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Finite Genus Solutions to a Hierarchy of Integrable Semi-Discrete Equations

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Abstract. Resorting to the discrete zero-curvature equation and the Lenard recursion equations, a hierarchy of integrable semi-discrete nonlinear evolution equations is derived from a 3×3 matrix spectral problem with three potentials. Based on the characteristic polynomial of the Lax matrix for the hierarchy, a trigonal curve is introduced, and the properties of the corresponding three-sheeted Riemann surface are studied, including the genus, three kinds of Abelian differentials, Riemann theta functions. The asymptotic properties of the Baker-Akhiezer function and fundamental meromorphic functions defined on the trigonal curve are analyzed with the established theory of trigonal curves. As a result, finite genus solutions of the whole integrable semi-discrete nonlinear evolution hierarchy are obtained.

AMS subject classifications: 37K10, 37K20, 14H42, 37K40

Key words: Integrable semi-discrete nonlinear evolution equation, trigonal curve, Baker-Akhiezer function, finite genus solution.

1. Introduction

Finite genus solutions of soliton equations have been a concerning topic over the past few decades, which reveal the inherent structure mechanism of solitons and describe the quasi-periodic behavior of nonlinear phenomena. Since the early 1970s, various methods have been developed to solve soliton equations, such as the inverse scattering method, Darboux transformation, Riemann-Hilbert approach and algebro-geometric methods — cf. [1, 6, 16–19, 27, 29, 30, 40] and references therein. Among these methods, the algebro-geometric method is a powerful tool to construct finite genus solutions of soliton equations associated with 2×2 matrix spectral problems based on the theory of hyperelliptic curves. It has been successfully applied to the KdV, nonlinear Schrödinger, mKdV, sine-Gordon, Toda lattice, and Ablowitz-Ladik equations and others [2, 3, 5, 8, 9, 12, 14, 15, 24, 28, 31, 34, 36]. The main tools used in this method include the theory of hyperelliptic curves, Riemann

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theta functions, Abel differentials, Abel map, and Abel-Jacobi inversion. When considering higher-order matrix spectral problems, the corresponding algebraic curve becomes non-hyperelliptic, which brings great complexity. Consequently, the investigations of finite genus solutions of soliton equations associated with 3×3 matrix spectral problem are relatively rare compared with 2×2 case. Nonetheless, some progress has been made in [4, 7, 13, 32, 33, 35, 37–39], certain finite genus solutions of the Boussinesq equation related to a third-order differential operator were found as special solutions of the Kadomtsev-Petviashvili equation or by the reduction theory of Riemann theta functions. Dickson *et al*. [10, 11] proposed a framework to yield finite genus solutions of the entire Boussinesq hierarchy based on the classical Burchnall-Chaundy polynomial, the Baker-Akhiezer function and the theory of trigonal curves. Based on this, Geng *et al*. [20] further developed an effective way to introduce algebraic curves associated with higher order matrix spectral problems and applied it to construct finite genus solutions of soliton equation hierarchies related to 3×3 matrix spectral problems, such as the modified Boussinesq, Kaup-Kupershmidt, coupled mKdV hierarchies [20, 21, 23, 26, 41].

In this paper, our main purpose is to derive an integrable hierarchy of semi-discrete nonlinear evolution equations associated with 3×3 matrix spectral problem and construct its finite genus solutions based on the theory of trigonal curves. The first member in the hierarchy is the discrete 3-field system

$$\begin{aligned}
u_{n,t} &= -(1-u_n v_n)[u_{n+2}(1-u_{n+1}v_{n+1}) - u_{n+1}w_n],\\
v_{n,t} &= (1-u_n v_n)[v_{n-2}(1-u_{n-1}v_{n-1}) - v_{n-1}w_{n-1}],\\
w_{n,t} &= w_n[u_{n+2}v_n(1-u_{n+1}v_{n+1}) - u_{n+1}v_{n-1}(1-u_n v_n)],
\end{aligned} \tag{1.1}$$

where $u_n = u(n,t)$, $u_{n,t} = \partial_t u_n$, $n \in \mathbb{Z}, t \in \mathbb{R}$. The same notation is used for $v(n,t)$ and $w(n,t)$. System (1.1) is reduced to

$$\begin{aligned}
u_{n,t} &= u_{n+2}(u_n v_n - 1)(1-u_{n+1}v_{n+1}),\\
v_{n,t} &= v_{n-2}(1-u_n v_n)(1-u_{n-1}v_{n-1})
\end{aligned} \tag{1.2}$$

for $w_n = 0$. In Ref. [22], finite genus solutions of the entire discrete integrable hierarchy of (1.2) are constructed. Due to the addition of one potential function, the properties of the trigonal curve corresponding to system (1.1) at origin become more complicated than the ones for (1.2). Therefore, constructing finite genus solutions to the hierarchy of (1.1) requires more effort. In particular, we need to introduce three Abelian differentials of the third kind to characterize the Baker-Akhiezer function and the meromorphic functions involved.

This paper is organized as follows. In Section 2, we construct the discrete integrable hierarchy from a 3×3 matrix spectral problem with three potentials by resorting to the Lenard recursion equations and zero-curvature equations. Section 3 introduces the stationary Baker-Akhiezer function ψ_3, two meromorphic functions ϕ_{13} and ϕ_{23} carrying the data of the divisor and a trigonal curve of genus $m-1$ with two infinite points and three zero points. The analytic properties of ϕ_{13}, ϕ_{23} and ψ_3 are studied by using the theory of trigonal curves. In Section 4, we present the explicit Riemann theta function representations of the stationary Baker-Akhiezer function, the meromorphic functions, and in particular, that of the solutions $u(n)$, $v(n)$ and $w(n)$ for the entire stationary discrete hierarchy. Section 5 extends the algebro-geometric analysis of Sections 3 and 4 to the time-dependent case. Finally, finite genus solutions for the entire discrete hierarchy are obtained.

2. Discrete Integrable Hierarchy

In this section, we construct a discrete integrable hierarchy by using the zero-curvature and Lenard equations. Assume that $u=u(n,t), v=v(n,t), w=w(n,t)$ are potentials with $(n,t)\in\mathbb{Z}\times\mathbb{R}$ satisfying $uvw\neq 0$. The shift operator E, its inverse, difference operators Δ and Δ_m are defined as follows:

$$
\begin{aligned}
&Ef(n)=f(n+1), && E^{-1}f(n)=f(n-1),\\
&\Delta f(n)=f(n+1)-f(n), && \Delta_m=\sum_{j=0}^{m}E^{j}f(n),
\end{aligned}
$$

where $f(n)$ is a lattice function, $n\in\mathbb{Z}$ is a lattice variable, and m is an arbitrary positive integer. We usually write $f=f(n)$, $f^{\pm}=E^{\pm 1}f$ for notational convenience.

Consider the 3×3 matrix spectral problem with three potentials

$$
E\psi=U\psi,\quad \psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix},\quad U=\begin{pmatrix}0&\lambda&0\\ 1&w&u\\ v&0&1\end{pmatrix}, \tag{2.1}
$$

where λ is a constant spectral parameter. We first introduce the sequences $\hat{g}_j, \check{g}_j$ and $\acute{g}_j$ by

$$
\begin{aligned}
&K\hat{g}_j=J\hat{g}_{j+1}, \quad \hat{g}_j=(\hat{a}_j,\hat{b}_j,\hat{c}_j,\hat{d}_j,\hat{e}_j)^T, \quad j\geq 0,\\
&K\check{g}_j=J\check{g}_{j+1}, \quad \check{g}_j=(\check{a}_j,\check{b}_j,\check{c}_j,\check{d}_j,\check{e}_j)^T, \quad j\geq 0,\\
&K\acute{g}_j=J\acute{g}_{j+1}, \quad \acute{g}_j=(\acute{a}_j,\acute{b}_j,\acute{c}_j,\acute{d}_j,\acute{e}_j)^T, \quad j\geq 0
\end{aligned} \tag{2.2}
$$

with the conditions

$$
\hat{g}_j|_{(u,v,w)=0}=\check{g}_j|_{(u,v,w)=0}=\acute{g}_j|_{(u,v,w)=0}, \quad j\geq 1.
$$

