East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2025-103.181125

Vol. **xx**, No. x, pp. 1-13
xxx 2026

Integrable mKdV Models as Reductions of AKNS Integrable Systems via Dual Similarity Transformations

Wen-Xiu Ma [1–4,*]

[1] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China.*
[2] *Research Center of Astrophysics and Cosmology, Khazar University, Baku 1096, Azerbaijan.*
[3] *Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA.*
[4] *Material Science Innovation and Modelling, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa.*

Received 14 June 2025; Accepted (in revised version) 18 November 2025.

Abstract. This paper explores integrable modified Korteweg-de Vries (mKdV) models using dual similarity transformations. Two representative examples involving distinct similarity transformations are presented, along with their corresponding reduced Ablowitz-Kaup-Newell–Segur matrix spectral problems. These examples illustrate how reduced matrix mKdV integrable models can be systematically generated through such transformations.

AMS subject classifications: 37K10, 35Q53, 37K40

Key words: Lax pair, AKNS matrix spectral problem, zero-curvature equation, integrable hierarchy, similarity transformation.

1. Introduction

The Lax pair formulation in soliton theory is a fundamental tool for constructing integrable models [18]. To gain deeper insights into the structure of integrable systems, especially multi-component ones, it is essential to explore a wide range of illustrative examples. These examples help classify integrable models within the Lax pair framework. A natural starting point is to consider matrix spectral problems built from matrix loop algebras [48]. Lax pairs often lead to bi-Hamiltonian structures [41], which ensure the existence of infinitely many commuting symmetries and conservation laws, key features of integrability. Furthermore, the Lax pair formulation provides a basis for solving Cauchy problems via the inverse scattering transform [4, 8].

*Corresponding author. *Email address:* `wma3@usf.edu` (W.X. Ma)

Similarity transformations have proven effective in deriving reduced matrix spectral problems and their associated integrable models [13, 23, 25, 42], with notable examples including nonlocal integrable models involving reflection-type symmetries [3]. Starting from the matrix Ablowitz-Kaup-Newell-Segur (AKNS) spectral problems, classical equations such as the nonlinear Schrödinger (NLS) and modified Korteweg-de Vries equations can be obtained as reduced models. More recently, dual similarity transformations have led to novel reduced integrable models [27]. A key challenge in this framework is to carefully handle the reductions imposed on the potential matrices by both transformations, while maintaining the invariance of the zero-curvature condition [29]. A recent comprehensive classification of lower-order nonlocal integrable models associated with the matrix AKNS framework has identified three types of nonlocal NLS models and two types of nonlocal mKdV models [28].

The inverse scattering transform remains a powerful and effective tool for solving Cauchy problems, including those arising in nonlocal integrable models [2, 40]. In addition to this method, various other analytical techniques have proven valuable for studying reduced integrable models, particularly in the construction of soliton solutions. Classical approaches such as the Hirota bilinear method, Bäcklund transformations, Darboux transformations, and the Riemann-Hilbert method continue to play important roles. Furthermore, several mathematical frameworks have been developed to address nonlocal reductions of integrable models (see, e.g., [14–16, 28, 44, 45, 50]).

The aim of this study is to construct dual similarity transformations of distinct forms and to derive the corresponding reduced integrable mKdV models based on the matrix AKNS spectral problems. In Section 2, we revisit the matrix AKNS spectral problems and the associated integrable mKdV hierarchies. A general framework is presented for implementing dual similarity transformations and obtaining reduced integrable models. Section 3 explores two representative cases that yield consistent similarity transformations, one involving off-diagonal matrices and the other involving diagonal matrices. These transformations are then applied to the matrix AKNS spectral problems to generate novel reduced integrable mKdV models. The examples underscore the structural diversity and richness of the reduced matrix AKNS integrable systems. The closing section synthesizes the main contributions and offers final remarks.

2. Dual Similarity Reductions of AKNS Integrable Models

2.1. Revisiting matrix AKNS hierarchies: Toward reduced integrable models

Within the AKNS framework for integrable models, the dependent variable $u = u(p, q)$ is a column vector composed of two matrix-valued potentials, defined as follows:

$$p = p(x,t) = (p_{jk})_{m\times n}, \quad q = q(x,t) = (q_{kj})_{n\times m},$$

where m and n are two natural numbers. For each $r \geq 0$, a pair of matrix AKNS spectral problems is given by

$$-i\phi_x = U\phi, \quad -i\phi_t = V^{[r]}\phi. \tag{2.1}$$

Here the Lax pairs is defined as

$$U = U(u,\lambda) = \lambda\Lambda + P, \quad V^{[r]} = V^{[r]}(u,\lambda) = \lambda^r\Omega + Q^{[r]},$$

where the matrices involved are defined by

$$\Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix}, \quad P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix},$$

and

$$\Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix}, \quad Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}.$$

In this formulation, I_k denotes the identity matrix of size k, λ is the spectral parameter, and α_1,α_2 and β_1,β_2 are two pairs of arbitrary but distinct constants. The matrix $Q^{[0]}$ is understood to be the zero matrix of order $m+n$. The Laurent series

$$W = \sum_{s\geq 0} \lambda^{-s} W^{[s]} = \sum_{s\geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \tag{2.2}$$

solves the stationary zero-curvature equation

$$W_x = i[U,W] \tag{2.3}$$

with the initial condition $W^{[0]} = \Omega$. This unique series expansion is used to formulate the Lax pairs of hierarchies of commuting integrable models (see, e.g., [57, 58]).

