East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.110122.040522

Vol. **12**, No. 4, pp. 912-927
November 2022

Stochastic Global Momentum-Preserving Schemes for Two-Dimensional Stochastic Partial Differential Equations

Mingzhan Song[1], Songhe Song[2], Wei Zhang[1] and Xu Qian[2,*]

[1]*Xi'an Satellite Control Center, Xi'an 710600, P.R.China.*
[2]*Department of Mathematics, College of Liberal Arts and Sciences, National University of Defense Technology, Changsha 410073, P.R.China.*

Received 11 January 2022; Accepted (in revised version) 4 May 2022.

Abstract. In this paper, the global momentum conservation laws and the global momentum evolution laws are presented for the two-dimensional stochastic nonlinear Schrödinger equation with multiplicative noise and the two-dimensional stochastic Klein-Gordon equation with additive noise, respectively. In order to preserve the global momenta or their changing trends in numerical simulation, the schemes are constructed by using a stochastic multi-symplectic formulation. It is shown that under periodic boundary conditions, the schemes have discrete global momentum conservation laws or the discrete global momentum evolution laws. Numerical experiments confirm global momentum-preserving properties of the schemes and their mean square convergence in the time direction.

AMS subject classifications: 65M06, 65P10

Key words: Stochastic global momentum-preserving scheme, stochastic nonlinear Schrödinger equation, global momentum conservation law, stochastic Klein-Gordon equation, global momentum evolution law.

1. Introduction

In recent decades, more and more researchers believe that structure-preserving methods can be used to construct long-time high precision numerical schemes for the Hamiltonian systems. Therefore, symplectic and multi-symplectic methods [20], discrete gradient method [4], average vector field method [7], Hamiltonian boundary value method [3,22], discrete variational derivative method [14] and local structure-preserving method [4,15] are actively studied and widely applied. On the other hand, stochastic Hamiltonian systems and various numerical methods for their solution provide a more accurate description of physical systems [18], and it is worth noting that the methods preserving the invariants of stochastic Hamiltonian systems became a popular tool in numerical simulations [9,16,26].

*Corresponding author. *Email addresses:* `smz161619@163.com` (M. Song), `qianxu@nudt.edu.cn` (X. Qian)

Nonlinear Schrödinger and nonlinear wave equations play an important role in mathematical physics, and the corresponding energy-preserving and momentum-preserving schemes of their solution have attracted considerable attention recently [5,15,23]. If stochastic noise of the external environment is present, the unique solvability of stochastic nonlinear Schrödinger equation (SNLSE) and stochastic Klein-Gordon equation (SKGE) is discussed in [2,12,13] and [1,6], respectively. Hong *et al.* [8,17] proposed a series of stochastic structure-preserving schemes for the numerical simulations of the SNLSE. Meanwhile, finite difference method [11], finite element method [10], spectral Galerkin method [24] and stochastic conformal method [21] have been applied to solve stochastic wave equation.

In this work, we mainly deal with the construction of numerical schemes which preserve the global momenta or the changing trends of two-dimensional stochastic systems. To the best of authors' knowledge, this topic is still not well studied. Note that SNLSE with multiplicative noise has the form

$$\mathrm{d}_t\psi - \mathrm{i}\left(\psi_{xx}+\psi_{yy}+a|\psi|^2\psi\right)\mathrm{d}t = \mathrm{i}\varepsilon_1\psi\circ\mathrm{d}\beta, \quad (x,y)\in[x_0,x_1]\times[y_0,y_1], \tag{1.1}$$

where $\psi(x,y,t)$ is a complex-valued function and $\beta(t)$ an independent standard Brownian motion defined on a complete filtered probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t>0},\mathbf{P})$. Here, the symbol $\circ$ stands for the Stratonovich product and ε_1 represents the size of the stochastic noise. If $\psi(x,t)=p(x,t)+\mathrm{i}q(x,t)$, the global momenta of the system (1.1) are defined by

$$I_1(p,q) := \int_{y_0}^{y_1}\int_{x_0}^{x_1}\frac{1}{2}(pq_x - qp_x)\,\mathrm{d}x\mathrm{d}y,$$

$$I_2(p,q) := \int_{y_0}^{y_1}\int_{x_0}^{x_1}\frac{1}{2}\left(pq_y - qp_y\right)\mathrm{d}x\mathrm{d}y.$$

Besides, the stochastic Klein-Gordon equation (SKGE) with additive noise has the form

$$\mathrm{d}_t u_t - \left(u_{xx}+u_{yy}-cu^3\right)\mathrm{d}t = \varepsilon_2\mathrm{d}W, \quad (x,y)\in[x_0,x_1]\times[y_0,y_1], \tag{1.2}$$

where $u(x,y,t)$ is a real-valued function and $W=W(t,\omega)$ a Q-Wiener process on separable Hilbert space $\mathbb{L}^2([x_0,x_1]\times[y_0,y_1];\mathbb{R})$ with a covariance operator $\mathscr{Q}$ (symmetric, nonnegative and finite trace operator), defined on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t>0},\mathbf{P})$. Let $\{\eta_m : m\in N_+\}$ be the eigenvalues of the operator $\mathscr{Q}$ on the orthonormal basis $\{e_m\}$. Then there exists a sequence of independent $\mathscr{F}_t$-Brownian motions $\{\beta_m\}$ such that

$$W(t,x,y,\omega) = \sum_{m=1}^{\infty}\sqrt{\eta_m}e_m(x,y)\beta_m(t,\omega),$$

where $\sum_{m\in N_+}\eta_m<\infty$. Writing $v=u_t$, we define the global momenta of the system (1.2) by

$$\tilde{I}_1(u) := \int_{y_0}^{y_1}\int_{x_0}^{x_1} vu_x\mathrm{d}x\mathrm{d}y,$$

$$\tilde{I}_2(u) = \int_{y_0}^{y_1}\int_{x_0}^{x_1} vu_y\mathrm{d}x\mathrm{d}y.$$