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Piecewise Linear Maximum Entropy Method for Fredholm Integral Equations with Weakly Singular Kernel

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Abstract. Approximate solution of Fredholm integral equations with certain types of weakly singular algebraic kernels is obtained by the piecewise linear maximum entropy method. During implementation of the method, two-dimension numerical integration is reduced to one-dimension numerical integration, which allows to lower the computational cost. The results of numerical experiments are consistent with theoretical findings and show the effectiveness of the method.

AMS subject classifications: 65R20, 94A17

Key words: Fredholm integral equation, piecewise linear maximum entropy method, weakly singular kernel.

1. Introduction

Consider the integral equation

$$f(x) - \int_0^1 k(x,t) f(t) dt = g(x), \quad x \in [0,1], \tag{1.1}$$

where $k(x,t)$ is algebraic or logarithmic weakly singular kernel and $g(x)$ and $f(x)$ are given and unknown functions, respectively. Such second kind Fredholm integral equations (SKFIEs) often appear in science and engineering. In this work, we assume that the right-hand side $g(x)$ is a continuous function and consider the Eqs. (1.1) with two specific kernels

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— viz. the equations

$$f(x)-\int_0^1 |x-t|^{-\alpha} f(t)dt = g(x), \quad 0<\alpha<1, \tag{1.2}$$

$$f(x)-\int_0^1 \ln|x-t| f(t)dt = g(x). \tag{1.3}$$

Due to the weak singularity of the kernel function, it is difficult to solve the integral equation. Numerical methods for second kind Fredholm integral equations with weakly singular kernel include spectral Jacobi-collocation method [19], Lagrange interpolation method [17], Galerkin method [14], wavelet method [15], etc. The collocation method for Eq. (1.2) studied in [19] is based Gauss-Jacobi quadrature formula. The barycentric Lagrange interpolation formula [17] was developed for Eq. (1.2), and transformed the integral equation into a matrix equation. The Galerkin method [14] using Legendre polynomials as the basis functions was applied to Eqs. (1.2), (1.3), and the convergence rate of the numerical solutions was given. In the reproducing kernel Hilbert space method [2] adopted for solving Eq. (1.3), the approximate solution was represented by finite number of terms of a series.

The concept of entropy first appeared in thermodynamics, representing the degree of chaos in a thermodynamic system. The maximum entropy principle was proposed by Jaynes [7, 8], and was widely used in the function recovery [1], data modeling [16], invariant measures [6, 10], solving integral equations [9, 13, 18] and other fields. The maximum entropy principle with polynomials was proposed to solve SKFIE [13]. The piecewise linear maximum entropy method (PLMEM) has been also used to find approximate solution of SKFIE and second kind Vollterа integral equations — cf. [9, 18]. However, the maximum entropy method in [9, 18] employs time expensive double numerical integration. In this work, we use a piecewise linear maximum entropy method and obtain an approximate form without singularity of the Eqs. (1.2) and (1.3). This method substantially reduces the computational cost of numerical integration.

The paper is arranged as follows. The maximum entropy method is introduced in Section 2. In Section 3, we apply the PLMEM to Fredholm integral equations (1.2) and (1.3) with weakly singular algebraic and logarithmic kernels. The results of numerical experiments presented in Section 4 show the efficiency of the method. The conclusion is given in Section 5.

2. Maximum Entropy Method

Let Σ be the Lebesgue σ algebra on $[0,1]$, μ the Lebesgue measure, and $([0,1],\Sigma,\mu)$ the corresponding measure space. The generalized Boltzmann entropy of f is defined by

$$H(f) = -\int_0^1 f(x)\ln f(x)d\mu(x) + \int_0^1 f(x)d\mu(x) \tag{2.1}$$

with $f(x)\ln f(x) := 1$ if $f(x)=0$.

The modified Boltzmann entropy $H(f)$ has the following properties — cf. [3]:

1. For any non-negative function $f \in L^1(0,1)$, the Boltzmann entropy $H(f)$ is either finite or $-\infty$.

2. H is proper upper semi-continuous and concave. Moreover, it is strictly concave on its effective domain consisting of all non-negative functions $f \in L^1(0,1)$ with $H(f) > -\infty$.

3. For any $\alpha > -\infty$, the upper level set $\{f \in L^1(0,1) : f \geqslant 0, H(f) \geqslant \alpha\}$ is weakly compact in $L^1(0,1)$.