The compatibility of the two matrix spectral problems in (2.1), i.e., the zero-curvature equations

$$U_t - V_x^{[r]} + i\big[U, V^{[r]}\big] = 0, \quad r \geq 0,$$

present the matrix AKNS hierarchy of integrable models

$$p_t = i\alpha b^{[r+1]}, \quad q_t = -i\alpha c^{[r+1]}, \quad r \geq 0, \tag{2.4}$$

where $\alpha = \alpha_1 - \alpha_2$. The classical AKNS integrable hierarchy with scalar potentials p and q corresponds to the simplest case, $m = n = 1$ [1]. Every system within the matrix AKNS integrable hierarchy possesses a bi-Hamiltonian structure and admits infinitely many symmetries and conserved quantities (see, e.g., [19, 49, 59]).

When $r = 2s+1$, with $s \geq 1$, the matrix AKNS integrable hierarchy in (2.4) results in the matrix mKdV integrable hierarchy. Particularly, setting $s = 1$ yields the Lax matrix $V^{[3]}$

$$V^{[3]} = \lambda^3\Omega + \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}\big(P^2 + iP_x\big) - \frac{\beta}{\alpha^3}\big(i[P,P_x] + P_{xx} + 2P^3\big), \tag{2.5}$$

where $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. This corresponds to the first nontrivial integrable system – the matrix mKdV integrable model

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_x qp), \quad q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_x pq + 3qpq_x),$$

where $\beta = \beta_1 - \beta_2$. These equations serve as foundational examples for the subsequent analysis. We also note that numerous higher-order matrix AKNS integrable models can be systematically derived using similar techniques (see, e.g., [26]).

2.2. Dual similarity transformations of different forms

We are concentrated on the case $m = n$, which correspond to two square potential matrices, p and q. Furthermore, we assume that

$$\alpha_1 = -\alpha_2 = \frac{1}{2}, \quad \beta_1 = -\beta_2 = -\frac{1}{2},$$

to simplify the subsequent computations. Taking two invertible block matrices of different forms

$$\Delta = \begin{bmatrix} 0 & \Delta_1 \\ \Delta_2 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

where Σ is assumed to be symmetric, the dual similarity transformations are given by

$$\begin{aligned} &\Delta U(\lambda)\Delta^{-1} = -U^T(\lambda) = -\big(U(\lambda)\big)^T, \\ &\Sigma U(\lambda)\Sigma^{-1} = -U^T(-\lambda) = -\big(U(-\lambda)\big)^T, \end{aligned} \tag{2.6}$$

where A^{-1} and A^T denote the inverse and transpose of a matrix A, respectively. The above similarity transformations impose the following constraints on the potential matrix P:

$$\Delta P \Delta^{-1} = -P^T, \quad \Sigma P \Sigma^{-1} = -P^T, \tag{2.7}$$

which equivalently gives rise to

$$p^T = -\Delta_2 p \Delta_1^{-1}, \quad q^T = -\Delta_1 q \Delta_2^{-1}, \tag{2.8}$$

and

$$p^T = -\Sigma_2 q \Sigma_1^{-1}, \quad q^T = -\Sigma_1 p \Sigma_2^{-1}, \tag{2.9}$$

respectively. Under the symmetric condition of Σ and the compatibility condition

$$\Delta_1^{-1}\Sigma_1 = \Sigma_2^{-1}\Delta_2, \tag{2.10}$$

the constraints on p and q imposed by the dual similarity transformations in (2.6) become consistent. The resulting reduced AKNS matrix spectral problem takes the form

$$-i\phi_x = U\phi, \quad U = \begin{bmatrix} \frac{1}{2}\lambda I_n & p \\ -\Sigma_2^{-1}p^T\Sigma_1 & -\frac{1}{2}\lambda I_n \end{bmatrix}, \tag{2.11}$$

where p must satisfy the first constraint in (2.8), or alternatively,

$$-i\phi_x = U\phi, \quad U = \begin{bmatrix} \frac{1}{2}\lambda I_n & -\Sigma_1^{-1}q^T\Sigma_2 \\ q & -\frac{1}{2}\lambda I_n \end{bmatrix}, \tag{2.12}$$

where q must satisfy the second constraint in (2.8).

2.3. Reduced integrable hierarchies of matrix mKdV type

Since

$$\Delta W(\lambda)\Delta^{-1}\big|_{\lambda=\infty} = -\big(W(\lambda)\big)^T\big|_{\lambda=\infty} = -\Omega,$$
$$\Sigma W(\lambda)\Sigma^{-1}\big|_{\lambda=\infty} = \big(W(-\lambda)\big)^T\big|_{\lambda=\infty} = \Omega,$$

it follows from the uniqueness of solutions to the stationary zero-curvature equation that

$$\Delta W(\lambda)\Delta^{-1} = -W^T(\lambda) = -\big(W(\lambda)\big)^T,$$
$$\Sigma W(\lambda)\Sigma^{-1} = W^T(-\lambda) = \big(W(-\lambda)\big)^T,$$

where W, defined by (2.2), is the solution to the stationary zero-curvature equation (2.3) with $W^{[0]} = \Omega$.

