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A New Multilevel Homotopic Adaptive Finite Element Algorithm for Convection-Dominated Diffusion-Reaction Problems

Jie Zhang[1], Guozhu Yu[1,*] and Xiaoping Xie[2]

[1] *School of Mathematics, Southwest Jiaotong University, Chengdu 610031, China.*
[2] *School of Mathematics, Sichuan University, Chengdu 610064, China.*

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Abstract. The multilevel homotopic adaptive finite element method (MHAFEM) was developed for solving convection-dominated diffusion-reaction problems on triangular meshes, where the mesh adaptation (refinement and mesh smoothing/moving) is based on an optimal interpolation error estimate in L^p norm. In this paper, we follow the framework of the MHAFEM to give a new multilevel homotopic adaptive finite element algorithm. This new MHAFEM executes mesh refinement based on an a posteriori error estimator instead of the interpolation error estimate. We apply this algorithm to 1D and 2D example problems using 1D linear elements and 2D quadrilateral transition elements, respectively. Numerical results demonstrate the remarkable efficiency of the algorithm.

AMS subject classifications: 65N22, 65N30

Key words: Convection-dominated diffusion-reaction problem, multilevel homotopic adaptive finite element method, quadrilateral transition element, layer.

1. Introduction

Convection-diffusion-reaction equations describe transport phenomena of various physical quantities in many real-world scenarios such as mass, heat, and energy. High convection often takes place alongside low diffusion. Due to the small diffusion coefficient, solutions of convection-dominated diffusion-reaction problems may have layers of small width where derivatives of the solutions change dramatically. Standard finite element methods such as continuous linear elements often exhibit large spurious oscillations in the layer regions, and layers may trigger oscillations throughout the whole computational domain that render the finite element approximation useless when the meshes are not fine enough.

There are roughly three strategies in the finite element analysis of convection-diffusion problems to avoid the non-physical oscillations or resolve the layers. The first one is to

*Corresponding author. *Email addresses:* jie_m_zhang@163.com (J. Zhang), yuguozhumail@163.com (G. Yu), xpxie@scu.edu.cn (X. Xie)

use certain stabilization method — e.g. the streamline upwind Petrov-Galerkin (SUPG) method [8], the Galerkin least-square method [4,10,37], the residual-free bubble method [5,7,23,24], the variational multiscale approach [35,36,38], the multiscale finite element method [18, 29], the subgrid stabilization method [6, 27, 28] and the nonlinear subgrid method (also called the dynamic diffusion method) [2, 19, 40, 45–47, 52], discontinuous Galerkin (DG) method [3, 9, 16, 30, 56], hybridizable DG method [15, 17, 25], and weak Galerkin method [11,26,55].

The second strategy is to apply Shishkin type meshes [44,48,49]. A common belief is that once the layer is fully resolved, then the corresponding finite element method holds the optimal or quasi-optimal convergence rate [42, 43]. By means of Shishkin meshes, Linß [43] considered a model singularly perturbed convection-diffusion problem whose solution contains exponential boundary and corner layers and compared several numerical methods. Numerical results showed that standard continuous linear and bilinear finite element methods work well under the Shishkin meshes. Zhang [57] used the bilinear finite element method on a Shishkin mesh for 2D convection-diffusion problems and even established a superconvergence result under certain regularity assumptions. However, Shishkin type meshes usually require a prior information to determine the location and width of the interior and boundary layers in advance.

The third strategy is to adopt an adaptive mesh adjustment based on a posteriori or interpolation error estimate. Local mesh refinement (h-adaption) based on the a posteriori error estimation is a commonly used tool, and the resulting adaptive procedure consists of loops of the form

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE.}$$

Note that the a posteriori error estimation (ESTIMATE) [1,19–22] is an essential ingredient of adaptivity. Verfürth [53] firstly proposed a posteriori error estimator for the standard Galerkin approximation and the SUPG discretization of the convection-diffusion equations with dominant convection, and showed that the estimator is robust when the local Peclét number is not very large. Later, John [39] numerically compared several a posteriori error estimators for the convection-diffusion equations with respect to the reliable estimation of the global error and with respect to the accuracy of the computed solutions on adaptively refined meshes, and showed that no error estimator works satisfactorily for all numerical examples. When the diffusion coefficient is small, the resulting adaptive mesh refinement may put the refined mesh at a wrong place. We mention that the moving mesh method (r-adaption) based on minimizing certain interpolation error is another way to obtain an adaptive mesh. Huang [33] adapted meshes through equidistributing the interpolation error and investigated the convergence for one-dimensional singularly perturbed two-point boundary value problems without turning points. For multiple dimensions, Huang [34] obtained interpolation error estimates in L^2 and H^1 norms, while Chen *et al.* [14] presented an optimal interpolation error estimate in L^p norm, $1 \leq p \leq \infty$.

By combining the streamline diffusion finite element method, anisotropic mesh adaptation and the homotopy of the diffusion coefficient [13], Chen *et al.* [50] developed a multilevel homotopic adaptive finite element method for convection dominated problems on