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# Dynamics of the Center of Mass in Arbitrary-Angle Rotating Bose-Einstein Condensates

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**Abstract.** This paper investigates the dynamics of the center of mass in three-dimensional arbitrary-angle rotating Bose-Einstein condensates (ARotBECs), which are governed by the Gross-Pitaevskii equation (GPE) with an arbitrary-angle angular momentum rotation term. The second-order ordinary differential equations (ODEs) which govern the motion of the center of mass of ARotBECs are analytically solved. Subsequently, a novel numerical scheme, which is mass conservative — i.e. the Lagrangian multiplier-based Crank-Nicolson leap-frog (LagM-CNLF) method, integrated with Fourier pseudo-spectral spatial discretization, is proposed to efficiently and accurately simulate the GPE with arbitrary-angle rotation term. Finally, distinct motion patterns of the center of mass are systematically categorized based on analytical solutions and rigorously validated through direct numerical simulations of the GPE, demonstrating robust consistency between theoretical predictions and numerical computational results. The dynamics of quantized vortices is also studied to show the effectiveness of the LagM-CNLF method.

**AMS subject classifications**: 65M22, 65M70

**Key words**: Bose-Einstein condensate, arbitrary-angle rotation, Gross-Pitaevskii equation, the center of mass, Lagrangian multiplier method.

## 1. Introduction

Since the landmark creation of Bose-Einstein condensates (BECs) in dilute bosonic atomic gases in 1995 [2, 12, 17], this macroscopic quantum state has become an indispensable platform for exploring quantum behaviors. The introduction of rotation into BEC systems not only poses new theoretical and experimental challenges but also opens new frontiers, revealing exotic quantum vortex patterns and generating cross-disciplinary impacts spanning astrophysics, atomic physics, quantum optics, and superfluid hydrodynamics [1, 13, 18–21, 25, 28].

Conventional experiments on rotational BECs typically focus on rotation along the $z$ axis, whereas practical applications may increasingly demand complex scenarios requiring arbit-

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rary-angle rotational configurations. For example, Prasad *et al.* [24] extended the Thomas-Fermi regime to harmonically trapped dipolar BECs polarized by a continuously rotating field, enabling arbitrary-angle orientations of the rotation axis. Recently, a BEC confined by a rotating harmonic trap whose rotation axis is not aligned with any of its principal axes has been investigated [23]. BECs with arbitrary angle rotation are a topic of interest in the field of quantum physics. When a BEC is rotated at an arbitrary angle, it exhibits unique and intriguing properties.

In this paper, we consider the arbitrary-angle rotating BEC. At temperatures $(T)$ significantly lower than the critical temperature $(T_c)$, the macroscopic behavior of an arbitrary-angle rotating BEC can be effectively characterized by a complex-valued macroscopic wave function $\psi(\boldsymbol{x},t)$. For $t \geq 0$, the evolution of wave function is described by the following three-dimensional dimensionless Gross-Pitaevskii equation [4]:

$$\begin{aligned} &i\partial_t\psi(\boldsymbol{x},t) = \left(-\frac{1}{2}\Delta + V(\boldsymbol{x}) + \beta|\psi(\boldsymbol{x},t)|^2 - \boldsymbol{\Omega}\cdot\boldsymbol{L}\right)\psi(\boldsymbol{x},t), \\ &\psi(\boldsymbol{x},t=0) = \psi_0(\boldsymbol{x}), \quad \boldsymbol{x} = (x,y,z)^\top \in \mathbb{R}^3, \end{aligned} \tag{1.1}$$

where $\psi_0(\boldsymbol{x})$ is the initial value. The constant $\beta$ associated with the $s$-scattering length $a_s$, characterizes the atom-atom interaction strength (positive for repulsive and negative for attractive interaction). $V(\boldsymbol{x})$ is a dimensionless external trapping potential and here chosen as the harmonic potential

$$V(\boldsymbol{x}) = \left(\gamma_x^2x^2 + \gamma_y^2y^2 + \gamma_z^2z^2\right)/2 \tag{1.2}$$

with $\gamma_x,\gamma_y$ and $\gamma_z$ being the trapping frequencies in each spatial direction. The angular velocity $\boldsymbol{\Omega} = (\omega_x,\omega_y,\omega_z)^\top \in \mathbb{R}^3$. The rotating frame is characterized by the inner product of angular velocity $\boldsymbol{\Omega}$ and momentum $\boldsymbol{L}$ as $\boldsymbol{\Omega}\cdot\boldsymbol{L}$, where the angular momentum is defined as

$$\boldsymbol{L} = (L_x,L_y,L_z)^\top := \boldsymbol{x}\times\boldsymbol{p} = \boldsymbol{x}\times(-i\nabla)$$

with each component given explicitly

$$L_x = -i(y\partial_z - z\partial_y), \quad L_y = -i(z\partial_x - x\partial_z), \quad L_z = -i(x\partial_y - y\partial_x).$$

The above GPE (1.1) conserves the total mass — i.e.

$$\mathcal{N}(t) := \|\psi(\cdot,t)\|^2 = \int_{\mathbb{R}^3}|\psi(\boldsymbol{x},t)|^2\mathrm{d}\boldsymbol{x} = \|\psi_0\|^2 \equiv 1, \quad t \geq 0,$$

and the energy per particle

$$\begin{aligned} \mathcal{E}\big(\psi(\cdot,t)\big) &:= \int_{\mathbb{R}^3}\left(\frac{1}{2}|\nabla\psi|^2 + V(\boldsymbol{x})|\psi|^2 + \frac{\beta}{2}|\psi|^4 - \psi^*(\boldsymbol{\Omega}\cdot\boldsymbol{L})\psi\right)\mathrm{d}\boldsymbol{x} \\ &= \mathcal{E}(\psi_0), \quad t \geq 0, \end{aligned} \tag{1.3}$$

where $\psi^*$ is the complex conjugate of $\psi$.