Consequently, for each $s \geq 0$, we obtain the following relations from the dual similarity transformations:

$$\begin{aligned} \Delta V^{[2s+1]}(\lambda)\Delta^{-1} &= -V^{[2s+1]T}(\lambda) = -\big(V^{[2s+1]}(\lambda)\big)^T, \\ \Sigma V^{[2s+1]}(\lambda)\Sigma^{-1} &= -V^{[2s+1]T}(-\lambda) = -\big(V^{[2s+1]}(-\lambda)\big)^T, \end{aligned} \tag{2.13}$$

where $V^{[2s+1]} = (\lambda^{2s+1}W)_+$ represents the polynomial part of $\lambda^{2s+1}W$, as previously defined. Further, we have the following invariance property:

$$\Delta\big(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}]\big)(\lambda)\Delta^{-1} = -\Big(\big(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}]\big)(\lambda)\Big)^T,$$
$$\Sigma\big(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}]\big)(\lambda)\Sigma^{-1} = -\Big(\big(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}]\big)(-\lambda)\Big)^T.$$

This implies that the matrix AKNS integrable models, defined by (2.4) with $r = 2s+1$, become the following reduced mKdV integrable models:

$$p_t = 2ib^{[2s+2]}\big|_{q=-\Sigma_2^{-1}p^T\Sigma_1}, \quad s \geq 0, \tag{2.14}$$

where p satisfies the first constraint in (2.8), or alternatively,

$$q_t = -2ic^{[2s+2]}\big|_{p=-\Sigma_1^{-1}q^T\Sigma_2}, \quad s \geq 0, \tag{2.15}$$

where q satisfies the second constraint in (2.8). The corresponding temporal matrix spectral problems are

$$-i\phi_t = V^{[2s+1]}\big|_{q=-\Sigma_2^{-1}p^T\Sigma_1}\phi, \quad s \geq 0, \tag{2.16}$$

and alternatively,

$$-i\phi_t = V^{[2s+1]}\big|_{p=-\Sigma_1^{-1}q^T\Sigma_2}\phi, \quad s \geq 0. \tag{2.17}$$

Together with (2.11) and (2.12), these equations describe the reduced mKdV integrable hierarchy.

Based on the Lax operator algebra theory (see, e.g., [22]), these reduced integrable models form an Abelian algebra, which guarantees an infinite sequence of commuting symmetries. Since three of Δ_1, Δ_2, Σ_1, and Σ_2 are arbitrary, selecting specific forms for these matrices enables the construction of a wide range of mKdV integrable models. A key requirement is that both Σ_1 and Σ_2 must be symmetric. These models serve as representative examples of reduced matrix AKNS integrable models. However, when $r = 2s$ with $s \geq 0$, the similarity properties defined in (2.13) no longer hold, and consequently, such reductions are not valid for those systems.

3. Two Representative Cases

In this section, we examine two representative cases by choosing distinct sets of dual similarity transformations. Each case demonstrates a reduced matrix AKNS spectral problem with arbitrary constants, along with its corresponding mKdV integrable model. We focus on the specific scenario where $m = n = 3$, with the spectral matrix given by

$$U = U(u,\lambda) = \begin{bmatrix} \frac{1}{2}\lambda I_3 & p \\ q & -\frac{1}{2}\lambda I_3 \end{bmatrix},$$

where the potential matrix p satisfies the first constraint in (2.8), and q is determined by either the first or second constraint in (2.9). In our analysis, we first select specific forms for Δ_1, Δ_2 ad Σ_1, and then construct Σ_2 using the compatibility condition given in (2.10).

Example 3.1. We begin by examining the first case through the selection of the following specific pairs of matrices:

$$\Delta_1 = \begin{bmatrix} \delta_1 & \delta_3 & 0 \\ 0 & \delta_2 & 0 \\ 0 & \delta_3 & \delta_1 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ \delta_3 & \delta_2 & \delta_3 \\ 0 & 0 & \delta_1 \end{bmatrix},$$

$$\Sigma_1 = \begin{bmatrix} 0 & 0 & \sigma_1 \\ 0 & \sigma_2 & 0 \\ \sigma_1 & 0 & 0 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} 0 & \frac{\delta_1\delta_3}{\sigma_1} & \frac{\delta_1^2}{\sigma_1} \\ \frac{\delta_1\delta_3}{\sigma_1} & \frac{2\delta_3^2}{\sigma_1} + \frac{\delta_2^2}{\sigma_2} & \frac{\delta_1\delta_3}{\sigma_1} \\ \frac{\delta_1^2}{\sigma_1} & \frac{\delta_1\delta_3}{\sigma_1} & 0 \end{bmatrix},$$

where δ_1, δ_2 and σ_1, σ_2 are arbitrary non-zero constants, while δ_3 is totally arbitrary. Note that Σ_1 and Σ_2 are symmetric. After specifying these matrices, the dual similarity transformations in (2.6) yield the explicit expressions for p and q

$$p = \begin{bmatrix} 0 & p_2 & p_1 \\ \frac{\delta_3 p_1 - \delta_1 p_2}{\delta_2} & \frac{\delta_3 p_1 - \delta_1 p_2 + \delta_2 p_3}{\delta_1 \delta_2} & p_3 \\ -p_1 & -\frac{\delta_3 p_1 + \delta_2 p_3}{\delta_1} & 0 \end{bmatrix},$$