The starting points of the sequences are

$$
\hat{g}_0=\begin{pmatrix}uv^-\\ -u\\ -v^-\\ uv^{--}(1-u^-v^-)-uv^-w^-\\ 1\end{pmatrix},\quad \check{g}_0=\begin{pmatrix}1\\ 0\\ 0\\ -\Delta_1^{-1}(w+uv^-)\\ 0\end{pmatrix},\quad \acute{g}_0=\begin{pmatrix}0\\ 0\\ 0\\ 1\\ 0\end{pmatrix},
$$

where $\Delta_1\Delta_1^{-1}=\Delta_1^{-1}\Delta_1=1$. The initial conditions mean to identify constants of summation as zero. Additionally, two matrix operators K and J are defined by

$$K=\begin{pmatrix} (u\Delta w+Eu)E & E^2-wE & 0 & uE^2 & 0\\ -E^{-1}v & 0 & w^- -E^{-1} & -v & 0\\ w\Delta wE & 0 & 0 & w\Delta E & 0\\ 0 & -v & uE & 0 & \Delta\\ (\Delta w+Euv^-)E-uv^-E^{-1} & v(E-w)E & u(w^- -E^{-1}) & \Delta\Delta_1 & 0 \end{pmatrix},$$

$$J=\begin{pmatrix} 0 & 1 & 0 & 0 & u\\ 0 & 0 & -E & 0 & -vE\\ -\Delta\Delta_1 & -v^+E^2 & u & 0 & 0\\ 0 & -v & uE & 0 & \Delta\\ (\Delta w+Euv^-)E-uv^-E^{-1} & v(E-w)E & u(w^- -E^{-1}) & \Delta\Delta_1 & 0 \end{pmatrix}.$$

Then $\hat{g}_j, \check{g}_j$ and $\acute{g}_j$ are uniquely determined by the recursion equation (2.2) up to a term in Ker$J$, which we always assume to be zero. For example, the first several members read as

$$\hat{g}_1=\begin{pmatrix} \hat{a}_1\\ -u\hat{e}_1-(u^{++}s^+ +u^+w)s\\ -v^-\hat{e}_1-E^{-2}(v^-s+vw)s^+\\ \hat{d}_1\\ -uv^-w^- -\Delta_1 uv^{--}s^- \end{pmatrix},\quad \acute{g}_1=\begin{pmatrix} -uv^-\\ u\\ v^-\\ uv^{--}s^- +uv^-w^-\\ 0 \end{pmatrix},$$

$$\check{g}_1=\begin{pmatrix} (E^2-1)^{-1}\left(-w\Delta(\check{d}_0^+ +w)+u\check{c}_1-v^+\check{b}_1^{++}\right)\\ u\Delta w+u^+ -u^2v^- +u\check{d}_0^{++}-u\check{e}_1\\ v^{--}+v^-(\check{d}_0^- -\check{e}_1)\\ \check{d}_1\\ uv^- \end{pmatrix},$$

where

$$\begin{aligned}
\hat{a}_1 &= -(1+E^2)uv^{--}s^-s^{--}-u^+v^-ws-uv^{--}w^{--}s^- +uv^-\Delta_1 uv^{--}s+(uv^-)^2w^-,\\
\hat{d}_1 &= -(v^{--}s^- +v^-w^-)(v^-u^+u+u^{++}s^+ +wu^+)s-v^{----}us^{---}s^{--}s^-\\
&\quad -v^{---}(u^-v^{--}+w^- +w^{---})us^-s^{--}-(uv^{--}s^-)^2\\
&\quad -v^{--}u(u^-v^{--}w^{--}+2v^-w^-u+w^-w^{--})s^- -\left(uv^-w^-\right)^2,\\
\check{d}_1 &= (E^2-1)^{-1}\left(u\check{c}_1^- -uw^-\check{c}_1+v\left(w\check{b}_1^+ -\check{b}_1^{++}\right)+uv^-\check{a}_1^- -u^+v\check{a}_1^{++}-\Delta w\check{a}_1^+\right)
\end{aligned}$$

with $s=uv-1$.

To obtain a discrete integrable hierarchy, let us first focus on the stationary discrete zero-curvature equation

$$(EV)U-UV=0,\quad V=\begin{pmatrix} V_{11} & \lambda V_{12} & \lambda V_{13}\\ V_{21} & V_{22} & V_{23}\\ V_{31} & \lambda V_{32} & \lambda V_{33} \end{pmatrix}, \tag{2.3}$$

which is equivalent to

$$\begin{aligned}
&EV_{12}+vEV_{13}-\lambda V_{21}=0,\\
&\lambda(EV_{11}-V_{22})+wEV_{12}=0,\\
&uEV_{12}+EV_{13}-\lambda V_{23}=0,\\
&\lambda EV_{31}+(wE-1)V_{32}-vV_{12}=0,\\
&uEV_{32}+\Delta V_{33}-vV_{13}=0,\\
&EV_{22}-V_{11}+vEV_{23}-wV_{21}-uV_{31}=0,\\
&u(EV_{22}-V_{33})+EV_{23}-V_{13}-wV_{23}=0,\\
&v(EV_{33}-V_{11})+EV_{32}-V_{31}=0,\\
&\lambda EV_{21}+w\Delta V_{22}-V_{12}-uV_{32}=0.
\end{aligned} \tag{2.4}$$

Each entry of the Lax matrix V, denoted by $V_{ij}=V_{ij}(a,b,c,d,e)$, is assumed to have the following polynomial dependence on the spectral parameter λ and lattice functions a,b,c,d,e:

$$\begin{aligned}
&V_{11}=d, && V_{12}=a, && V_{13}=b,\\
&V_{21}=a^{+}+vb^{+}, && V_{22}=d^{+}+wa^{+}, && V_{23}=ua^{+}+b^{+},\\
&V_{31}=v^{-}a^{-}+c^{-}-w^{-}c, && V_{32}=c, && V_{33}=e.
\end{aligned} \tag{2.5}$$

Substituting (2.5) into (2.4) yields

$$\begin{aligned}
&(uw^{+}+u^{+})a^{++}-uwa^{+}+b^{++}-wb^{+}+ud^{++}-\lambda(b+ue)=0,\\
&-v^{-}a^{-}+w^{-}c-c^{-}-vd+\lambda(c^{+}+ve^{+})=0,\\
&w\Delta(wa^{+}+d^{+})+\lambda(a^{++}-a+v^{+}b^{++}-uc)=0,\\
&\Delta e+uc^{+}-vb=0,\\
&\Delta wa^{+}+u^{+}va^{++}-uv^{-}a^{-}+v(b^{++}-wb^{+})+uw^{-}c-uc^{-}+d^{++}-d=0.
\end{aligned} \tag{2.6}$$

Expand a,b,c,d and e into Laurent series in λ, i.e.

$$a=\sum_{j\geq 0}a_j\lambda^{-j},\quad b=\sum_{j\geq 0}b_j\lambda^{-j},\quad c=\sum_{j\geq 0}c_j\lambda^{-j},\quad d=\sum_{j\geq 0}d_j\lambda^{-j},\quad e=\sum_{j\geq 0}e_j\lambda^{-j}.$$

Then the Eqs. (2.6) is equivalent to the recurrence equation

$$KG_j=JG_{j+1},\quad JG_0=0,\quad j\geq 0,$$

where $G_j=(a_j,b_j,c_j,d_j,e_j)^T$. Since the equation $JG_0=0$ has a solution — viz.

$$G_0=\alpha_0\hat{g}_j+\beta_0\check{g}_j+\gamma_0\acute{g}_j,$$

the vector G_j can be expressed as

$$\begin{aligned}
G_j=&\,\alpha_0\hat{g}_j+\beta_0\check{g}_j+\gamma_0\acute{g}_j+\alpha_1\hat{g}_{j-1}+\beta_1\check{g}_{j-1}+\gamma_1\acute{g}_{j-1}\\
&+\cdots+\alpha_j\hat{g}_0+\beta_j\check{g}_0+\gamma_j\acute{g}_0,\quad j\geq 0,
\end{aligned} \tag{2.7}$$

where α_j, β_j and γ_j are arbitrary constants. Thus, the stationary zero-curvature equation (2.3) is solved. Using Eq. (2.7), we define the Lax matrix

$$V^{(q)} = (\lambda^q V)_+ = \begin{pmatrix} V_{11}^{(q)} & \lambda V_{12}^{(q)} & \lambda V_{13}^{(q)} \\ V_{21}^{(q)} & V_{22}^{(q)} & V_{23}^{(q)} \\ V_{31}^{(q)} & \lambda V_{32}^{(q)} & \lambda V_{33}^{(q)} \end{pmatrix}, \tag{2.8}$$

$$V_{ij}^{(q)} = V_{ij}(a^{(q)}, b^{(q)}, c^{(q)}, d^{(q)}, e^{(q)}),$$

where

$$(a^{(q)}, b^{(q)}, c^{(q)}, d^{(q)}, e^{(q)})^T = \sum_{j=0}^{q} G_j \lambda^{q-j}, \quad q \geq 0.$$

