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An Energy Stable TFPM-Based Petrov-Galerkin Scheme for Solving the Allen-Cahn Equation

Yueran Wang[1], Wenli Yang[2] and Zhongyi Huang[1,*]

[1]*Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.*
[2]*School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China.*

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Abstract. An energy stable tailored finite point method based Petrov-Galerkin scheme to solve Allen-Cahn equation is proposed. In time discretization, we present both first-order and second-order semi-discrete schemes based on stabilized and convex-splitting techniques, which satisfy unconditional energy stability. We prove the maximum bound preserving principle for first-order schemes. Due to nonlinearity, the well-posedness of weak formulations based on semi-discrete schemes are demonstrated. As the nature of singularly perturbation in semi-discrete level remains when ϵ is extremely small, we establish a specified Petrov-Galerkin scheme which leads to a unified way for space discretization. To this end, we set up nonlinear solvers which are proved to be stable and convergent. Then we construct our Petrov-Galerkin scheme, which is built upon problem-dependent test function space. The stability and second-order convergence of this scheme are rigorously proved in one dimension. In order to compute test functions, specialized TFPM schemes are incorporated into the scheme. Numerical experiments show the accuracy, efficiency, and the good performance of the method on uniform meshes even when mesh size h is much larger than ϵ.

AMS subject classifications: 65M60, 74A50, 65M12

Key words: Allen-Cahn equation, energy stability, nonlinear solvers, Petrov-Galerkin scheme, tailored finite point method.

1. Introduction

In this paper, we consider the numerical methods for solving the following Allen-Cahn equation:

$$\frac{\partial u}{\partial t} = \epsilon \Delta u - f(u), \quad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

*Corresponding author. *Email addresses:* yr-wang20@mails.tsinghua.edu.cn (Y. Wang), yangwl19@cumt.edu.cn (W. Yang), zhongyih@tsinghua.edu.cn (Z. Huang)

$$u(x,0)=u_0(x), \qquad x\in\Omega, \tag{1.2}$$
$$u(x,t)=0, \qquad (x,t)\in\partial\Omega\times[0,T], \tag{1.3}$$

where $\Omega\subseteq\mathbf{R}^d$ is a bounded domain and $f(u)=F'(u)$, which is nonlinear in general, corresponds with the free energy density $F(u)$. The parameter ϵ is referred to as the interface width. It is sufficiently small in practice, which leads to essential problems in numerical computations.

This equation was originally introduced by Allen and Cahn [1] in order to describe the phase separation of polycrystaline material. Nowadays, Allen-Cahn equation has been utilized across widespread applications, including crystal growth [5, 29, 30], phase transition [14], interface diffusion [2], N-phase flows [46], and image analysis [4, 32, 35]. It is well-known that Allen-Cahn equation can be viewed as the L^2 gradient flow with respect to the bulk free energy

$$E(u)=\int_\Omega\Big(\frac{\epsilon}{2}|\nabla u|^2+F(u)\Big)\,dx.$$

As a direct consequence, the solution to (1.1)-(1.3) enjoys the property of energy dissipation — i.e.

$$\frac{d}{dt}E\big(u(t)\big)=-\int_\Omega\left|\frac{\partial u}{\partial t}\right|^2 dt\le 0.$$

Another important property satisfied by Allen-Cahn equation is the maximum bound preserving (MBP) principle, which states that if the initial condition u_0 is bounded — i.e. $|u_0(x)|\le 1$ for all $x\in\Omega$, then

$$|u(x,t)|\le 1, \quad (x,t)\in\Omega\times(0,T].$$

Given their significance, the law of energy decay and MBP principle should be taken into account when designing numerical schemes.

As one of the most important equations among the realm of phase-field models, a variety of scholars have made valuable contributions to the development of numerical methods for solving this type of problems over the years. Due to the presence of nonlinear term, the numerical schemes using explicit treatment usually impose strict requirements on the time step to ensure discrete energy dissipation. To loosen these requirements, various methods based on adding stabilization terms with specific parameters have been proposed [16, 25, 39, 49]. Note that the resulting schemes still conditionally comply with the energy dissipation law. Convex splitting methods [13, 15, 31] reformulate the total free energy into the difference of two convex functionals, which leads to unconditional discrete energy stable schemes. Recently, a new group of methods inspired by invariant energy quadratization methods [45, 48] and called scalar auxiliary variable (SAV) methods has been developed — cf. [37, 38]. The idea of SAV method is to introduce a variable that depends only on space coordinates. This will lead to constant coefficient linear system at each time step which is easy to solve. Nevertheless, the discrete energy law fulfilled by SAV schemes is in the sense of a modified energy that often differs from the original one. We re-