$$q = \begin{bmatrix} -\frac{\delta_3 \sigma_1 \sigma_2 p_3}{\delta_1^2 \delta_2} & -\frac{\sigma_1 \sigma_2 p_3}{\delta_1^2} & q_{13} \\ \frac{\sigma_1 \sigma_2 p_3}{\delta_1 \delta_2} & 0 & \frac{\sigma_1 \sigma_2 (\delta_3 p_1 - \delta_1 p_2)}{\delta_1 \delta_2^2} \\ \frac{\sigma_1 (\delta_2 \sigma_1 p_1 - \delta_3 \sigma_2 p_3)}{\delta_1^2 \delta_2} & \frac{\sigma_1 \sigma_2 (\delta_1 p_2 - \delta_3 p_1)}{\delta_1^2 \delta_2} & \frac{\delta_3 \sigma_1 \sigma_2 (\delta_1 p_2 - \delta_3 p_1)}{\delta_1^2 \delta_2^2} \end{bmatrix},$$

where

$$q_{13} = \frac{\sigma_1 (\delta_1 \delta_3 \sigma_2 p_2 - \delta_2^2 \sigma_1 p_1 - \delta_3^2 \sigma_2 p_1)}{\delta_1^2 \delta_2^2}.$$

Consequently, the reduced matrix mKdV integrable model system, with $u = (p_1, p_2, p_3)^T$, can be expressed as follows:

$$\begin{aligned} p_{1,t} &= p_{1,xxx} + \frac{3\sigma_1}{\delta_1^2 \delta_2} \Big\{ \big(2\delta_2 \sigma_1 p_1^2 - 3\delta_3 \sigma_2 p_1 p_3 + 2\delta_1 \sigma_2 p_2 p_3\big) p_{1,x} \\ &\qquad + \sigma_2 p_1 \big[\delta_1 p_3 p_{2,x} + (\delta_1 p_2 - \delta_3 p_1) p_{3,x}\big] \Big\}, \\ p_{2,t} &= p_{2,xxx} + \frac{3\sigma_1}{\delta_1^2 \delta_2} \Big\{ p_2 \big[(\delta_2 \sigma_1 p_1 - \delta_3 \sigma_2 p_3) p_{1,x} + \sigma_2 (\delta_1 p_2 - \delta_3 p_1) p_{3,x}\big] \\ &\qquad + \big(\delta_2 \sigma_1 p_1^2 - 2\delta_3 \sigma_2 p_1 p_3 + 3\delta_1 \sigma_2 p_2 p_3\big) p_{2,x} \Big\}, \\ p_{3,t} &= p_{3,xxx} + \frac{3\sigma_1}{\delta_1^2 \delta_2} \Big[\big(\delta_2 \sigma_1 p_1 p_3 - \delta_3 \sigma_2 p_3^2\big) p_{1,x} + \delta_1 \sigma_2 p_3^2 p_{2,x} \\ &\qquad + \big(\delta_2 \sigma_1 p_1^2 - 3\delta_3 \sigma_2 p_1 p_3 + 3\delta_1 \sigma_2 p_2 p_3\big) p_{3,x} \Big], \end{aligned}$$

where $\delta_1, \delta_2, \sigma_1$ and σ_2 are arbitrary nonzero constants, whereas δ_3 is arbitrary and not required to be nonzero.

Fixing

$$\delta_1 = \delta_2 = 1, \quad \delta_3 = -1, \quad \sigma_1 = \sigma_2 = 1,$$

we obtain the following two square potential matrices:

$$p = \begin{bmatrix} 0 & p_2 & p_1 \\ -p_1 - p_2 & p_1 + p_2 - p_3 & p_3 \\ -p_1 & p_1 - p_3 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} p_3 & -p_3 & -2p_1 - p_2 \\ p_3 & 0 & -p_1 - p_2 \\ p_1 + p_3 & p_1 + p_2 & -p_1 - p_2 \end{bmatrix}.$$

The corresponding integrable simplified mKdV model system is given by

$$\begin{aligned}
p_{1,t} &= p_{1,xxx} + 3\big(2p_1^2 + 3p_1p_3 + 2p_2p_3\big)p_{1,x} + 3p_1p_3p_{2,x} + 3p_1(p_1+p_2)p_{3,x},\\
p_{2,t} &= p_{2,xxx} + 3(p_1+p_3)p_2p_{1,x} + 3\big(p_1^2 + 2p_1p_3 + 3p_2p_3\big)p_{2,x} + 3(p_1+p_2)p_2p_{3,x},\\
p_{3,t} &= p_{3,xxx} + 3(p_1+p_3)p_3p_{1,x} + 3p_3^2p_{2,x} + 3\big(p_1^2 + 3p_1p_3 + 3p_2p_3\big)p_{3,x}.
\end{aligned}$$