Let $\{g_i(x)\}_{i=0}^n \subset L^\infty(0,1)$ and $\{m_i\}_{i=0}^n$ be given functions and real numbers, respectively. The maximum entropy problem is the following optimization problem:

$$\begin{aligned} &\max\{H(f)\} \\ &\text{s.t.} \quad f \in L^1(0,1), \quad f \geq 0, \\ &\int_0^1 f(x)g_i(x)dx = m_i, \quad i = 0,\dots,n. \end{aligned} \tag{2.2}$$

Note that for a given function f, the numbers m_i, $i = 0,\dots,n$ defined by

$$m_i := \int_0^1 f(x)g_i(x)dx \tag{2.3}$$

are called the moments of f with respect to the system $\{g_i(x)\}_{i=0}^n$.

The solution to (2.3) is generally not unique. We need the following lemma for solving the above maximum entropy optimization problem.

Lemma 2.1. *For any two nonnegative functions $f, g \in L^1(0,1)$, the inequality*

$$\int_0^1 f(x)dx - \int_0^1 f(x)\ln f(x)dx \leq \int_0^1 g(x)dx - \int_0^1 f(x)\ln g(x)dx$$

holds.

Proof. In the Gibbs inequality

$$u - u\ln u \leq v - u\ln v, \quad u, v \geq 0$$

we set $u = f(x)$, $v = g(x)$ and integrate the result. □

More exactly, the following theorem holds.

Theorem 2.1 (cf. Song *et al.* [18]). *The optimization problem* (2.2) *has a unique solution — viz.*

$$f_n(x) = e^{\sum_{i=0}^n \lambda_i g_i(x)},$$

where $\lambda_0, \lambda_1, \dots, \lambda_n$ are constants satisfying the constraints

$$\int_0^1 e^{\sum_{i=0}^n \lambda_i g_i(x)} g_j(x)dx = m_j, \quad j = 0,1,\dots,n. \tag{2.4}$$

3. PLMEM for Eqs. (1.2) and (1.3)

Divide the interval $[0,1]$ into n equal subintervals $[x_{i-1}, x_i]$, where $x_i = i \times h$, $i = 0,1,2,\ldots,n$ and $h = 1/n$. Let $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$ be a set of piecewise linear basis functions defined on $[0,1]$ such that

$$\phi_i(x_j) = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad j \neq i. \end{cases}$$

Then ϕ_i can be written as

$$\phi_i(x) = w\left(\frac{x - x_i}{h}\right) = \begin{cases} \dfrac{x - x_{i-1}}{h}, & x_{i-1} \le x < x_i, \\ \dfrac{x_{i+1} - x}{h}, & x_i \le x < x_{i+1}, \\ 0, & x \notin [x_{i-1}, x_{i+1}], \end{cases} \tag{3.1}$$

where

$$w(x) = \begin{cases} 1 + x, & -1 \le x \le 0, \\ 1 - x, & 0 < x \le 1, \\ 0, & x \notin [-1, 1] \end{cases}$$

is the standard piecewise linear basis function. Let Δ_n denote the set of all continuous piecewise linear functions defined on $[0,1]$. They can be written as linear combinations of the basis functions $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$, i.e. $\Delta_n = \text{span}\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$.

Consider the integral equation (1.1). Multiplying both sides of this equation by $\phi_i(x)$ and integrating over $[0,1]$, we get

$$\int_0^1 f(x)\phi_i(x)dx - \int_0^1 \left(\int_0^1 k(x,t)f(t)dt\right)\phi_i(x)dx = \int_0^1 g(x)\phi_i(x)dx. \tag{3.2}$$

Changing the order of integration in the second term in the left-hand side of (3.2) yields

$$\begin{aligned} &\int_0^1 \left(\int_0^1 k(x,t)f(t)dt\right)\phi_i(x)dx \\ =& \int_0^1 \int_0^1 k(x,t)\phi_i(x)dx f(t)dt \\ =& \int_0^1 \widetilde{g}_i(t)f(t)dt, \end{aligned}$$

where

$$\widetilde{g}_i(t) = \int_0^1 k(x,t)\phi_i(x)dx.$$

Consequently, we write the Eq. (3.2) as

$$\int_0^1 f(x)\phi_i(x)dx - \int_0^1 \widetilde{g}_i(x)f(x)dx = \int_0^1 g(x)\phi_i(x)dx,$$

or

$$\int_0^1 \big(\phi_i(x) - \widetilde{g}_i(x)\big)f(x)dx = \int_0^1 g(x)\phi_i(x)dx. \tag{3.3}$$

Let

$$g_i(x) = \phi_i(x) - \widetilde{g}_i(x), \quad m_i = \int_a^b g(x)\phi_i(x)dx. \tag{3.4}$$

Then (3.3) can be now written as

$$\int_0^1 f(x)g_i(x)dx = m_i, \quad i = 0, 1, \ldots, n.$$

Therefore, the solution of the integral equation (1.1) can be approximated by the solution of the maximum entropy problem (2.2) and we can derive an approximate solution of the Fredholm integral equation (1.1) by applying Theorem 2.1.