Assuming that $V^{(q)}$ satisfies the stationary zero curvature equation

$$(EV^{(q)})U - UV^{(q)} = 0, \tag{2.9}$$

one obtains the stationary hierarchy

$$KG_q = JG_{q+1} = 0.$$

Next, we turn to the time-dependent case. Let ψ satisfy the discrete spectral problem (2.1) and the auxiliary problem

$$\psi_{t_r} = \widetilde{V}^{(r)}\psi, \quad \widetilde{V}^{(r)} = \begin{pmatrix} \widetilde{V}_{11}^{(r)} & \lambda \widetilde{V}_{12}^{(r)} & \lambda \widetilde{V}_{13}^{(r)} \\ \widetilde{V}_{21}^{(r)} & \widetilde{V}_{22}^{(r)} & \widetilde{V}_{23}^{(r)} \\ \widetilde{V}_{31}^{(r)} & \lambda \widetilde{V}_{32}^{(r)} & \lambda \widetilde{V}_{33}^{(r)} \end{pmatrix}, \tag{2.10}$$

where

$$\begin{aligned} &\widetilde{V}_{ij}^{(r)} = V_{ij}(\tilde{a}^{(r)}, \tilde{b}^{(r)}, \tilde{c}^{(r)}, \tilde{d}^{(r)}, \tilde{e}^{(r)}), \\ &(\tilde{a}^{(r)}, \tilde{b}^{(r)}, \tilde{c}^{(r)}, \tilde{d}^{(r)}, \tilde{e}^{(r)})^T = \sum_{j \geq 0} \tilde{G}_j \lambda^{r-j} \end{aligned} \tag{2.11}$$

with $\tilde{G}_j = (\tilde{a}_j, \tilde{b}_j, \tilde{c}_j, \tilde{d}_j, \tilde{e}_j)^T$ determined by

$$\widetilde{G}_j = \tilde{\alpha}_0 \hat{g}_j + \tilde{\alpha}_1 \hat{g}_{j-1} + \cdots + \tilde{\alpha}_j \hat{g}_0, \quad j \geq 0. \tag{2.12}$$

The constants $\{\tilde{\alpha}_j\}$ in (2.12) are independent of $\{\alpha_j\}$. Then the compatibility of Eqs. (2.1) and (2.10) yields the discrete zero-curvature equation, $U_{t_r} = (E\widetilde{V}^{(r)})U - U\widetilde{V}^{(r)}$, which is equivalent to the discrete integrable hierarchy

$$(u_{t_r}, v_{t_r}, w_{t_r})^T = X_r, \quad r \geq 0 \tag{2.13}$$

with the vector fields

$$X_j = \mathscr{P}(K\widetilde{G}_j) = \mathscr{P}(J\widetilde{G}_{j+1}), \quad j \geq 0,$$

where $\mathscr{P}$ is the projective map $(\gamma^1,\gamma^2,\gamma^3,\gamma^4,\gamma^5)^T \to (\gamma^1,\gamma^2,\gamma^3)^T$. The first nontrivial member in the hierarchy (2.13) is as follows:

$$\begin{aligned} u_{t_0} &= -\tilde{\alpha}_0(1-uv)\big(u^{++}(1-u^+v^+)-u^+w\big),\\ v_{t_0} &= \tilde{\alpha}_0(1-uv)\big(v^{--}(1-u^-v^-)-v^-w^-\big),\\ w_{t_0} &= \tilde{\alpha}_0 w\Delta(1-uv)u^+v^-, \end{aligned}$$

which is exactly Eq. (1.1) while $\tilde{\alpha}_0=1$, $t_0=t$.

3. Stationary Baker-Akhiezer Function

In this section, we first introduce the corresponding trigonal curve $\mathscr{K}_{m-1}$ with the help of the Lax matrix $V^{(q)}$ for the discrete integrable hierarchy. Then the stationary Baker-Akhiezer function and the fundamental meromorphic functions on $\mathscr{K}_{m-1}$ are studied.

It can be directly checked that $yI-V^{(q)}$ also satisfies the stationary zero-curvature equation (2.9). Hence, the characteristic polynomial of Lax matrix $V^{(q)}$, $\mathscr{F}_m(\lambda,y)=\det(yI-V^{(q)})$, is independent of the variable n with the expansion

$$\det(yI-V^{(q)}) = y^3 - R_m(\lambda)y^2 + S_m(\lambda)y - T_m(\lambda),$$

where R_m, S_m and T_m are constant coefficient polynomials of λ. According to the q-th stationary zero curvature equation (2.9), we can set $d_q=\rho$ and $wa_q^+=\sigma$, where ρ and σ are two constants satisfying $\rho\sigma(\rho+\sigma)\neq 0$. Then one obtains from (2.5) and (2.8) that

$$\begin{aligned} R_m(\lambda) &= \mathrm{tr}\,(V^{(q)}) = V_{11}^{(q)}+V_{22}^{(q)}+\lambda V_{33}^{(q)} = \alpha_0\lambda^{q+1}+(\alpha_1+2\gamma_0)\lambda^q+\cdots+2\rho+\sigma,\\ S_m(\lambda) &= \begin{vmatrix} V_{11}^{(q)} & \lambda V_{12}^{(q)}\\ V_{21}^{(q)} & V_{22}^{(q)}\end{vmatrix} + \begin{vmatrix} V_{22}^{(q)} & V_{23}^{(q)}\\ \lambda V_{32}^{(q)} & \lambda V_{33}^{(q)}\end{vmatrix} + \begin{vmatrix} V_{11}^{(q)} & \lambda V_{13}^{(q)}\\ V_{31}^{(q)} & \lambda V_{33}^{(q)}\end{vmatrix}\\ &= \big(2\alpha_0\gamma_0-\beta_0^2\big)\lambda^{2q+1}+\cdots+\rho^2+\rho\sigma, \end{aligned} \tag{3.1}$$

$$T_m(\lambda) = \det(V^{(q)}) = \begin{vmatrix} V_{11}^{(q)} & \lambda V_{12}^{(q)} & \lambda V_{13}^{(q)}\\ V_{21}^{(q)} & V_{22}^{(q)} & V_{23}^{(q)}\\ V_{31}^{(q)} & \lambda V_{32}^{(q)} & \lambda V_{33}^{(q)}\end{vmatrix} = \lambda\big(-\alpha_0\beta_0^2\lambda^{3q+1}+\cdots\big).$$

This immediately leads to a trigonal curve of degree m by the equation

$$\mathscr{K}_{m-1}:\ \mathscr{F}_m(\lambda,y) = y^3 - R_m y^2 + S_m y - T_m = 0, \tag{3.2}$$

where $m=3q+2$ as $\alpha_0\beta_0\neq 0$. For $q\geq 1$, these curves are non-hyperelliptic. In what follows, we always assume $\alpha_0\beta_0\neq 0$ for the rest of the paper. By (3.1) and (3.2), the trigonal curve $\mathscr{K}_{m-1}$ can be compacted by joining two different infinite points P_{∞_1} and the double branch point P_{∞_2}. The compactification of the curve $\mathcal{K}_{m-1}$ is still denoted by the same symbol for convenience. The discriminant of the curve (3.2) is

$$\begin{aligned}\Delta(\lambda) &= 4S_m^3 - R_m^2 S_m^2 + 4R_m^3 T_m - 18R_m S_m T_m - 27T_m^2 \\ &= -4\alpha_0^4 \beta_0^2 \lambda^{6q+5} + \cdots - \rho^2\sigma^2(\rho+\sigma)^2.\end{aligned}$$

Thus, by the Riemann-Hurwitz formula the genus of $\mathcal{K}_{m-1}$ is $3q+1$. The trigonal curve $\mathcal{K}_{m-1}$ becomes a three-sheeted compact Riemann surface of genus $m-1$ if it is nonsingular and irreducible. Points P on $\mathcal{K}_{m-1}$ are denoted as $P=(\lambda, y)$ satisfying $\mathcal{F}_m(\lambda, y)=0$ together with P_{∞_1} and P_{∞_2}. In addition, one infers from Eq. (3.2) that $\mathcal{K}_{m-1}$ has three different points at $\lambda=0$, which are denoted by P_{0_1}, P_{0_2} and P_{0_3}. Introducing local coordinates $\zeta_{Q_0}: P \mapsto (\lambda-\lambda_0)$ near the point $Q_0=(\lambda_0, y_0) \in \mathcal{K}_{m-1}$, where Q_0 is neither a branch point nor a infinite point; $\zeta_{P_{\infty_1}}: P \mapsto \lambda^{-1}$ near P_{∞_1}; $\zeta_{P_{\infty_2}}: P \mapsto \lambda^{-1/2}$ near P_{∞_2}; $\zeta_{P_{0_j}}: P \mapsto \lambda$ near P_{0_j} for $j=1,2,3$ and similar at other branch points of $\mathcal{K}_{m-1}$.