Example 3.2. Secondly, we investigate the other scenario by introducing the following two specific matrix pairs:

$$\Delta_1 = \begin{bmatrix} \delta_1 & 0 & 0 \\ \delta_3 & \delta_2 & \delta_3 \\ 0 & 0 & \delta_1 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} \delta_1 & \delta_3 & 0 \\ 0 & \delta_2 & 0 \\ 0 & \delta_3 & \delta_1 \end{bmatrix},$$

$$\Sigma_1 = \begin{bmatrix} 0 & 0 & \sigma_1 \\ 0 & \sigma_2 & 0 \\ \sigma_1 & 0 & 0 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} \dfrac{\delta_3^2}{\sigma_2} & \dfrac{\delta_2\delta_3}{\sigma_2} & \dfrac{\delta_1^2}{\sigma_1} + \dfrac{\delta_3^2}{\sigma_2} \\ \dfrac{\delta_2\delta_3}{\sigma_2} & \dfrac{\delta_2^2}{\sigma_2} & \frac{\delta_2\delta_3}{\sigma_2} \\ \dfrac{\delta_1^2}{\sigma_1} + \dfrac{\delta_3^2}{\sigma_2} & \dfrac{\delta_2\delta_3}{\sigma_2} & \dfrac{\delta_3^2}{\sigma_2} \end{bmatrix},$$

where δ_1, δ_2 and σ_1, σ_2 are arbitrary non-zero constants, while δ_3 is arbitrary and not required to be non-zero. Note that we have simply swapped Δ_1 and Δ_2 compared to the previous example, and once again, both Σ_1 and Σ_2 are symmetric. With these matrices in place, the dual similarity transformations described in (2.6) provide the explicit formulas for p and q

$$p = \begin{bmatrix} \dfrac{\delta_3 p_2}{\delta_2} & p_2 & p_1 \\ -\dfrac{\delta_1 p_2}{\delta_2} & 0 & p_3 \\ -\dfrac{\delta_1 p_1 + \delta_3 p_3}{\delta_1} + \dfrac{\delta_3 p_2}{\delta_2} & -\dfrac{\delta_2 p_3}{\delta_1} & -\dfrac{\delta_3 p_3}{\delta_1} \end{bmatrix},$$

$$q = \begin{bmatrix} 0 & -\dfrac{\sigma_1\sigma_2 p_3}{\delta_1^2} & \dfrac{\sigma_1^2(\delta_3 p_2 - \delta_2 p_1)}{\delta_1^2\delta_2} \\ q_{21} & \dfrac{\delta_3\sigma_1\sigma_2(\delta_2 p_3 - \delta_1 p_2)}{\delta_1^2\delta_2^2} & q_{23} \\ \dfrac{\sigma_1^2(\delta_2 p_1 - \delta_3 p_2)}{\delta_1^2\delta_2} & \dfrac{\sigma_1\sigma_2 p_2}{\delta_1\delta_2} & 0 \end{bmatrix},$$

where

$$q_{21} = \frac{\sigma_1[\delta_1\delta_2\sigma_2 p_3 - \delta_3\sigma_1(\delta_2 p_1 - \delta_3 p_2)]}{\delta_1^2\delta_2^2},$$

$$q_{23}=\frac{\sigma_1(\delta_2\delta_3\sigma_1p_1-\delta_1^2\sigma_2p_2-\delta_3^2\sigma_1p_2)}{\delta_1^2\delta_2^2}.$$

Consequently, the associated reduced matrix mKdV integrable model for the vector $u=(p_1,p_2,p_3)^T$ takes the form

$$\begin{aligned}
p_{1,t}=p_{1,xxx}+\frac{3\sigma_1}{\delta_1^2\delta_2^2}\Big[&\big(2\delta_2^2\sigma_1p_1^2-3\delta_2\delta_3\sigma_1p_1p_2+\delta_3^2\sigma_1p_2^2+2\delta_1\delta_2\sigma_2p_2p_3\big)p_{1,x}\\
&-\big(\delta_2\delta_3\sigma_1p_1^2-\delta_3^2\sigma_1p_1p_2-\delta_1\delta_2\sigma_2p_1p_3\big)p_{2,x}\\
&+\delta_1\delta_2\sigma_2p_1p_2p_{3,x}\Big],\\
p_{2,t}=p_{2,xxx}+\frac{3\sigma_1}{\delta_1^2\delta^2}\Big\{&\delta_2\sigma_1(\delta_2p_1-\delta_3p_2)p_2p_{1,x}\\
&+\big[\delta_2^2\sigma_1p_1^2+3\delta_2(\delta_1\sigma_2p_3-\delta_3\sigma_1p_1)p_2+2\delta_3^2\sigma_1p_2^2\big]p_{2,x}\\
&+\delta_1\delta_2\sigma_2p_2^2p_{3,x}\Big\},\\
p_{3,t}=p_{3,xxx}+\frac{3\sigma_1}{\delta_1^2\delta_2^2}\Big\{&\delta_2\sigma_1(\delta_2p_1-\delta_3p_2)p_3p_{1,x}\\
&+\big[\delta_1\delta_2\sigma_2p_3-\delta_3\sigma_1(\delta_2p_1-\delta_3p_2)\big]p_3p_{2,x}\\
&+\big[3\delta_1\delta_2\sigma_2p_2p_3+\sigma_1(\delta_2p_1-\delta_3p_2)^2\big]p_{3,x}\Big\},
\end{aligned}$$

where $\delta_1,\delta_2,\sigma_1,\sigma_2$ are arbitrary non-zero constants, while δ_3 is completely arbitrary and not necessarily non-zero.