In summary, the algorithm using the PLMEM for solving Fredholm integral equations can be summarized as follows:

Step 1. Compute moments m_i and moment functions $g_i(x)$ in Eq. (3.4).

Step 2. Solve the nonlinear equations (2.4) to get the constants $\{\lambda_i\}_{i=0}^n$.

Step 3. Substitute the Lagrange multipliers $\{\lambda_i\}_{i=0}^n$ into $f_n(x)$ in Theorem 2.1 to obtain the approximate solution.

Computing $g_i(x)$ requires numerical integration, and substituting $g_i(x)$ into Eq. (2.4) necessitates an additional one, thus resulting in double numerical integrations for Fredholm integral equations (1.2) and (1.3). However, analytic expressions of $g_i(x)$ can be derived. Then only single numerical integration is needed in Eq. (2.4).

3.1. PLMEM for Eq. (1.2)

Considering the Fredholm integral equation with weakly singular algebraic kernel

$$f(x) - \int_0^1 |x-t|^{-\alpha} f(t)dt = g(x), \quad 0 < \alpha < 1,$$

we represent the integral term as

$$
\begin{aligned}
\widetilde{g}_i(t) &= \int_0^1 |x-t|^{-\alpha}\phi_i(x)dx \\
&= \int_0^t (t-x)^{-\alpha}\phi_i(x)dx + \int_t^1 (x-t)^{-\alpha}\phi_i(x)dx \\
&= \overline{f}_i(t) + \overline{g}_i(t),
\end{aligned}
$$

where

$$
\overline{f}_i(t) = \int_0^t (t-x)^{-\alpha}\phi_i(x)dx, \quad \overline{g}_i(t) = \int_t^1 (x-t)^{-\alpha}\phi_i(x)dx. \tag{3.5}
$$

To obtain analytic representations of the functions in (3.5), we substitute functions $\phi_i(x)$ of (3.1) into (3.5) and consider four integrals with variable limit — viz.

$$
\begin{aligned}
A_1(t,s,x_i) &= \int_s^{x_i} (x-t)^{-\alpha}\phi_i(x)dx \\
&= \int_s^{x_i} \left(\frac{x-t}{h(x-t)^\alpha} + \frac{t-x_{i-1}}{h(x-t)^\alpha}\right)dx \\
&= \left(\frac{1}{h(2-\alpha)}(x-t)^{2-\alpha} + \frac{t-x_{i-1}}{h(1-\alpha)}(x-t)^{1-\alpha}\right)\Bigg|_s^{x_i}, \quad s \in [x_{i-1}, x_i), \\
A_2(t,s,x_{i+1}) &= \int_s^{x_{i+1}} (x-t)^{-\alpha}\phi_i(x)dx \\
&= \int_s^{x_{i+1}} \left(\frac{x_{i+1}-t}{h(x-t)^\alpha} - \frac{x-t}{h(x-t)^\alpha}\right)dx \\
&= \left(\frac{x_{i+1}-t}{h(1-\alpha)}(x-t)^{1-\alpha} - \frac{1}{h(2-\alpha)}(x-t)^{2-\alpha}\right)\Bigg|_s^{x_{i+1}}, \quad s \in [x_i, x_{i+1}), \\
A_3(t,x_{i-1},s) &= \int_{x_{i-1}}^s (t-x)^{-\alpha}\phi_i(x)dx \\
&= \int_{x_{i-1}}^s -\left(\frac{t-x}{h(t-x)^\alpha} + \frac{x_{i-1}-t}{h(t-x)^\alpha}\right)dx \\
&= \left(-\frac{1}{h(2-\alpha)}(t-x)^{2-\alpha} - \frac{x_{i-1}-t}{h(1-\alpha)}(t-x)^{1-\alpha}\right)\Bigg|_{x_{i-1}}^s, \quad s \in (x_{i-1}, x_i], \\
A_4(t,x_i,s) &= \int_{x_i}^s (t-x)^{-\alpha}\phi_i(x)dx \\
&= \int_{x_i}^s \left(\frac{x_{i+1}-t}{h(t-x)^\alpha} + \frac{t-x}{h(t-x)^\alpha}\right)dx
\end{aligned}
$$