In the following, we introduce the stationary Baker-Akhiezer function $\psi(P,n,n_0)$

$$\begin{aligned}&E\psi(P,n,n_0) = U\big(u(n),v(n),w(n);\lambda(P)\big)\psi(P,n,n_0),\\ &V^{(q)}\big(u(n),v(n),w(n);\lambda(P)\big)\psi(P,n,n_0) = y(P)\psi(P,n,n_0),\\ &\psi_3(P,n_0,n_0)=1, P\in\mathcal{K}_{m-1}\backslash\{P_{\infty_1},P_{\infty_2}\}, \quad n,n_0\in\mathbb{Z}.\end{aligned} \tag{3.3}$$

The fundamental meromorphic functions are defined by

$$\phi_{13}(P,n) = \frac{\psi_1(P,n,n_0)}{\psi_3(P,n,n_0)}, \quad P\in\mathcal{K}_{m-1}, \quad n,n_0\in\mathbb{Z}, \tag{3.4}$$

$$\phi_{23}(P,n) = \frac{\psi_2(P,n,n_0)}{\psi_3(P,n,n_0)}, \quad P\in\mathcal{K}_{m-1}, \quad n,n_0\in\mathbb{Z}. \tag{3.5}$$

Through some straightforward algebraic calculations, one obtains from Eqs. (3.3)-(3.5) that

$$\psi_3(P,n,n_0) = \begin{cases}\prod\limits_{n'=n_0}^{n-1}\big(v(n')\phi_{13}(P,n')+1\big), & n\geq n_0+1,\\ 1, & n=n_0,\\ \prod\limits_{n'=n}^{n_0-1}\big(v(n')\phi_{13}(P,n')+1\big)^{-1}, & n\leq n_0-1,\end{cases} \tag{3.6}$$

and

$$\begin{aligned}\phi_{13}(P,n) &= \frac{yV_{12}^{(q)}(\lambda,n)+C_m(\lambda,n)}{yV_{32}^{(q)}(\lambda,n)+A_m(\lambda,n)}\\ &= \frac{\lambda F_{m-1}(\lambda,n)}{y^2V_{12}^{(q)}(\lambda,n) - y\left(C_m(\lambda,n)+V_{12}^{(q)}(\lambda,n)R_m(\lambda)\right)+D_m(\lambda,n)}\\ &= \frac{y^2V_{32}^{(q)}(\lambda,n) - y\left(A_m(\lambda,n)+V_{32}^{(q)}(\lambda,n)R_m(\lambda)\right)+B_m(\lambda,n)}{E_{m-1}(\lambda,n)},\end{aligned} \tag{3.7}$$

$$\begin{aligned}\phi_{23}(P,n) &= \frac{yV_{21}^{(q)}(\lambda,n)+\mathscr{C}_m(\lambda,n)}{yV_{31}^{(q)}(\lambda,n)+\mathscr{A}_m(\lambda,n)} \\ &= \frac{H_{m-1}(\lambda,n)}{y^2V_{21}^{(q)}(\lambda,n)-y\left(\mathscr{C}_m(\lambda,n)+V_{21}^{(q)}(\lambda,n)R_m(\lambda)\right)+\mathscr{D}_m(\lambda,n)} \\ &= \frac{y^2V_{31}^{(q)}(\lambda,n)-y\left(\mathscr{A}_m(\lambda,n)+V_{31}^{(q)}(\lambda,n)R_m(\lambda)\right)+\mathscr{B}_m(\lambda,n)}{-\lambda E_{m-1}(\lambda,n)},\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}A_m &= V_{12}^{(q)}V_{31}^{(q)}-V_{11}^{(q)}V_{32}^{(q)}, \\ B_m &= \lambda V_{31}^{(q)}\left(V_{12}^{(q)}V_{33}^{(q)}-V_{13}^{(q)}V_{32}^{(q)}\right)+\lambda V_{32}^{(q)}\left(V_{22}^{(q)}V_{33}^{(q)}-V_{23}^{(q)}V_{32}^{(q)}\right), \\ C_m &= \lambda\left(V_{13}^{(q)}V_{32}^{(q)}-V_{12}^{(q)}V_{33}^{(q)}\right), \\ D_m &= V_{12}^{(q)}\left(V_{11}^{(q)}V_{22}^{(q)}-\lambda V_{12}^{(q)}V_{21}^{(q)}\right)+\lambda V_{13}^{(q)}\left(V_{11}^{(q)}V_{32}^{(q)}-V_{12}^{(q)}V_{31}^{(q)}\right),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathscr{A}_m &= \lambda V_{21}^{(q)}V_{32}^{(q)}-V_{22}^{(q)}V_{31}^{(q)}, \\ \mathscr{B}_m &= \lambda V_{31}^{(q)}\left(V_{11}^{(q)}V_{33}^{(q)}-V_{13}^{(q)}V_{31}^{(q)}\right)+\lambda V_{32}^{(q)}\left(\lambda V_{21}^{(q)}V_{33}^{(q)}-V_{23}^{(q)}V_{31}^{(q)}\right), \\ \mathscr{C}_m &= V_{23}^{(q)}V_{31}^{(q)}-\lambda V_{21}^{(q)}V_{33}^{(q)}, \\ \mathscr{D}_m &= V_{21}^{(q)}\left(V_{11}^{(q)}V_{22}^{(q)}-\lambda V_{12}^{(q)}V_{21}^{(q)}\right)+V_{23}^{(q)}\left(V_{22}^{(q)}V_{31}^{(q)}-\lambda V_{21}^{(q)}V_{32}^{(q)}\right),\end{aligned} \tag{3.10}$$

$$\begin{aligned}E_{m-1} &= \lambda V_{21}^{(q)}\left(V_{32}^{(q)}\right)^2+V_{31}^{(q)}V_{32}^{(q)}\left(V_{11}^{(q)}-V_{22}^{(q)}\right)-V_{12}^{(q)}\left(V_{31}^{(q)}\right)^2, \\ F_{m-1} &= \left(V_{12}^{(q)}\right)^2V_{23}^{(q)}+V_{12}^{(q)}V_{13}^{(q)}\left(\lambda V_{33}^{(q)}-V_{22}^{(q)}\right)-\lambda\left(V_{13}^{(q)}\right)^2V_{32}^{(q)}, \\ H_{m-1} &= \lambda\left(V_{21}^{(q)}\right)^2V_{13}^{(q)}+V_{21}^{(q)}V_{23}^{(q)}\left(\lambda V_{33}^{(q)}-V_{11}^{(q)}\right)-V_{31}^{(q)}\left(V_{23}^{(q)}\right)^2.\end{aligned} \tag{3.11}$$

There are some interrelationships among above polynomials $A_m, B_m, C_m, D_m, \mathscr{A}_m, \mathscr{B}_m, \mathscr{C}_m, \mathscr{D}_m, E_{m-1}, F_{m-1}, H_{m-1}, R_m, S_m, T_m$, for example

$$\begin{aligned}&\lambda V_{32}^{(q)}F_{m-1} = V_{12}^{(q)}D_m-\left(V_{12}^{(q)}\right)^2S_m-(C_m)^2-V_{12}^{(q)}C_mR_m, \\ &\lambda A_mF_{m-1} = \left(V_{12}^{(q)}\right)^2T_m+C_mD_m,\end{aligned} \tag{3.12}$$

$$\begin{aligned}&V_{12}^{(q)}E_{m-1} = V_{32}^{(q)}B_m-\left(V_{32}^{(q)}\right)^2S_m-(A_m)^2-V_{32}^{(q)}A_mR_m, \\ &C_mE_{m-1} = \left(V_{32}^{(q)}\right)^2T_m+A_mB_m,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&V_{31}^{(q)}H_{m-1} = V_{21}^{(q)}\mathscr{D}_m-\left(V_{21}^{(q)}\right)^2S_m-\mathscr{C}_m^2-V_{21}^{(q)}\mathscr{C}_mR_m, \\ &\mathscr{A}_mH_{m-1} = \left(V_{21}^{(q)}\right)^2T_m+\mathscr{C}_m\mathscr{D}_m,\end{aligned} \tag{3.14}$$

$$
\begin{aligned}
&-\lambda V_{21}^{(q)}E_{m-1}=V_{31}^{(q)}\mathscr{B}_m-\left(V_{31}^{(q)}\right)^2S_m-\mathscr{A}_m^2-V_{31}^{(q)}\mathscr{A}_mR_m,\\
&-\lambda\mathscr{C}_mE_{m-1}=\left(V_{31}^{(q)}\right)^2T_m+\mathscr{A}_m\mathscr{B}_m.
\end{aligned}
\tag{3.15}
$$