Selecting

$$\delta_1=\delta_2=1,\quad \delta_3=-1,\quad \sigma_1=\sigma_2=1,$$

we obtain

$$p=\begin{bmatrix}-p_2 & p_2 & p_1\\ -p_2 & 0 & p_3\\ -p_1-p_2+p_3 & -p_3 & p_3\end{bmatrix},\quad q=\begin{bmatrix}0 & -p_3 & -p_1-p_2\\ p_1+p_2+p_3 & p_2-p_3 & -p_1-2p_2\\ p_1+p_2 & p_2 & 0\end{bmatrix},$$

and the corresponding integrable simplified mKdV model system is given as follows:

$$\begin{aligned}
p_{1,t}&=p_{1,xxx}+3\big(2p_1^2+3p_1p_2+p_2^2+2p_2p_3\big)p_{1,x}+3p_1\big[(p_1+p_2+p_3)p_{2,x}+p_2p_{3,x}\big],\\
p_{2,t}&=p_{2,xxx}+3\big(p_1^2+3p_1p_2+2p_2^2+3p_2p_3\big)p_{2,x}+3p_2\big[(p_1+p_2)p_{1,x}+p_2p_{3,x}\big],\\
p_{3,t}&=p_{3,xxx}+3\big[(p_1+p_2)^2+3p_2p_3\big]p_{3,x}+3p_3\big[(p_1+p_2)p_{1,x}+(p_1+p_2+p_3)p_{2,x}\big].
\end{aligned}$$

It is worth noting that the computation of the Lax matrix $V^{[3]}$, as defined in (2.5), is straightforward in both examples. This matrix serves as the temporal component of the Lax pairs corresponding to the reduced mKdV integrable models. Furthermore, one can derive the complete integrable hierarchies described by (2.14) (or (2.15)), along with their associated Lax pairs given by (2.11) and (2.16) (or (2.12) and (2.17)).

4. Concluding Remarks

This paper presents two representative examples of dual similarity transformations and applies them to matrix AKNS spectral problems to construct reduced matrix mKdV integrable models. The associated reduced spectral problems and resulting integrable systems are explicitly derived with the aid of Maple. By employing dual similarity transformations of distinct forms, a systematic methodology emerges for generating new integrable models. These constructions also contribute to a broader understanding of the structure and classification of mKdV-type integrable systems (see, e.g., [24, 29, 32]).

By applying various dual similarity transformations to zero-curvature equations, a broad spectrum of integrable reductions can be obtained (see, e.g., [47, 54–56]). Certain reduced mKdV-type integrable systems of this kind can be solved by means of the inverse scattering transform [53]. The examples presented in this study underscore the versatility and depth of reduced Lax pairs in constructing integrable models. A key step involves designing diagonal and off-diagonal block matrices within these transformations, which fundamentally shape the structure of the resulting systems. Furthermore, these techniques contribute to ongoing developments in integrable models associated with higher-order matrix spectral problems, as discussed in [10–12, 31, 34–36]. Comparative investigations with other integrable systems may provide deeper insights into the algebraic and geometric underpinnings of such models.

Rogue waves, lump solutions, and soliton waves are well-known and intriguing nonlinear wave phenomena (see, e.g., [5, 6, 9, 17, 20, 21, 30, 33, 43, 46, 51, 52]). An exciting direction for future research is to investigate such phenomena within the framework of the reduced integrable models presented in this work. Analytical techniques such as the inverse scattering transform and the Riemann–Hilbert method serve as powerful tools for this purpose. It would also be of significant interest to explore the existence and structure of Darboux transformations associated with these models. Given the distinct structure of reduced integrable systems [37–39], their nonlinear wave dynamics could offer new opportunities for applications in applied mathematics and engineering sciences.

In summary, this research establishes a robust framework for the formulation and analysis of integrable models. The developed models are expected to provide valuable new insights into the classification of multi-component integrable systems within the zero-curvature formulation. The structural results presented here may contribute meaningfully to future developments and applications of integrable models in both the physical and engineering sciences.

Acknowledgments

The work was supported in part by the Ministry of Science and Technology of China (Grant Nos. G2021016032L, G2023016011L) and by the National Natural Science Foundation of China (Grant Nos. 12271488, 11975145).

References

[1] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Stud. Appl. Math. **53**(4), 249–315 (1974).

[2] M.J. Ablowitz and Z.H. Musslimani, *Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation*, Nonlinearity **29**(3), 915–946 (2016).

[3] M.J. Ablowitz and Z.H. Musslimani, *Integrable nonlocal nonlinear equations*, Stud. Appl. Math. **139**(1), 7–59 (2017).

[4] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM (1981).

[5] B.D.V. Campos, *Characterization of rational solutions of a KdV-like equation*, Math. Comput. Simul. **201**, 396–416 (2022).

[6] J.Y. Chu, Y.Q. Liu and W.X. Ma, *Integrability and multiple-rogue and multi-soliton wave solutions of the 3+1-dimensional Hirota-Satsuma-Ito equation*, Mod. Phys. Lett. B **39**(21), Paper No. 2550060 (2025).

[7] A. Das, *Integrable Models*, World Scientific (1989).

[8] P.G. Drazin and R.S. Johnson, *Solitons: An Introduction*, Cambridge University Press (1983).

[9] D. Gao, W.X. Ma and X. Lü, *Wronskian solution, Bäcklund transformation and Painlevé analysis to a (2+1)-dimensional Konopelchenko-Dubrovsky equation*, Z. Naturforsch. **79a**(9), 887–895 (2024).