$$= \left(\frac{x_{i+1} - t}{h(1-\alpha)} (t-x)^{1-\alpha} + \frac{1}{h(2-\alpha)} (t-x)^{2-\alpha} \right) \Big|_{x_i}^{s}, \quad s \in (x_i, x_{i+1}].$$

Now we can provide analytic expressions for the functions $\bar{g}_i(t)$ and $\bar{f}_i(t)$ in (3.5). Since the basis functions $\{\phi_i(x)\}$ have local support, we have four cases.

1. For $t \leqslant x_{i-1}$,

$$\begin{aligned}
\bar{g}_i(t) &= \int_t^1 (x-t)^{-\alpha} \phi_i(x) dx \\
&= \int_{x_{i-1}}^{x_i} (x-t)^{-\alpha} \phi_i(x) dx + \int_{x_i}^{x_{i+1}} (x-t)^{-\alpha} \phi_i(x) dx \\
&= A_1(t, x_{i-1}, x_i) + A_2(t, x_i, x_{i+1}), \\
\bar{f}_i(t) &= 0.
\end{aligned}$$

2. For $x_{i-1} < t \leqslant x_i$,

$$\begin{aligned}
\bar{g}_i(t) &= \int_t^1 (x-t)^{-\alpha} \phi_i(x) dx \\
&= \int_t^{x_i} (x-t)^{-\alpha} \phi_i(x) dx + \int_{x_i}^{x_{i+1}} (x-t)^{-\alpha} \phi_i(x) dx \\
&= A_1(t, t, x_i) + A_2(t, x_i, x_{i+1}), \\
\bar{f}_i(t) &= \int_0^t (t-x)^{-\alpha} \phi_i(x) dx \\
&= \int_{x_{i-1}}^t (t-x)^{-\alpha} \phi_i(x) dx \\
&= A_3(t, x_{i-1}, t).
\end{aligned}$$

3. For $x_i < t \leqslant x_{i+1}$,

$$\begin{aligned}
\bar{g}_i(t) &= \int_t^1 (x-t)^{-\alpha} \phi_i(x) dx \\
&= \int_t^{x_{i+1}} (x-t)^{-\alpha} \phi_i(x) dx \\
&= A_2(t, t, x_{i+1}), \\
\bar{f}_i(t) &= \int_0^t (t-x)^{-\alpha} \phi_i(x) dx \\
&= \int_{x_{i-1}}^{x_i} (t-x)^{-\alpha} \phi_i(x) dx + \int_{x_i}^t (t-x)^{-\alpha} \phi_i(x) dx \\
&= A_3(t, x_{i-1}, x_i) + A_4(t, x_i, t).
\end{aligned}$$

4. For $t \geqslant x_{i+1}$,

$$\begin{aligned}\bar{g}_i(t) &= 0,\\ \bar{f}_i(t) &= \int_0^t (t-x)^{-\alpha}\phi_i(x)dx\\ &= \int_{x_{i-1}}^{x_i}(t-x)^{-\alpha}\phi_i(x)dx + \int_{x_i}^{x_{i+1}}(t-x)^{-\alpha}\phi_i(x)dx\\ &= A_3(t,x_{i-1},x_i)+A_4(t,x_i,x_{i+1}).\end{aligned}$$

Having obtained analytic representation of $\bar{g}_i(x)$ and $\bar{f}_i(x)$, we can write analytic representation of $g_i(x)$ in (3.4) for Fredholm integral equation with weakly singular algebraic kernel.