From Eqs. (3.11), one infers that E_{m-1},F_{m-1},H_{m-1} are polynomials with respect to λ of degree $m-1$. Therefore, they can be represented in the form

$$
\begin{aligned}
&E_{m-1}(\lambda,n)=\alpha_0^2\beta_0\big(v^-(n)\big)^2\prod_{j=1}^{m-1}\big(\lambda-\mu_j(n)\big),\\
&F_{m-1}(\lambda,n)=-\alpha_0^2\beta_0u(n)\prod_{j=1}^{m-1}\big(\lambda-\nu_j(n)\big),\\
&H_{m-1}(\lambda,n)=-\alpha_0^2\beta_0u^+(n)\big(1-u(n)v(n)\big)\prod_{j=1}^{m-1}\big(\lambda-\xi_j(n)\big),
\end{aligned}
\tag{3.16}
$$

where $\{\mu_j\}_{j=1}^n,\{\nu_j\}_{j=1}^n,\{\xi_j\}_{j=1}^n$ are zeros of E_{m-1},F_{m-1},H_{m-1}, respectively. Define

$$
\begin{aligned}
\hat{\mu}_j(n)=\big(\mu_j(n),y(\hat{\mu}_j(n))\big)&=\left(\mu_j(n),-\frac{A_m(\mu_j(n),n)}{V_{32}^{(q)}(\mu_j(n),n)}\right)\\
&=\left(\mu_j(n),-\frac{\mathscr{A}_m(\mu_j(n),n)}{V_{31}^{(q)}(\mu_j(n),n)}\right),\quad j=1,\dots,m-1,\\
\hat{\nu}_j(n)=\big(\nu_j(n),y(\hat{\nu}_j(n))\big)&=\left(\nu_j(n),-\frac{C_m(\nu_j(n),n)}{V_{12}^{(q)}(\nu_j(n),n)}\right),\quad j=1,\dots,m-1,\\
\hat{\xi}_j(n)=\big(\xi_j(n),y(\hat{\xi}_j(n))\big)&=\left(\xi_j(n),-\frac{\mathscr{C}_m(\xi_j(n),n)}{V_{21}^{(q)}(\xi_j(n),n)}\right),\quad j=1,\dots,m-1.
\end{aligned}
$$

It follows from (3.12)-(3.15) that $\{\hat{\mu}_j(n)\}_{j=1,\dots,m-1}\subset\mathscr{K}_{m-1}$, $\{\hat{\nu}_j(n)\}_{j=1,\dots,m-1}\subset\mathscr{K}_{m-1}$ and $\{\hat{\xi}_j(n)\}_{j=1,\dots,m-1}\subset\mathscr{K}_{m-1}$.

Next, we will give the asymptotic expansions for ϕ_{13} and ϕ_{23} near $P_{\infty_1},P_{\infty_2},P_{0_1},P_{0_2}$ and P_{0_3}. A direct calculation from Eqs. (3.3) yields

$$
\begin{aligned}
&\big(v(n)\phi_{13}(P,n)+1\big)\phi_{13}^+(P,n)=\lambda\phi_{23}(P,n),\\
&\big(v(n)\phi_{13}(P,n)+1\big)\phi_{23}^+(P,n)=\phi_{13}(P,n)+w(n)\phi_{23}(P,n)+u(n).
\end{aligned}
\tag{3.17}
$$

Lemma 3.1. *(I) Let $\zeta=\lambda^{-1}$ and $\zeta=\lambda^{-1/2}$ be the local coordinates near P_{∞_1} and P_{∞_2}, respectively. Then*

$$
\phi_{13}(P,n)\underset{\zeta\to0}{=}\begin{cases}-u(n)+\mathscr{O}(\zeta) & \text{as}\quad P\to P_{\infty_1},\\[4pt] \dfrac{1}{v^-(n)}\zeta^{-1}+\mathscr{O}(1) & \text{as}\quad P\to P_{\infty_2},\end{cases}
\tag{3.18a}
$$

$$\phi_{23}(P,n) \underset{\zeta\to 0}{=} \begin{cases} -u^+(n)\big(1-u(n)v(n)\big)\zeta + \mathscr{O}(\zeta^2) & as \quad P \to P_{\infty_1}, \\ \dfrac{1}{v^-(n)} + \mathscr{O}(\zeta) & as \quad P \to P_{\infty_2}. \end{cases} \tag{3.18b}$$

(II) Let $\zeta = \lambda$ be the local coordinates near P_{0_1}, P_{0_2}, and P_{0_3}. We have

$$\begin{aligned} &\phi_{13}(P,n) \underset{\zeta\to 0}{=} \begin{cases} \kappa(n) + \mathscr{O}(\zeta) & as \quad P \to P_{0_1}, \\ -\dfrac{1}{v(n)} + \mathscr{O}(\zeta) & as \quad P \to P_{0_2}, \\ \chi^-(n)\zeta + \mathscr{O}(\zeta^2) & as \quad P \to P_{0_3}, \end{cases} \\ &\phi_{23}(P,n) \underset{\zeta\to 0}{=} \begin{cases} \kappa(n)w^-(n)\zeta^{-1} + \mathscr{O}(1) & as \quad P \to P_{0_1}, \\ \dfrac{1-u(n)v(n)}{v(n)w(n)} + \mathscr{O}(\zeta) & as \quad P \to P_{0_2}, \\ \chi(n) + \mathscr{O}(\zeta) & as \quad P \to P_{0_3}, \end{cases} \end{aligned} \tag{3.19}$$

where $\kappa(n)$ and $\chi(n)$ satisfy the equations

$$\kappa^+(n)\big(v(n)\kappa(n)+1\big) = w^-(n)\kappa(n), \quad \chi^+(n) = w(n)\chi(n) + u(n).$$

Moreover, the divisors of ϕ_{13} and ϕ_{23} are given by

$$\big(\phi_{13}(P,n)\big) = \mathscr{D}_{P_{0_3},\hat{\nu}_1(n),\ldots,\hat{\nu}_{m-1}(n)}(P) - \mathscr{D}_{P_{\infty_2},\hat{\mu}_1(n),\ldots,\hat{\mu}_{m-1}(n)}(P), \tag{3.20}$$

$$\big(\phi_{23}(P,n)\big) = \mathscr{D}_{P_{\infty_1},\hat{\xi}_1(n),\ldots,\hat{\xi}_{m-1}(n)}(P) - \mathscr{D}_{P_{0_1},\hat{\mu}_1(n),\ldots,\hat{\mu}_{m-1}(n)}(P). \tag{3.21}$$

Proof. Inserting the following two sets of ansatz:

$$\begin{aligned} &\phi_{13}(P,n) \underset{\zeta\to 0}{=} \sum_{j\geq 0}\kappa_{1,j}(n)\zeta^j, \quad \phi_{23}(P,n) \underset{\zeta\to 0}{=} \sum_{j\geq 1}\chi_{1,j}(n)\zeta^j, \quad P \to P_{\infty_1}, \\ &\phi_{13}(P,n) \underset{\xi\to 0}{=} \sum_{j\geq -1}\kappa_{2,j}(n)\xi^j, \quad \phi_{23}(P,n) \underset{\xi\to 0}{=} \sum_{j\geq 0}\chi_{2,j}(n)\xi^j, \quad P \to P_{\infty_2}, \end{aligned} \tag{3.22}$$

into Eqs. (3.17), respectively, and comparing the coefficients of the same powers of ζ yield

$$\begin{aligned} &\kappa_{1,0} = -u, \quad \kappa_{1,1} = -u^{++}s^+s - u^+ws, \\ &\chi_{1,1} = u^+s, \quad \chi_{1,2} = s^+s\big(u^{++}(w^+ + u^+v) + u^{+++}s^{++}\big) + (u^+)^2vws, \\ &v\sum_{i=0}^{k}\kappa_{1,i}\kappa^+_{1,k-i} + \kappa^+_{1,k} = \chi_{1,k+1}, \quad v\sum_{i\geq 1}^{k}\kappa_{1,k-i}\chi^+_{1,i} + \chi^+_{1,k} = \kappa_{1,k} + w\chi_{1,k}, \quad k \geq 2, \end{aligned} \tag{3.23}$$

and

$$\kappa_{2,-1} = \frac{1}{v^-}, \quad \kappa_{2,0} = \frac{1}{v^-}\left(\Delta_1^{-1}(w^- + u^-v^{--}) - \frac{v^{--}}{v^-}\right),$$

$$\chi_{2,0}=\frac{1}{v^-},\quad \chi_{2,1}=\frac{1}{v^-}\left(w^-+u^-v^{--}-\frac{v^{--}}{v^-}\right),$$

$$v^-\chi_{2,2}=v^-\kappa_{2,0}^+(1+v\kappa_{2,0})+\Delta_1 v^-\kappa_{2,1},$$

$$v\sum_{i=0}^{k}\kappa_{2,i}\kappa_{2,k-i}+\kappa_{2,k}^+=\chi_{2,k+1},\quad v\sum_{i=-1}^{k}\kappa_{2,i}\chi_{2,k-i}+\chi_{2,k}^+=\kappa_{2,k}+w\chi_{2,k},\quad k\geq 2.$$

This proves (3.18). Similarly, we can prove (3.19). The relations (3.18) and (3.19) imply that $\phi_{13}(P,n)$ has a simple zero P_{0_3} and a simple pole P_{∞_2}. Additionally, it follows from (3.8), (3.16) that $\{\hat{v}_1(n),\ldots,\hat{v}_{m-1}(n)\}$ and $\{\hat{\mu}_1(n),\ldots,\hat{\mu}_{m-1}(n)\}$ are $m-1$ zeros and $m-1$ poles of ϕ_{13}, respectively. This proves (3.20) and (3.21) hold for the same reason. ☐