[10] X.G. Geng, M.X. Jia, B. Xue and Y.Y. Zhai, *Application of tetragonal curves to coupled Boussinesq equations*, Lett. Math. Phys. **114**, Paper No. 30 (2024).

[11] X.G. Geng and X. Zeng, *Algebro-geometric quasi-periodic solutions to the Satsuma-Hirota hierarchy*, Phys. D. **448**, Paper No. 133738 (2023).

[12] V.S. Gerdjikov, *Nonlinear evolution equations related to Kac-Moody algebra $A_r^{(1)}$: Spectral aspects*, Turk. J. Math. **46**, 1821–1844 (2022).

[13] V.S. Gerdjikov and N.A. Kostov, *Reductions of multicomponent mKdV equations on symmetric spaces of DIII-type*, SIGMA **4**, Paper No. 029 (2008).

[14] M. Gürses and A. Pekcan, *Nonlocal nonlinear Schrödinger equations and their soliton solutions*, J. Math. Phys. **59**(5), Paper No. 051501 (2018).

[15] M. Gürses and A. Pekcan, *Nonlocal modified KdV equations and their soliton solutions by Hirota method*, Commun. Nonlinear Sci. Numer. Simul. **67**, 427–448 (2019).

[16] J.L. Ji and Z.N. Zhu, *On a nonlocal modified Korteweg-de Vries equation: Integrability, Darboux transformation and soliton solutions*, Commun. Nonlinear Sci. Numer. Simul. **42**, 699–708 (2017).

[17] M. Kaplan and N. Raza, *Construction of complexiton-type solutions using bilinear form of Hirota-type*, Int. J. Nonlinear Sci. Numer. Simul. **24**(1), 349–357 (2023).

[18] P.D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21**(5), 467–490 (1968).

[19] T.S. Liu and T.C. Xia, *Multi-component generalized Gerdjikov-Ivanov integrable hierarchy and its Riemann-Hilbert problem*, Nonlinear Anal. Real World Appl. **68**, Paper No. 103667 (2022).

[20] H.C. Ma, Y.X. Bai and A.P. Deng, *General M-lump, high-order breather, and localized interaction solutions to (2+1)-dimensional generalized Bogoyavlensky-Konopelchenko equation*, Front. Math. China **17**(5), 943–960 (2022).

[21] H.C. Ma, S.P. Yue, Y.D. Gao and A.P. Deng, *Lump solution, breather soliton and more soliton solutions for a (2+1)-dimensional generalized Benjamin-Ono equation*, Qual. Theory Dyn. Syst. **22**(2), Paper No. 72 (2023).

[22] W.X. Ma, *The algebraic structures of isospectral Lax operators and applications to integrable equations*, J. Phys. A: Math. Gen. **25**(20), 5329–5343 (1992).

[23] W.X. Ma, *Application of the Riemann-Hilbert approach to the multicomponent AKNS integrable hierarchies*, Nonlinear Anal.: Real World Appl. **47**, 1–17 (2019).

[24] W.X. Ma, *Nonlocal integrable mKdV equations by two nonlocal reductions and their soliton solutions*, J. Geom. Phys. **177**, Paper No. 104522 (2022).

[25] W.X. Ma, *A novel kind of reduced integrable matrix mKdV equations and their binary Darboux transformations*, Mod. Phys. Lett. B **36**(20), Paper No. 2250094 (2022).

[26] W.X. Ma, *Matrix integrable fifth-order mKdV equations and their soliton solutions*, Chin. Phys. B **32**(2), Paper No. 020201 (2023).

[27] W.X. Ma, *Sasa-Satsuma type matrix integrable hierarchies and their Riemann-Hilbert problems and soliton solutions*, Phys. D **446**, Paper No. 133672 (2023).

[28] W.X. Ma, *Nonlocal integrable equations in soliton theory*, in: *Nonlinear and Modern Mathematical Physics*, S. Manukure and W.X. Ma (Eds), pp. 251–266, Springer Nature (2024).

[29] W.X. Ma, *Real reduced matrix mKdV integrable hierarchies under two local group reductions*, East Asian J. Appl. Math. **14**(2), 281–292 (2024).

[30] W.X. Ma, *Lump waves and their dynamics of a spatial symmetric generalized KP model*, Rom. Rep. Phys. **76**(3), Paper No. 108 (2024).

[31] W.X. Ma, *A combined generalized Kaup-Newell soliton hierarchy and its hereditary recursion operator and bi-Hamiltonian structure*, Theor. Math. Phys. **221**(1), 1603–1614 (2024).

[32] W.X. Ma, *Matrix mKdV integrable hierarchies via two identical group reductions*, Mathematics **13**(9), Paper No. 1438 (2025).

[33] W.X. Ma, *Dispersion-governed lump waves in a generalized Calogero-Bogoyavlenskii-Schiff-like model with spatially symmetric nonlinearity*, Axioms **14**(12), Paper No. 869 (2025).

[34] W.X. Ma, *A soliton hierarchy derived from a fourth-order matrix spectral problem possessing four fields*, Chaos Solitons Fractals **195**, Paper No. 116309 (2025).

[35] W.X. Ma, *A combined integrable hierarchy with four potentials and its recursion operator and bi-Hamiltonian structure*, Indian J. Phys. **99**(3), 1063–1069 (2025).