3.2. PLMEM for Eq. (1.3)

Considering the Fredholm integral equation with weakly singular logarithmic kernel

$$f(x) - \int_0^1 \ln|x-t| f(t)dt = g(x), \quad 1 \le x \le 1,$$

we represent the integral term as

$$\begin{aligned}\widetilde{g}_i(t) &= \int_0^1 \ln|x-t|\phi_i(x)dx\\ &= \int_0^t \ln(t-x)\phi_i(x)dx + \int_t^1 \ln(x-t)\phi_i(x)dx\\ &= \overline{f}_i(t)+\overline{g}_i(t),\end{aligned}$$

where

$$\overline{f}_i(t) = \int_0^t \ln(t-x)\phi_i(x)dx, \quad \overline{g}_i(t) = \int_t^1 \ln(x-t)\phi_i(x)dx. \tag{3.6}$$

To obtain an analytic representation for the functions in (3.6), we substitute functions $\phi_i(x)$ of (3.1) into (3.6) and consider four double integrals with variable limit — viz.

$$\begin{aligned}L_1(t,s,x_i) &= \int_s^{x_i} \phi_i(x)\ln(x-t)dx\\ &= \frac{1}{h}\left[\int_s^{x_i}(x-t)\ln(x-t)dx + \int_s^{x_i}(t-x_{i-1})\ln(x-t)dx\right], \quad s\in[x_{i-1},x_i).\end{aligned}$$

By integration by parts, we can get

$$L_1(t,s,x_i) = \frac{(x-t)^2}{2h}\left(\ln(x-t)-\frac{1}{2}\right)\Bigg|_s^{x_i} + \frac{t-x_{i-1}}{h}\Big((x-t)\ln(x-t)-x\Big)\Bigg|_s^{x_i}.$$

Similarly, we write

$$
\begin{aligned}
L_2(t,s,,x_{i+1}) &= \int_s^{x_{i+1}} \phi_i(x)\ln(x-t)dx \\
&= -\frac{1}{h}\left[\int_s^{x_{i+1}} (x-t)\ln(x-t)dx - \int_s^{x_{i+1}} (t-x_{i+1})\ln(x-t)dx\right] \\
&= -\frac{(x-t)^2}{2h}\left(\ln(x-t)-\frac{1}{2}\right)\Bigg|_s^{x_{i+1}} - \frac{t-x_{i+1}}{h}\big((x-t)\ln(x-t)-x\big)\Bigg|_s^{x_{i+1}}, \\
& s\in[x_i,x_{i+1}), \\
L_3(t,x_{i-1},s) &= \int_{x_{i-1}}^{s} \phi_i(x)\ln(t-x)dx \\
&= -\frac{1}{h}\left[\int_{x_{i-1}}^{s} (t-x)\ln(t-x)dx + \int_{x_{i-1}}^{s} (x_{i-1}-t)\ln(t-x)dx\right] \\
&= \frac{(t-x)^2}{2h}\left(\ln(t-x)-\frac{1}{2}\right)\Bigg|_{x_{i-1}}^{s} + \frac{x_{i-1}-t}{h}\big((t-x)\ln(t-x)+x\big)\Bigg|_{x_{i-1}}^{s}, \\
& s\in(x_{i-1},x_i], \\
L_4(t,x_i,s) &= \int_{x_i}^{s} \phi_i(x)\ln(t-x)dx \\
&= \frac{1}{h}\left[\int_{x_i}^{s} (t-x)\ln(t-x)dx + \int_{x_i}^{s} (x_{i+1}-t)\ln(t-x)dx\right] \\
&= -\frac{(t-x)^2}{2h}\left(\ln(t-x)-\frac{1}{2}\right)\Bigg|_{x_i}^{s} - \frac{x_{i+1}-t}{h}\big((t-x)\ln(t-x)+x\big)\Bigg|_{x_i}^{s}, \\
& s\in(x_i,x_{i+1}].
\end{aligned}
$$

Now we can provide analytic expressions for the functions $\bar{g}_i(t)$ and $\bar{f}_i(t)$ in (3.6). Since the basis functions $\{\phi_i(x)\}$ have local support, we have four cases.

1. For $t \leqslant x_{i-1}$,

$$
\begin{aligned}
\bar{g}_i(t) &= \int_t^1 \ln(x-t)\phi_i(x)dx \\
&= \int_{x_{i-1}}^{x_i} \ln(x-t)\phi_i(x)dx + \int_{x_i}^{x_{i+1}} \ln(x-t)\phi_i(x)dx \\
&= L_1(t,x_{i-1},x_i) + L_2(t,x_i,x_{i+1}), \\
\bar{f}_i(t) &= 0.
\end{aligned}
$$

