East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2025-029.190725

Vol. **16**, No. 2, pp. 413-426
May 2026

The Quasi-Reversibility Regularization for Source Inversion of a Linear Kuramoto-Sivashinsky Equation

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Received 4 February 2025; Accepted (in revised version) 19 July 2025.

Abstract. The source inversion problem for the linear Kuramoto-Sivashinsky equation is studied. To address the ill-posedness of the source term inversion, a quasi-reversibility regularization method is used for the stable reconstruction of the source term. In the case of noisy measurement data, the convergence (error estimate) of regularized solutions with respect to noise level is analyzed under a priori choice of the regularization parameter. Subsequently, a modified Morozov discrepancy principle is proposed to select regularization parameters for the a posteriori principle. The convergence (error estimate) of regularized solutions is established under this a posteriori principle. Numerical examples are provided to verify the effectiveness of the proposed quasi-reversibility regularization method.

AMS subject classifications: 35R30, 65M32

Key words: Kuramoto-Sivashinsky equation, source term inversion, quasi-reversibility regularization, ill-posed problem, convergence.

1. Introduction

The Kuramoto-Sivashinsky (KS) equation is an important partial differential equation. This equation was proposed by Kuramoto *et al.* [11–13] and by Sivashinsky [18]. The KS equation has been widely used to describe various physical phenomena such as turbulence, flame propagation, and surface erosion caused by ion sputtering in amorphous materials [4, 19]. However, studies on the inverse problems of the KS equation are relatively limited. Relevant works [3, 6, 15] have explored boundary control and null control problems, reflecting ideas and methods for inverse problems; while works [1, 16] focus on the inversion of the anti-diffusion and principal coefficients in the KS equation, showing the

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Lipschitz stability of coefficient inversion. Inverse source problems aiming to determine unknown source terms based on measurable information [2,7,8,10,20–24,26], are very common in applied sciences and engineering. However, very often the problems are ill-posed, so that small measurement errors may lead to significant fluctuations in the solution. For example, determining the intensity of a pollution source by measuring the concentration of specific pollutants [20] is crucial in environmental protection. This paper focuses on the KS equation to study the quasi-reversibility regularization method for its source inversion problem, as well as the convergence analysis of the regularized solution.

Let $Q_T = (0, L) \times (0, T)$, where $L, T > 0$. Consider the following linear KS equation with the initial and boundary conditions:

$$\begin{aligned} & y_t + y_{xxxx} + k y_{xx} - q y = f(x), && (x,t) \in Q_T, \\ & y(0,t) = y(L,t) = 0, \;\; y_{xx}(0,t) = y_{xx}(L,t) = 0, && t \in (0,T), \\ & y(x,0) = h(x), && x \in [0, L], \end{aligned} \tag{1.1}$$

where the constant $k > 0$ is the anti-diffusion parameter. Here, we consider the following source inversion problem: Given coefficients k, q, and initial condition $h(x)$, reconstruct the source term $f(x)$ from the final value $y(x, T) = g(x)$, $x \in [0, L]$.

It is well known that the inverse source problems for partial differential equations are often ill-posed [21–24], so that small measurement errors can have drastic impact on solutions. In fact, it is easily verified that if $k > 0$, $q \geq 0$, and $\lambda_n^4 - k\lambda_n^2 - q \neq 0$, then the problem (1.1) has the solution

$$y(x,t) = \sum_{n=1}^{\infty} h_n e^{-(\lambda_n^4 - k\lambda_n^2 - q)t} \phi_n(x) + \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n^4 - k\lambda_n^2 - q} \left(1 - e^{-(\lambda_n^4 - k\lambda_n^2 - q)t}\right) \phi_n(x),$$

where

$$\lambda_n = \frac{n\pi}{L}, \quad \phi_n(x) = \sqrt{\frac{2}{L}} \sin(\lambda_n x), \quad h_n = \langle h, \phi_n \rangle, \quad f_n = \langle f, \phi_n \rangle, \quad g_n = \langle g, \phi_n \rangle$$

and $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(0, L)$. Expanding the final value $y(x, T) = g(x)$ in the Fourier series with respect to the system $\{\phi_n(x)\}_{n=1}^{\infty}$ gives

$$f(x) = \sum_{n=1}^{\infty} \frac{\lambda_n^4 - k\lambda_n^2 - q}{1 - e^{-(\lambda_n^4 - k\lambda_n^2 - q)T}} \left(g_n - h_n e^{-(\lambda_n^4 - k\lambda_n^2 - q)T}\right) \phi_n(x). \tag{1.2}$$

In practice, the data obtained are not noise-free $g(x)$ but noisy measurement data $g^\delta(x)$ such that

$$\|g^\delta - g\|_{L^2(0,L)} \leq \delta,$$

where δ is the noise level. Consequently, the data used for computations in formula (1.2) are measured data $g^\delta(x)$, i.e. $g_n^\delta = \langle g^\delta, \phi_n \rangle$. In this case, the error in $g^\delta(x)$ is severely amplified by the fact that the coefficients

$$c_n = \frac{\lambda_n^4 - k\lambda_n^2 - q}{1 - e^{-(\lambda_n^4 - k\lambda_n^2 - q)T}}$$