[36] W.X. Ma, *A matrix integrable Hamiltonian hierarchy and its two integrable reductions*, Int. J. Geom. Methods Mod. Phys. **22**(3), Paper No. 2450293 (2025).

[37] W.X. Ma, *An application of dual group reductions to the AKNS integrable mKdV model*, Mod. Phys. Lett. B **39**(35), Paper No. 2550233 (2025).

[38] W.X. Ma, *Integrable matrix modified Korteweg-de Vries equations derived from reducing AKNS Lax pairs*, Rom. J. Phys. **70**(7-8), Paper No. 110 (2025).

[39] W.X. Ma, *Reduced matrix integrable hierarchies via group reduction involving off-diagonal block matrices*, Commun. Theor. Phys. **78**(1), Paper No. 015001 (2026).

[40] W.X. Ma, Y.H. Huang and F.D. Wang, *Inverse scattering transforms for non-local reverse-space matrix non-linear Schrödinger equations*, Eur. J. Appl. Math. **33**(6), 1062–1082 (2022).

[41] F. Magri, *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. **19**(5), 1156–1162 (1978).

[42] A.V. Mikhailov, *The reduction problem and the inverse scattering method*, Phys. D **3**(1&2), 73–I17 (1981).

[43] J.G. Rao, J.S. He and B.A. Malomed, *Resonant collisions between lumps and periodic solitons in the Kadomtsev-Petviashvili I equation*, J. Math. Phys. **63**(1), Paper No. 013510 (2022).

[44] C.Q. Song, D.Y. Liu and L.Y. Ma, *Soliton solutions of a novel nonlocal Hirota system and a nonlocal complex modified Korteweg-de Vries equation*, Chaos Solitons Fractals **181**, Paper No. 114707 (2024).

[45] C.Q. Song, D.M. Xiao and Z.N. Zhu, *Solitons and dynamics for a general integrable nonlocal coupled nonlinear Schrödinger equation*, Commun. Nonlinear Sci. Numer. Simul. **45**, 13–28 (2017).

[46] S. Stalin, R. Ramakrishnan and M. Lakshmanan, *Dynamics of nondegenerate vector solitons in a long-wave-short-wave resonance interaction system,* Phys. Rev. E **105**(4), Paper No. 044203 (2022).

[47] T.A. Sulaiman, U. Younas, A. Yusuf, M. Younis, M. Bilal and Shafqat-Ur-Rehman, *Extraction of new optical solitons and MI analysis to three coupled Gross-Pitaevskii system in the spinor Bose-Einstein condensate*, Mod. Phys. Lett. B **35**(06), Paper No. 2150109 (2021).

[48] G.Z. Tu, *On Liouville integrability of zero-curvature equations and the Yang hierarchy*, J. Phys. A: Math. Gen. **22**(13), 2375–2392 (1989).

[49] H.F. Wang and Y.F. Zhang, *A kind of generalized integrable couplings and their bi-Hamiltonian structure*, Int. J. Theoret. Phys. **60**(5), 1797–1812 (2021).

[50] X. Wang, D.L. Du and H. Wang, *A nonlocal finite-dimensional integrable system related to the nonlocal mKdV equation*, Theor. Math. Phys. **218**(3), 370–387 (2024).

[51] A.-M. Wazwaz, *Breather wave solutions for an integrable (3+1)-dimensional combined pKP-BKP equation*, Chaos Solitons Fractals **182**, Paper No. 114886 (2024).

[52] J.J. Yang, S.F. Tian and Z.Q. Li, *Riemann-Hilbert method and multi-soliton solutions of an extended modified Korteweg-de Vries equation with N distinct arbitrary-order poles*, J. Math. Anal. Appl. **511**(2), Paper No. 126103 (2022).

[53] J.J. Yang, S.F. Tian and Z.Q. Li, *Inverse scattering problem for the matrix modified Korteweg-de Vries equation with finite density type initial data*, Phys. D **455**, Paper No. 133911 (2023).

[54] R.S. Ye, Y. Zhang and W.X. Ma, *Bound states of dark solitons in N-coupled complex modified Korteweg-de Vries equations*, Acta Appl. Math. **178**(1), Paper No. 7 (2022).

[55] U. Younas, T.A. Sulaiman and J. Ren, *Diversity of optical soliton structures in the spinor Bose-Einstein condensate modeled by three-component Gross-Pitaevskii system*, Int. J. Mod. Phys. B **37**(01), Paper No. 2350004 (2023).

[56] F. Yu and L. Li, *Vector dark and bright soliton wave solutions and collisions for spin-1 Bose-Einstein condensate*, Nonlinear Dyn. **87**(4), 2697–2713 (2017).

[57] J. Zhang, C.P. Zhang and Y.N. Cui, *Bi-integrable and tri-integrable couplings of a soliton hierarchy associated with SO(3)*, Adv. Math. Phys. **2017**, Paper No. 9743475 (2017).

[58] Q.L. Zhao, Y.D. Zhong and X.Y. Li, *Explicit solutions to a hierarchy of discrete coupling Korteweg-de Vries equations*, J. Appl. Anal. Comput. **12**(4), 1353–1370 (2022).

[59] X.M. Zhu and J.B. Zhang, *The integrability of a new fractional soliton hierarchy and its application*, Adv. Math. Phys. **2022**, Paper No. 2200092 (2022).