2. For $x_{i-1} < t \leqslant x_i$,

$$\begin{aligned}\bar{g}_i(t) &= \int_t^1 \ln(x-t)\phi_i(x)dx\\ &= \int_t^{x_i} \ln(x-t)\phi_i(x)dx + \int_{x_i}^{x_{i+1}} \ln(x-t)\phi_i(x)dx\\ &= L_1(t,t,x_i) + L_2(t,x_i,x_{i+1}),\\ \bar{f}_i(t) &= \int_0^t \ln(t-x)\phi_i(x)dx\\ &= \int_{x_{i-1}}^{t} \ln(t-x)\phi_i(x)dx\\ &= L_3(t,x_{i-1},t).\end{aligned}$$

3. For $x_i < t \leqslant x_{i+1}$,

$$\begin{aligned}\bar{g}_i(t) &= \int_t^1 \ln(x-t)\phi_i(x)dx\\ &= \int_t^{x_{i+1}} \ln(x-t)\phi_i(x)dx\\ &= L_2(t,t,x_{i+1}),\\ \bar{f}_i(t) &= \int_0^t \ln(t-x)\phi_i(x)dx\\ &= \int_{x_{i-1}}^{x_i} \ln(t-x)\phi_i(x)dx + \int_{x_i}^{t} \ln(t-x)\phi_i(x)dx\\ &= L_3(t,x_{i-1},x_i) + L_4(t,x_i,t).\end{aligned}$$

4. For $t \geqslant x_{i+1}$,

$$\begin{aligned}\bar{g}_i(t) &= 0,\\ \bar{f}_i(t) &= \int_0^t \ln(t-x)\phi_i(x)dx\\ &= \int_{x_{i-1}}^{x_i} \ln(t-x)\phi_i(x)dx + \int_{x_i}^{x_{i+1}} \ln(t-x)\phi_i(x)dx\\ &= L_3(t,x_{i-1},x_i) + L_4(t,x_i,x_{i+1}).\end{aligned}$$

Having obtained analytic representation for $\bar{g}_i(x)$ and $\bar{f}_i(x)$, we can write analytic representation of $g_i(x)$ in (3.4) for Fredholm integral equation with weakly singular logarithmic kernel.

4. Convergence Analysis

In this section, we exploit the results in [3] to show the convergence and determine the convergence rate of the PLMEM for Fredholm integral equations with weakly singular kernels.

Let $K : L^1(0,1) \to L^1(0,1)$ be Fredholm integral operator

$$(Kf)(x) = \int_0^1 k(x,t)f(t)dt,$$

where the weakly singular kernel function k is measurable on the square $[0,1] \times [0,1]$. If I is the identity operator, then Eq. (1.1) can be written as

$$(I-K)f = g.$$

To study the convergence of the maximum entropy method, we need the following general theory for moment problems adapted to our case. Let $F : L^1(0,1) \to [-\infty, \infty)$ be a nonlinear functional with weakly compact upper level sets. Let $\{D_n\} \subset L^1(0,1)$ be a decreasing sequence of closed subsets such that $D_{n+1} \subset D_n$, $n \in \mathbb{N}$. We consider the optimization problem

$$\max\left\{F(f) : f \in D_n\right\} \tag{4.1}$$

and the limit problem

$$\max\left\{F(f) : f \in \bigcap_{n=1}^{\infty} D_n\right\}. \tag{4.2}$$

Let f_n and f^* be solution of (4.1) and (4.2), respectively.

Theorem 4.1 (cf. Borwein & Lewis [4]). *If* $F : L^1(0,1) \to [-\infty, \infty)$ *is strongly convex and the optimal value of* (4.2) *is finite, then the problems* (4.1) *and* (4.2) *have unique solutions* f_n *and* f^**, respectively, and* $\lim_{n\to\infty} \|f_n - f^*\|_{L^1} = 0$.

Let us recall that the Boltzmann entropy $H(f)$ defined by (2.1), is strongly convex [4]. In addition, the integral term $\int_0^1 f(x)dx$ is convex, lower semi-continuous, and bounded below. Therefore, the generalized Boltzmann entropy $H(f)$ is strongly convex. Now a nested partition of $[0,1]$ makes the feasible set of moment constraint (2.2) monotonically decreasing. Divide $[0,1]$ into 2^r, $r = 1,2,\ldots$ equal subintervals and denote by f_r the corresponding numerical solutions of the PLMEM. Let f^* be the solution of the corresponding limit problem $\max\{H(f) : f \in \bigcap_{n=1}^{\infty} D_n\}$. In fact, the unique solution f^* to the limit problem is the solution to the integral equation. Using Theorem 4.1, we get the following convergence theorem for $\{f_r\}$.

