

# Energy-Stable Full Discretization of the Modified Elastic Flow of Closed Curves

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**Abstract.** An energy-stable full discretization for the modified elastic flow of closed curves is proposed. This is a gradient flow of a modified elastic energy combining bending and Dirichlet energies. The minimization of Dirichlet energy can lead to improved mesh quality. Gradient flows for both isotropic and anisotropic cases are considered. We derive new evolution equations for the parameterization and curvature vector of curves in arbitrary codimension. The proposed formulation is discretized by a parametric finite element method in space and a first-order implicit scheme in time. We establish the unconditional energy stability for the fully discretized scheme. Additionally, the second-order accuracy of the BDF2 scheme is demonstrated. Numerical examples in two and three dimensions illustrate the efficiency, energy stability, and asymptotic mesh distribution of the method for simulating the modified elastic flow.

**AMS subject classifications:** 65M60, 65M12, 53E40, 35G20

**Key words:** Curve evolution, elastic flow, parametric finite element method, unconditional stability, Dirichlet energy.

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## 1. Introduction

The elastic energy functional is of great importance in various fields, including differential geometry, computer graphics, and mathematical modeling [10, 11, 26, 36]. Elastic flow describes the dynamic motion of a curve driven by elastic energy minimization. The equilibrium shapes associated with critical points of the elastic energy are known as elastic curves. In this paper, we present energy-stable numerical schemes for the simulation of elastic flow of closed curves with isotropic and anisotropic energies in  $\mathbb{R}^n$ ,  $n \in \mathbb{N}^+$  and  $n \geq 2$ .

**Isotropic case.** Recall that the classical isotropic elastic energy is defined by

$$E_{\tilde{\lambda}_1}(t) = E(t) + \tilde{\lambda}_1 L(t), \quad (1.1)$$

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where  $E(t) = \int_{\Gamma_t} |\boldsymbol{\kappa}|^2 ds/2$  and  $L(t) = \int_{\mathbb{I}} |\mathbf{x}_\rho| d\rho$  are respectively the bending energy and the length functional,  $\tilde{\lambda}_1 \geq 0$  is a constant,  $s$  denotes the arc-length of a curve  $\Gamma_t$ ,  $ds = |\mathbf{x}_\rho| d\rho$ ,  $\rho \in \mathbb{I} = [0, 1]$  is a time-independent variable, and  $\mathbb{I} = \mathbb{R}/\mathbb{Z}$  represents the periodic unit interval. Besides, the isotropic curvature vector  $\boldsymbol{\kappa}$  is defined by

$$\boldsymbol{\kappa} = \mathbf{x}_{ss}, \tag{1.2}$$

and  $\mathbf{x}(\cdot) = (x_1(\cdot), x_2(\cdot), \dots, x_n(\cdot)) : \mathbb{I} \rightarrow \mathbb{R}^n$  denotes the parameterization of a curve. Let  $\boldsymbol{\tau} = \mathbf{x}_s$  be the tangent vector and  $\nabla_s f := f_s - (f_s \cdot \boldsymbol{\tau})\boldsymbol{\tau}$  denote the normal projection of a vector-valued function  $f$ . The  $L^2$ -gradient flow of  $E_{\tilde{\lambda}_1}(t)$  is given by

$$\mathbf{x}_t = -\nabla_s^2 \boldsymbol{\kappa} - \frac{1}{2} |\boldsymbol{\kappa}|^2 \boldsymbol{\kappa} + \tilde{\lambda}_1 \boldsymbol{\kappa}.$$

It is called the isotropic elastic flow. Here we consider a modified isotropic elastic energy  $E_{\lambda_1}(t)$  penalized by the Dirichlet energy  $D(t)$  as follows:

$$E_{\lambda_1}(t) = E(t) + \lambda_1 D(t), \quad D(t) := \frac{1}{2} \int_{\mathbb{I}} |\mathbf{x}_\rho|^2 d\rho, \tag{1.3}$$

where  $\lambda_1 \geq 0$  is a constant. The minimization of  $D(t)$  is effective for improving the mesh quality in evolving interface problems, as shown in [17–19]. The  $L^2$ -gradient flow of the isotropic energy functional  $E_{\lambda_1}(t)$  is given by

$$\mathbf{x}_t = -\nabla_s^2 \boldsymbol{\kappa} - \frac{1}{2} |\boldsymbol{\kappa}|^2 \boldsymbol{\kappa} + \lambda_1 \mathbf{x}_{\rho s}. \tag{1.4}$$

We refer to (1.4) as the modified isotropic elastic flow. In the previous work [33], the authors illustrated that the set of stationary points of  $E_{\lambda_1}(t)$  is equivalent to that of the classical isotropic elastic energy  $E_{\tilde{\lambda}_1}(t)$ . This observation is beneficial for analyzing error estimates and improving the mesh quality, since it allows minimizing the Dirichlet energy. Therefore, we consider the energy functional (1.3) instead of (1.1).

**Anisotropic case.** The classical anisotropic elastic energy  $E_{\tilde{\lambda}_2}$  is defined as

$$E_{\tilde{\lambda}_2}(t) = E_\gamma(t) + \tilde{\lambda}_2 L_\gamma(t), \tag{1.5}$$

where  $E_\gamma(t) := \int_{\Gamma_t} |\boldsymbol{\kappa}_\gamma|^2 ds/2$  and  $L_\gamma(t) := \int_{\mathbb{I}} \gamma(\mathbf{x}_\rho) d\rho$  are respectively the anisotropic bending energy and the weighted length functional, and  $\tilde{\lambda}_2 \geq 0$  is a constant. Moreover, the anisotropic function  $\gamma \in C^2(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R}^+) \cap C(\mathbb{R}^n, \mathbb{R}^+)$  is convex, and positively homogeneous of degree one — i.e.

$$\gamma(\lambda \mathbf{p}) = \lambda \gamma(\mathbf{p}) \quad \text{for all } \mathbf{p} \in \mathbb{R}^n \quad \text{and all } \lambda \in \mathbb{R}^+.$$

This yields

$$\boldsymbol{\gamma}'(\mathbf{p}) \cdot \mathbf{p} = \gamma(\mathbf{p}), \quad \boldsymbol{\gamma}''(\mathbf{p})\mathbf{p} = \mathbf{0} \quad \text{for all } \mathbf{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \tag{1.6}$$