Lemma 3.2. *(I) Near $P_{\infty_j}\in\mathcal{K}_{m-1}$, in terms of the local coordinate $\zeta=\lambda^{-1/j}$, $j=1,2$, we have*

$$\psi_3(P,n,n_0)=\begin{cases}\mathcal{O}(1) & \text{as}\quad P\to P_{\infty_1},\\ \dfrac{v^-(n)}{v^-(n_0)}\zeta^{n_0-n}\big(1+\mathcal{O}(\zeta)\big) & \text{as}\quad P\to P_{\infty_2}.\end{cases}\tag{3.24}$$

(II) Near $P_{0_j}\in\mathcal{K}_{m-1}$, $j=1,2,3$, in terms of the local coordinate $\zeta=\lambda$, we have

$$\psi_3(P,n,n_0)=\begin{cases}\mathcal{O}(1) & \text{as}\quad P\to P_{0_1},\\ \Gamma(n,n_0)\zeta^{n-n_0}\big(1+\mathcal{O}(\zeta)\big) & \text{as}\quad P\to P_{0_2},\\ 1+\mathcal{O}(\zeta) & \text{as}\quad P\to P_{0_3},\end{cases}\tag{3.25}$$

where

$$\Gamma(n,n_0)=\begin{cases}\displaystyle\prod_{n'=n_0}^{n-1}\left(v^+(n')\frac{\big(u(n')v(n')-1\big)}{v(n')w(n')}\right) & \text{as}\quad n\geq n_0+1,\\ 1 & \text{as}\quad n=n_0,\\ \displaystyle\prod_{n'=n}^{n_0-1}\left(v^+(n')\frac{\big(u(n')v(n')-1\big)}{v(n')w(n')}\right)^{-1} & \text{as}\quad n\leq n_0-1.\end{cases}$$

The divisor $(\psi_3(P,n,n_0))$ of $\psi_3(P,n,n_0)$ is given by

$$\begin{aligned}\big(\psi_3(P,n,n_0)\big)=&\,\mathcal{D}_{\hat{\mu}_1(n),\hat{\mu}_2(n),\ldots,\hat{\mu}_{m-1}(n)}-\mathcal{D}_{\hat{\mu}_1(n_0),\hat{\mu}_2(n_0),\ldots,\hat{\mu}_{m-1}(n_0)}\\ &+(n-n_0)(\mathcal{D}_{P_{0_2}}-\mathcal{D}_{P_{\infty_2}}).\end{aligned}\tag{3.26}$$

Proof. From Lemma 3.1, we obtain (3.24) and (3.25). Using the relations (3.3)-(3.5) and (3.7), one computes

$$1+v\phi_{13}(P,n)=\frac{\lambda(1-uv)}{\lambda+vw\phi_{13}^+(P,n)+\lambda v\phi_{23}^+(P,n)}\underset{P\to\hat{\mu}_j(n)}{=}\frac{E_{m-1}^+(P)}{E_{m-1}(P)}\mathcal{O}(1).\tag{3.27}$$

Then, the meromorphic function $1+v\phi_{13}$ on $\mathcal{K}_{m-1}\backslash\{P_{\infty_1},P_{\infty_2},P_{0_1},P_{0_2},P_{0_3}\}$ has $m-1$ zeros $\hat{\mu}_1^+,\ldots,\hat{\mu}_{m-1}^+$ and $m-1$ poles $\hat{\mu}_1,\ldots,\hat{\mu}_{m-1}$. Hence, we get (3.26) by using (3.6), (3.24), (3.25) and (3.27). ☐

4. Finite Genus Solutions of Stationary Discrete Integrable Hierarchy

In this section, we construct the Riemann theta function representations of ϕ_{13}, ϕ_{23} and ψ_3, from which finite genus solutions of the stationary discrete integrable hierarchy are obtained.

Let us equip the Riemann surface $\mathscr{K}_{m-1}$ with canonical basis of cycles $a_1, \ldots, a_{m-1}$, $b_1, \ldots, b_{m-1}$ satisfying the following intersection numbers:

$$a_j \circ b_k = \delta_{jk}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j,k = 1, \ldots, m-1,$$

where δ_{ij} stands for the Kronecker symbol. For the present, we will choose our holomorphic differential basis as the following set:

$$\tilde{\omega}_l(P) = \frac{1}{3y^2 - 2yR_m + S_m} \begin{cases} \lambda^{l-1} d\lambda, & 1 \leq l \leq 2q+1, \\ y\lambda^{l-2q-2} d\lambda, & 2q+2 \leq l \leq m-1. \end{cases} \tag{4.1}$$

Define period matrices A and B as

$$A_{ij} = \int_{a_j} \tilde{\omega}_i, \quad B_{ij} = \int_{b_j} \tilde{\omega}_i, \quad i,j = 1, \ldots, m-1,$$

which are invertible [25]. Set $C = A^{-1}$, $\tau = A^{-1}B$. Taking the linear combinations

$$\omega_j = \sum_{l=1}^{m-1} C_{jl} \tilde{\omega}_l, \quad j = 1, \ldots, m-1,$$

we obtain a canonical basis of holomorphic differentials $\underline{\omega} = (\omega_1, \ldots, \omega_{m-1})$, normalized by the conditions

$$\int_{a_k} \omega_j = \delta_{jk}, \quad \int_{b_k} \omega_j = \tau_{jk}, \quad j,k = 1, \ldots, m-1.$$

The matrix τ is called the period matrix of the Riemann surface $\mathscr{K}_{m-1}$, which is symmetric and has a positive-definite imaginary part.

The Abelian differential of the third kind on $\mathscr{K}_{m-1} \backslash \{Q_1, Q_2\}$ is denoted by $\omega^{(3)}_{Q_1,Q_2}$. It satisfies the relations

$$\begin{aligned} \omega^{(3)}_{Q_1,Q_2} &\underset{\zeta \to 0}{=} \big(\zeta^{-1} + \mathcal{O}(1)\big) d\zeta \qquad \text{as} \quad P \to Q_1, \\ \omega^{(3)}_{Q_1,Q_2} &\underset{\zeta \to 0}{=} \big(-\zeta^{-1} + \mathcal{O}(1)\big) d\zeta \quad \text{as} \quad P \to Q_2. \end{aligned}$$

Here $\omega^{(3)}_{Q_1,Q_2}$ is normal if the normalized condition holds

$$\int_{a_k} \omega^{(3)}_{Q_1,Q_2} = 0, \quad k = 1, \ldots, m-1.$$

The relation of the normal Abelian differential of the third kind and the canonical holomorphic differential basis is given by

$$\frac{1}{2\pi \mathrm{i}}\int_{b_k}\omega^{(3)}_{Q_1,Q_2}=\int_{Q_2}^{Q_1}\omega_k,\quad k=1,\dots,m-1.$$

Theorem 4.1. *If $\{\tilde{\omega}_1,\dots,\tilde{\omega}_{m-1}\}$ is a holomorphic differential basis, then the normalized Abelian differentials of the third kind $\omega^{(3)}_{P_{0_3},P_{\infty_2}}$, $\omega^{(3)}_{P_{\infty_1},P_{0_1}}$ and $\omega^{(3)}_{P_{0_2},P_{\infty_2}}$ can be constructed by*

$$\begin{aligned}
\omega^{(3)}_{P_{0_3},P_{\infty_2}}&=\frac{(y-\rho)(y-\sigma-\rho)-\alpha_0\lambda^{q+1}y}{\lambda\mathscr{F}_{m,y}(\lambda,y)}d\lambda+\sum_{j=1}^{m-1}z_j^{(1)}\tilde{\omega}_j,\\
\omega^{(3)}_{P_{\infty_1},P_{0_1}}&=\frac{y(\rho-y)}{\lambda\mathscr{F}_{m,y}(\lambda,y)}d\lambda+\sum_{j=1}^{m-1}z_j^{(2)}\tilde{\omega}_j,\\
\omega^{(3)}_{P_{0_2},P_{\infty_2}}&=\frac{y(y-\alpha_0\lambda^{q+1}-\sigma-\rho)}{\lambda\mathscr{F}_{m,y}(\lambda,y)}d\lambda+\sum_{j=1}^{m-1}z_j^{(3)}\tilde{\omega}_j,
\end{aligned}\tag{4.2}$$

where constants $\{z_j^{(1)}\}_{j=1}^{m-1}$, $\{z_j^{(2)}\}_{j=1}^{m-1}$ and $\{z_j^{(3)}\}_{j=1}^{m-1}$ are determined by the normalized conditions

$$\int_{a_k}\omega^{(3)}_{P_{0_3},P_{\infty_2}}=\int_{a_k}\omega^{(3)}_{P_{\infty_1},P_{0_1}}=\int_{a_k}\omega^{(3)}_{P_{0_2},P_{\infty_2}}=0,\quad k=1,2,\dots,m-1.$$