Theorem 4.2. *If* $H(f^*) > -\infty$*, then* $\lim_{r\to\infty} \|f_r - f^*\|_{L^1} = 0$.

When the operator $I-K$ is invertible, we can give the convergence rate of the maximum entropy method, so next we give the condition for the invertibility of $I-K$.

Lemma 4.1 (cf. Yue & Wang [20]). *The integral operator*

$$(Kf)(x) = \int_0^1 k(x,t)f(t)dt$$

with a weakly singular kernel function $k(x,t) = |x-t|^{-\alpha}, 0 < \alpha < 1$, *is a compact on* $L^1(0,1)$.

Theorem 4.3 (cf. Kress [12]). *Let* $K : L^1(0,1) \to L^1(0,1)$ *be a compact linear operator. Then* $I - K$ *is injective if and only if it is surjective. If* $I - K$ *is injective (and therefore also bijective), then the inverse operator* $(I-K)^{-1} : L^1(0,1) \to L^1(0,1)$ *exists and bounded.*

Assume that the operator $I - K$ is injective and $\ln f^*$ is in the range of $I - K$. According to [4, Theorem 4.7], the convergence rate of the maximum entropy method depends on the minimum distance

$$d_r := \inf \left\{ \left\| \ln f^* - \sum_{i=0}^{2^r} \lambda_i g_i \right\|_{L^\infty} : \lambda_0, \lambda_1, \ldots, \lambda_{2^r} \in \mathbb{R} \right\}$$

from the function $\ln f^*$ to the linear subspace generated by all g_i. In our method $g_i = (I-K)\phi_i$, $i = 0, 1, \ldots, n$. Since

$$\begin{aligned} \ln f^* - \sum_{i=0}^{2^r} \lambda_i g_i &= \ln f^* - \sum_{i=0}^{2^r} \lambda_i (I-K)\phi_i \\ &= (I-K)\left[(I-K)^{-1} \ln f^* - \sum_{i=0}^{2^r} \lambda_i \phi_i \right], \end{aligned}$$

we can evaluate d_r as

$$d_r \leqslant \|I - K\|_{L^\infty} \inf \left\{ \left\| (I-K)^{-1} \ln f^* - \sum_{i=0}^{2^r} \lambda_i \phi_i \right\|_{L^\infty} : \lambda_0, \lambda_1, \ldots, \lambda_{2^r} \in \mathbb{R} \right\}.$$

Then the error estimates are obtained through a standard pointwise approximation result for continuous piecewise linear functions.

Theorem 4.4. *Suppose that* $f^*(x) \geqslant c$ *on* $[0,1]$ *for some* $c > 0$ *and* $(I-K)^{-1} f^*$ *is twice continuously differentiable on* $[0,1]$. *Then*

$$\|f_r - f^*\|_{L^1} = \mathcal{O}(h^2) = \mathcal{O}\left(\frac{1}{4^r}\right),$$

where $h = 1/2^r$ *is the length of the* 2^r *subintervals of* $[0,1]$.

Proof. It follows from [4, Theorem 4.7] that

$$\|f_r - f^*\|_{L^1} \leqslant d_r e^{d_r/2} = \mathcal{O}(d_r).$$

Denote $v=(I-K)^{-1}\ln f^*$ and let $v_r\in\triangle_{2^r}$ be the continuous piecewise linear interpolation of v, i.e.

$$v_r(x)=v(x_i),\quad i=0,1,\ldots,2^r.$$

According to the interpolation theory [5], $\|v-v_r\|_{L^\infty}=\mathscr{O}(h^2)$. Then

$$d_r\le\|I-K\|_{L^\infty}\|v-v_r\|_{L^\infty}=\mathscr{O}(h^2).$$ □

5. Numerical Examples

To test the effectiveness of the PLMEM for the Fredholm integral equations with weakly singular kernel, we consider a few examples. The results of numerical experiments are presented in Tables 1-3. We also compute the convergence rate of the method as

$$CR=\log_2\left(\|f_{r-1}-f^*\|_{L^1}/\|f_r-f^*\|_{L^1}\right),\quad r=3,4,\ldots,9.$$

The numerical results are consistent with the error analysis and convergence rate estimation in Section 4.

