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On the Periodic Solutions of the Davey-Stewartson Systems

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Abstract. The periodic, traveling wave solutions of all four versions of the Davey-Stewartson system (namely the focusing and the defocusing cases of both the Davey-Stewartson I and the Davey-Stewartson II equations) are derived and classified. For all four versions, these solutions are described in terms of elliptic functions. Special reductions and limiting cases, including harmonic limits, soliton limits, and one-dimensional solutions, are also explicitly discussed.

AMS subject classifications: 35Q51, 35B10, 35C08, 37K10

Key words: Davey-Stewartson, periodic solutions, solitons, elliptic functions.

1. Introduction and Background

Equations of nonlinear Schrödinger (NLS) type arise as physical models in a number of different physical contexts, ranging from water waves, to optics, Bose-Einstein condensates, plasmas and more [11,31,35,39]. The simplest and most well-known case is of course that of the cubic NLS equation itself in one spatial dimension. There are also many physical contexts, however, in which the system is not confined to just one coordinate, and two spatial dimensions are necessary to accurately describe the dynamics. In these situations, more general systems of equations of NLS type often arise, in which the dynamics of an NLS-type weakly nonlinear envelope to that of a "mean field" [10,14,54]. One such case is that of the equations that model the evolution of wave packets in shallow water [10], a special limit of which gives rise to the Davey-Stewartson system [20]. Similar systems of NLS-type equations with coupling to mean fields have also been derived in optical materials with quadratic nonlinearity [1,2].

Like the NLS equation [56,57], the Davey-Stewartson system of equations is also an integrable system [6]. As such, it possesses a deep mathematical structure — cf. [6], including the existence of a Lax pair, the Painlevé property [8,9,47,49], the amenability of its initial value problem to inverse scattering [26,27], the existence of a rich family of solutions,

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including solitons as well as exponentially localized objects called dromions [17,28], even more exotic solutions [48] and more. Because of this, the Davey-Stewartson system continues to a very active topic of study [12,29,36,37,42,52,53]. Nonlocal variants of the Davey-Stewartson system have also been a subject of study in recent years — cf. [25,43,44,50] and references therein.

Like most other integrable evolution equations, the one-dimensional NLS equation also admits a rich family of traveling wave periodic solutions [13,32–34], which are expressed in terms of elliptic functions. The well-known soliton solutions are simply the limiting case of this more general family of solutions. In turn, these solutions provide the starting point for further investigations such as stability [18, 19, 21] as well as dispersionless or semiclassical limits [12,36,37]. It would be safe to expect that a similar class of solutions also exists for the Davey-Stewartson system. Surprisingly, however, no such solutions have been presented in the literature to the best of our knowledge. Further compounding the issue is that four variants of the Davey-Stewartson system exist, and that different authors write the system in different ways, which can often create confusion. The present work aims at addressing this issue and presenting the periodic, traveling wave solutions of all four variants of the Davey-Stewartson system in a concise but self-contained manner.

This work is organized as follows. In Section 2 we introduce the four variants of the Davey-Stewartson and we briefly review their Lax pair, invariances, and reductions, and one-dimensional reductions. In Section 3 we derive the periodic, traveling wave solutions of the defocusing DSII system. In Section 4 we present various examples, and in Section 5 we discuss various distinguished limits, including one-dimensional reductions, the plane-wave and soliton limits, and trivial-phase solutions. In Section 6 we generalize the calculations to all four variants of the Davey-Stewartson system, and in Section 7 we end this work with a few concluding remarks.

2. Preliminaries: Davey-Stewartson Systems, Lax Pair, Symmetries and Reductions

The four variants of the Davey-Stewartson system. The general Davey-Stewartson equations are the system

$$iq_t + \frac{1}{2}(q_{xx} - \sigma q_{yy}) + \psi q = 0, \tag{2.1a}$$

$$\psi_{xx} + \sigma\psi_{yy} = -\nu\big(|q|^2\big)_{xx} + \sigma\nu\big(|q|^2\big)_{yy} \tag{2.1b}$$

for a complex-valued function q and a real-valued function ψ of x, y and t. The parameters $\sigma = \pm1$ and $\nu = \pm1$ determine the four possible variants of the system. Specifically, the values $\sigma = -1$ and $\sigma = 1$ denote respectively the so-called Davey-Stewartson I (DSI) and the Davey-Stewartson II (DSII) systems. Likewise, the values $\nu = -1$ and $\nu = 1$ identify the focusing and defocusing cases, although in this case the distinction is more ambiguous, since in this case one has focusing or defocusing behavior depending the particular spatial reduction is considered (see below for further details). For convenience, we list the four variants of the Davey-Stewartson system explicitly: