the existence and convergence of the regularized solution. Finally, the variational problem is solved by using the linearized iterative method combined with the piecewise linear finite element approximation. The Morozov's discrepancy principle is used to get a suitable stopping step. Numerical results for three examples in one-dimensional and two-dimensional cases demonstrate the effectiveness and stability of the proposed method.

Acknowledgments

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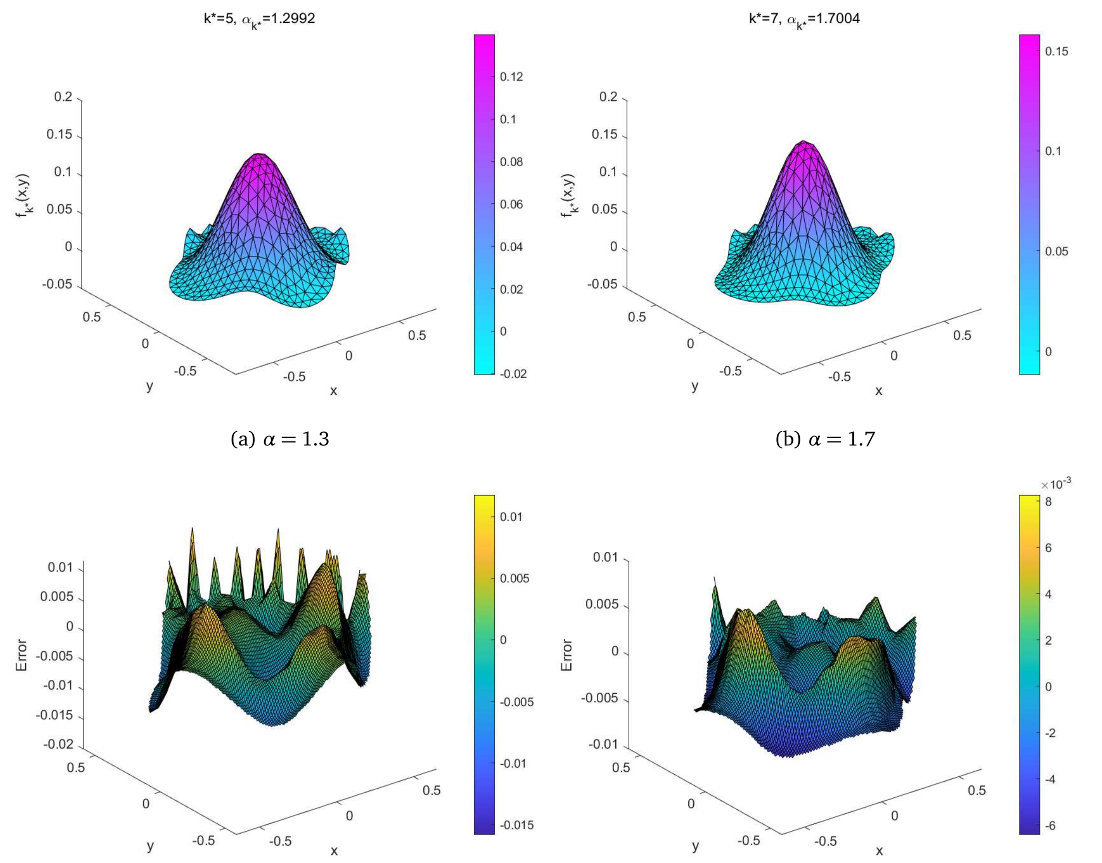

(a) $\alpha = 1.3$ (b) $\alpha = 1.7$

(c) $\alpha = 1.3$ (d) $\alpha = 1.7$

Figure 14: Example 7.3. Problem (P_2). Numerical solutions and absolute errors, $\varepsilon = 0.01$.

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