Example 5.1. Consider the integral equation

$$f(x)-\int_0^1|x-t|^{-1/2}f(t)dt=g(x),\quad x\in[0,1],$$

where

$$g(x)=1-\frac{\pi}{2}-2x^{-1/2}-2(1-x)^{1/2}-x\ln\left(1+(1-x)^{1/2}\right)$$
$$-(1-x)\ln(1+x^{1/2})+\frac{1}{2}x\ln x+\frac{1}{2}(1-x)\ln(1-x).$$

The exact solution is

$$f^*(x)=1+x^{1/2}+(1-x)^{1/2}.$$

The L^1 and L^∞ errors are shown in Table 1 for uniform partition of $[0,1]$ into n.

From the results in Table 1, we can see the L^1 error and L^∞ error of the numerical solution converge to zero. And when $n=512$, the L^1 error of the numerical solution of the integral equation is 1.9×10^{-5}, and the L^∞ error is 2.9×10^{-3}. In [19] , the best L^∞ error of the spectral collocation method to solve this equation is 10^{-2}.

Example 5.2. Consider the integral equation

$$\frac{3}{4}\sqrt{2}f(x)-\int_0^1|x-t|^{-1/2}f(t)dt=g(x),\quad x\in[0,1],$$

where

$$g(x)=3\big(x(1-x)\big)^{3/4}-\frac{3}{8}\pi\big(1+4x(1-x)\big).$$

Table 1: Examples 5.1. Errors.

n	L^1 Error	CR	L^∞ Error	CR
4	4.968e-02	-	1.124e-01	-
8	2.043e-02	1.28	1.232e-01	-0.13
16	6.421e-03	1.67	6.870e-02	0.84
32	1.962e-03	1.71	3.364e-02	1.03
64	6.057e-04	1.70	1.371e-02	1.29
128	1.890e-04	1.63	6.057e-03	1.18
256	5.974e-05	1.72	4.170e-03	0.54
512	1.917e-05	1.64	2.907e-03	0.52

Table 2: Examples 5.2. Errors.

n	L^1 Error	CR	L^∞ Error	CR
4	8.487e-02	-	3.840e-01	-
8	2.409e-02	1.82	1.476e-01	1.38
16	7.768e-03	1.63	4.041e-02	1.87
32	2.373e-02	1.71	1.921e-02	1.07
64	6.991e-04	1.76	1.054e-02	0.87
128	2.036e-04	1.78	5.241e-03	1.01
256	6.024e-05	1.76	3.360e-03	0.64
512	1.786e-05	1.75	2.107e-03	0.67

The exact solution is

$$f^*(x) = 2\sqrt{2}\big(x(1-x)\big)^{3/4}.$$

The L^1 errors L^∞ errors are shown in Table 2.

Table 2 shows the error of the numerical solution tends to zero. If $n = 512$, the L^1 error of numerical solution is 1.7×10^{-5}, while L^∞ error is 2.1×10^{-3}. In [11], the L^∞ error of the shifted Jacobi spectral Galerkin method to solve the equation is 2.27×10^{-3} and the L^∞ error of the iterated shifted Jacobi spectral Galerkin method to solve the equation is 3.62×10^{-4}.

Example 5.3. Consider the integral equation

$$f(x) - \int_0^1 \ln|x-t| f(t)dt = g(x), \quad x \in [0,1],$$

where

$$g(x) = x - \frac{1}{2}\left(x^2 \ln x + (1-x^2)\ln(1-x) - x - \frac{1}{2}\right).$$

The exact solution is $f^*(x) = x$. The L^1 and L^∞ errors are shown in Table 3.

Table 3: Examples 5.3. Errors.

n	L^1 Error	CR	L^∞ Error	CR
4	8.164e-03	-	5.006e-02	-
8	2.203e-03	1.89	2.226e-02	1.17
16	6.177e-04	1.83	9.980e-03	1.16
32	1.745e-04	1.82	4.405e-03	1.18
64	4.907e-05	1.83	1.803e-03	1.29
128	1.367e-05	1.84	8.538e-04	1.08
256	3.776e-06	1.86	4.221e-04	1.02
512	1.034e-06	1.87	2.096-04	1.01

Table 3 shows that the error tends to zero. If $n = 512$, the L^1 error is 1.0×10^{-6}, while L^∞ norm error is 2.0×10^{-4}. In [14], the best L^∞ error of the Legendre Galerkin method to solve this equation 2.3×10^{-4}.

6. Conclusion

We used the maximum entropy method to solve second kind Fredholm integral equations with two types of weakly singular kernels. This method can substantially reduce the cost of numerical integration. Numerical results show that the method has the same accuracy as certain spectral methods.

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