Proof. Recalling (4.1) that the Abelian differentials $\lambda\tilde{\omega}_{3q+1}$ is holomorphic on $\mathscr{K}_{m-1}\setminus\{P_{\infty_1},P_{\infty_2}\}$ and has simple poles at P_{∞_1} and P_{∞_2}, $\lambda^{-1}\tilde{\omega}_1$ is holomorphic on $\mathscr{K}_{m-1}\setminus\{P_{\infty_1},P_{\infty_2}\}$ and has simple poles at P_{0_1},P_{0_2} and P_{0_3}, $\lambda^{-1}\tilde{\omega}_{2q+2}$ is holomorphic on $\mathscr{K}_{m-1}\setminus\{P_{\infty_1},P_{\infty_2},P_{0_3}\}$ and has simple poles at P_{0_1} and P_{0_2}. Moreover, the Abelian differential $\lambda^{-1}\mathscr{F}^{-1}_{m,y}y^2d\lambda$ has simple poles at $P_{\infty_1},P_{0_1},P_{0_2}$. Taking the linear combination of these Abelian differentials and considering the definition of Abelian differentials of the third kind yield (4.2). ☐

It follows from Theorem 4.1 that

$$\omega^{(3)}_{P_{0_3},P_{\infty_2}}\underset{\zeta\to 0}{=}\begin{cases}\mathscr{O}(1)d\zeta & \text{as}\quad P\to P_{\infty_1},\\ \big(-\zeta^{-1}+\mathscr{O}(1)\big)d\zeta & \text{as}\quad P\to P_{\infty_2},\end{cases}$$

$$\omega^{(3)}_{P_{\infty_1},P_{0_1}}\underset{\zeta\to 0}{=}\begin{cases}\big(\zeta^{-1}+\mathscr{O}(1)\big)d\zeta & \text{as}\quad P\to P_{\infty_1},\\ \mathscr{O}(1)d\zeta & \text{as}\quad P\to P_{\infty_2},\end{cases}$$

$$\omega^{(3)}_{P_{0_2},P_{\infty_2}}\underset{\zeta\to 0}{=}\begin{cases}\mathscr{O}(1)d\zeta & \text{as}\quad P\to P_{\infty_1},\\ \big(-\zeta^{-1}+\mathscr{O}(1)\big)d\zeta & \text{as}\quad P\to P_{\infty_2},\end{cases}$$

$$\omega^{(3)}_{P_{0_3},P_{\infty_2}}\underset{\zeta\to 0}{=}\begin{cases}\mathscr{O}(1)d\zeta & \text{as}\quad P\to P_{0_1},\\ \mathscr{O}(1)d\zeta & \text{as}\quad P\to P_{0_2},\\ \big(\zeta^{-1}+\mathscr{O}(1)\big)d\zeta & \text{as}\quad P\to P_{0_3},\end{cases}$$

$$\omega^{(3)}_{P_{\infty_1},P_{0_1}} \underset{\zeta\to 0}{=} \begin{cases} \big(-\zeta^{-1}+\mathscr{O}(1)\big)d\zeta & \text{as} \quad P\to P_{0_1}, \\ \mathscr{O}(1)d\zeta & \text{as} \quad P\to P_{0_2}, \\ \mathscr{O}(1)d\zeta & \text{as} \quad P\to P_{0_3}, \end{cases}$$

$$\omega^{(3)}_{P_{0_2},P_{\infty_2}} \underset{\zeta\to 0}{=} \begin{cases} \mathscr{O}(1)d\zeta & \text{as} \quad P\to P_{0_1}, \\ \big(\zeta^{-1}+\mathscr{O}(1)\big)d\zeta & \text{as} \quad P\to P_{0_2}, \\ \mathscr{O}(1)d\zeta & \text{as} \quad P\to P_{0_3}. \end{cases}$$

Then we have

$$\begin{aligned} &\int_{Q_0}^{P} \omega^{(3)}_{P_{0_3},P_{\infty_2}} \underset{\zeta\to 0}{=} \begin{cases} \omega^{\infty_1}(P_{0_3},P_{\infty_2})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{\infty_1}, \\ -\ln\xi+\omega^{\infty_2}(P_{0_3},P_{\infty_2})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{\infty_2}, \end{cases} \\ &\int_{Q_0}^{P} \omega^{(3)}_{P_{\infty_1},P_{0_1}} \underset{\zeta\to 0}{=} \begin{cases} \ln\zeta+\omega^{\infty_1}(P_{\infty_1},P_{0_1})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{\infty_1}, \\ \omega^{\infty_2}(P_{\infty_1},P_{0_1})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{\infty_2}, \end{cases} \\ &\int_{Q_0}^{P} \omega^{(3)}_{P_{0_2},P_{\infty_2}} \underset{\zeta\to 0}{=} \begin{cases} \omega^{\infty_1}(P_{0_2},P_{\infty_2})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{\infty_1}, \\ -\ln\zeta+\omega^{\infty_2}(P_{0_2},P_{\infty_2})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{\infty_2}, \end{cases} \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\int_{Q_0}^{P} \omega^{(3)}_{P_{0_3},P_{\infty_2}} \underset{\zeta\to 0}{=} \begin{cases} \omega^{0_1}(P_{0_3},P_{\infty_2})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{0_1}, \\ \omega^{0_2}(P_{0_3},P_{\infty_2})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{0_2}, \\ \ln\zeta+\omega^{0_3}(P_{0_3},P_{\infty_2})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{0_3}, \end{cases} \\ &\int_{Q_0}^{P} \omega^{(3)}_{P_{\infty_1},P_{0_1}} \underset{\zeta\to 0}{=} \begin{cases} -\ln\zeta+\omega^{0_1}(P_{\infty_1},P_{0_1})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{0_1}, \\ \omega^{0_2}(P_{\infty_1},P_{0_1})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{0_2}, \\ \omega^{0_3}(P_{\infty_1},P_{0_1})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{0_3}, \end{cases} \\ &\int_{Q_0}^{P} \omega^{(3)}_{P_{0_2},P_{\infty_2}} \underset{\zeta\to 0}{=} \begin{cases} \omega^{0_1}(P_{0_2},P_{\infty_2})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{0_1}, \\ \ln\zeta+\omega^{0_2}(P_{0_2},P_{\infty_2})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{0_2}, \\ \omega^{0_3}(P_{0_2},P_{\infty_2})+\mathscr{O}(\zeta) & \text{as} \quad P\to P_{0_3}, \end{cases} \end{aligned} \tag{4.4}$$

where Q_0 is an appropriately chosen base point on $\mathscr{K}_{m-1}\backslash\{P_{\infty_1},P_{\infty_2},P_{0_1},P_{0_2},P_{0_3}\}$ and ω^{∞_l}, $l=1,2$, ω^{0_j}, $j=1,2,3$, are integration constants depending on the base points Q_0. The b-periods of $\omega^{(3)}_{P_{0_2},P_{\infty_2}}$ is denoted by

$$\underline{U}^{(3)}=\big(U^{(3)}_1,\ldots,U^{(3)}_{m-1}\big), \quad U^{(3)}_k=\frac{1}{2\pi \mathrm{i}}\int_{b_k}\omega^{(3)}_{P_{0_2},P_{\infty_2}}(P), \quad k=1,\ldots,m-1. \tag{4.5}$$

Let $\mathscr{T}_{m-1}$ be the period lattice $\{\underline{z}\in\mathbb{C}^{m-1} \mid \underline{z}=\underline{N}+\underline{M}\tau,\ \underline{N},\underline{M}\in\mathbb{Z}^{m-1}\}$. The complex torus $\mathscr{J}_{m-1}=\mathbb{C}^{m-1}/\mathscr{T}_{m-1}$ is called Jacobian variety of $\mathscr{K}_{m-1}$. The Abel map $\underline{\mathscr{A}}:\mathscr{K}_{m-1}\to\mathscr{J}_{m-1}$ is defined by

$$\underline{\mathscr{A}}(P)=\big(\mathscr{A}_1(P),\ldots,\mathscr{A}_{m-1}(P)\big)=\left(\int_{Q_0}^{P}\omega_1,\ldots,\int_{Q_0}^{P}\omega_{m-1}\right)(\mathrm{mod}\ \mathscr{T}_{m-1}).$$

A direct linear extension to the divisor group $\mathrm{Div}(\mathscr{K}_{m-1})$

$$\underline{\mathscr{A}}\left(\sum l_k P_k\right) = \sum l_k \underline{\mathscr{A}}(P_k).$$

Noting the nonspecial divisors

$$\mathscr{D}_{\underline{\hat{\mu}}(n)} = \sum_{k=1}^{m-1} \hat{\mu}_k(n), \quad \mathscr{D}_{\underline{\hat{\nu}}(n)} = \sum_{k=1}^{m-1} \hat{\nu}_k(n), \quad \mathscr{D}_{\underline{\hat{\xi}}(n)} = \sum_{k=1}^{m-1} \hat{\xi}_k(n),$$

we define Abel-Jacobi coordinates

$$\begin{aligned}
\underline{\rho}^{(1)}(n) &= \underline{\mathscr{A}}\left(\sum_{k=1}^{m-1} \hat{\mu}_k(n)\right) = \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\mu}(n)} \underline{\omega}, \\
\underline{\rho}^{(2)}(n) &= \underline{\mathscr{A}}\left(\sum_{k=1}^{m-1} \hat{\nu}_k(n)\right) = \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\nu}(n)} \underline{\omega}, \\
\underline{\rho}^{(3)}(n) &= \underline{\mathscr{A}}\left(\sum_{k=1}^{m-1} \hat{\xi}_k(n)\right) = \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\xi}(n)} \underline{\omega},
\end{aligned}$$

where $\underline{\rho}^{(1)}(n)$, $\underline{\rho}^{(2)}(n)$ and $\underline{\rho}^{(3)}(n)$ can be linearized in the following text.

Let $\theta(\underline{z})$ denote the Riemann theta function associated with $\mathscr{K}_{m-1}$, i.e.

$$\theta(\underline{z}) = \sum_{\underline{N} \in \mathbb{Z}^{m-1}} \exp\left\{2\pi \mathrm{i}\langle \underline{N}, \underline{z}\rangle + \pi \mathrm{i}\langle \underline{N}, \underline{N}\tau\rangle\right\},$$

where $\underline{z} = (z_1, \ldots, z_{m-1}) \in \mathbb{C}^{m-1}$, $\underline{N} = (N_1, \ldots, N_{m-1}) \in \mathbb{Z}^{m-1}$, the angle brackets denote the Euclidean inner product

$$\langle \underline{N}, \underline{z}\rangle = \sum_{j=1}^{m-1} N_j z_j, \quad \langle \underline{N}, \underline{N}\tau\rangle = \sum_{j,k=1}^{m-1} \tau_{jk} N_j N_k.$$

For the sake of brevity, we define the function $\underline{z} : \mathscr{K}_{m-1} \times \sigma^{m-1}\mathscr{K}_{m-1} \to \mathbb{C}^{m-1}$ by

$$\begin{aligned}
\underline{z}\big(P, \underline{\hat{\mu}}(n)\big) &= \underline{\Lambda} - \underline{\mathscr{A}}(P) + \underline{\rho}^{(1)}(n), \quad P \in \mathscr{K}_{m-1}, \\
\underline{z}\big(P, \underline{\hat{\nu}}(n)\big) &= \underline{\Lambda} - \underline{\mathscr{A}}(P) + \underline{\rho}^{(2)}(n), \quad P \in \mathscr{K}_{m-1}, \\
\underline{z}\big(P, \underline{\hat{\xi}}(n)\big) &= \underline{\Lambda} - \underline{\mathscr{A}}(P) + \underline{\rho}^{(3)}(n), \quad P \in \mathscr{K}_{m-1},
\end{aligned} \tag{4.6}$$

where $\underline{\hat{\mu}} = \{\hat{\mu}_1, \ldots, \hat{\mu}_{m-1}\} \in \sigma^{m-1}\mathscr{K}_{m-1}$, $\underline{\hat{\nu}} = \{\hat{\nu}_1, \ldots, \hat{\nu}_{m-1}\} \in \sigma^{m-1}\mathscr{K}_{m-1}$, and $\underline{\hat{\xi}} = \{\hat{\xi}_1, \ldots, \hat{\xi}_{m-1}\} \in \sigma^{m-1}\mathscr{K}_{m-1}$. Here $\sigma^{m-1}\mathscr{K}_{m-1}$ denotes the $(m-1)$-th symmetric power of $\mathscr{K}_{m-1}$, $\underline{\Lambda} = (\Lambda_1, \ldots, \Lambda_{m-1})$ is the vector of Riemann constants depending on the base point Q_0 by following expression:

$$\Lambda_j = \frac{1}{2}(1 + \tau_{jj}) - \sum_{\substack{i=1 \\ i \neq j}}^{m-1} \int_{a_i} \omega_i(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \ldots, m-1.$$

Lemma 4.1. *Assume that the curve K_{m-1} is nonsingular. Let $(n,n_0)\in\mathbb{Z}^2$, then*

$$\rho^{(1)}(n)=\rho^{(1)}(n_0)+(n-n_0)\big(\mathscr{A}(P_{\infty_2})-\mathscr{A}(P_{0_2})\big), \tag{4.7}$$
$$\rho^{(2)}(n)=\rho^{(2)}(n_0)+(n-n_0)\big(\mathscr{A}(P_{\infty_2})-\mathscr{A}(P_{0_2})\big), \tag{4.8}$$
$$\rho^{(3)}(n)=\rho^{(3)}(n_0)+(n-n_0)\big(\mathscr{A}(P_{\infty_2})-\mathscr{A}(P_{0_2})\big). \tag{4.9}$$

Proof. Applying Abel's theorem to (3.26) proves (4.7) and applying it to (3.20) and (3.21) results in

$$\rho^{(1)}(n)+\mathscr{A}(P_{\infty_2})=\rho^{(2)}(n)+\mathscr{A}(P_{0_3}), \tag{4.10}$$
$$\rho^{(1)}(n)+\mathscr{A}(P_{0_1})=\rho^{(3)}(n)+\mathscr{A}(P_{\infty_1}). \tag{4.11}$$

Substituting (4.7) into (4.10) and (4.11) yields (4.8) and (4.9), respectively. □

Using Lemma 4.1, (4.5) and (4.7) we can rewrite $\theta(\underline{z}(P_{0_j},\underline{\hat{\mu}}(n)))$, $\theta(\underline{z}(P_{0_j},\underline{\hat{\nu}}(n)))$, $\theta(\underline{z}(P_{0_j},\underline{\hat{\xi}}(n)))$, $\theta(\underline{z}(P_{\infty_l},\underline{\hat{\mu}}(n)))$, $\theta(\underline{z}(P_{\infty_l},\underline{\hat{\nu}}(n)))$ and $\theta(\underline{z}(P_{\infty_l},\underline{\hat{\xi}}(n)))$, respectively, as

$$\begin{aligned}
&\theta\big(\underline{z}(P_{0_j},\underline{\hat{\mu}}(n))\big)=\theta\left(\underline{\widetilde{M}}_j^{(1)}-\underline{U}^{(3)}n\right), \quad \theta\big(\underline{z}(P_{\infty_l},\underline{\hat{\mu}}(n))\big)=\theta\left(\underline{\widetilde{K}}_l^{(1)}-\underline{U}^{(3)}n\right),\\
&\theta\big(\underline{z}(P_{0_j},\underline{\hat{\nu}}(n))\big)=\theta\left(\underline{\widetilde{M}}_j^{(2)}-\underline{U}^{(3)}n\right), \quad \theta\big(\underline{z}(P_{\infty_l},\underline{\hat{\nu}}(n))\big)=\theta\left(\underline{\widetilde{K}}_l^{(2)}-\underline{U}^{(3)}n\right),\\
&\theta\big(\underline{z}(P_{0_j},\underline{\hat{\xi}}(n))\big)=\theta\left(\underline{\widetilde{M}}_j^{(3)}-\underline{U}^{(3)}n\right), \quad \theta\big(\underline{z}(P_{\infty_l},\underline{\hat{\xi}}(n))\big)=\theta\left(\underline{\widetilde{K}}_l^{(3)}-\underline{U}^{(3)}n\right),
\end{aligned}$$

where

$$\begin{aligned}
&\underline{\widetilde{M}}_j^{(s)}=\Lambda-\underline{\mathscr{A}}(P_{0_j})+\rho^{(s)}(n_0)+\underline{U}^{(3)}n_0, \quad j=1,2,3, \quad s=1,2,3,\\
&\underline{\widetilde{K}}_l^{(s)}=\Lambda-\underline{\mathscr{A}}(P_{\infty_l})+\rho^{(s)}(n_0)+\underline{U}^{(3)}n_0, \quad l=1,2, \qquad s=1,2,3.
\end{aligned}$$

Lemma 4.2. *Assume that $\mathscr{K}_{m-1}$ is nonsingular and irreducible. Let $P=(\lambda,y)\in\mathscr{K}_{m-1}\backslash\{P_{\infty_1},P_{\infty_2}\}$, $(n,n_0)\in\mathbb{Z}^2$. Suppose that $\mathscr{D}_{\underline{\hat{\mu}}(n)}$, $\mathscr{D}_{\underline{\hat{\nu}}(n)}$ and $\mathscr{D}_{\underline{\hat{\xi}}(n)}$ are nonspecial for $n\in\mathbb{Z}$. Then*

$$\begin{aligned}
\psi_3(P,n,n_0)=&\frac{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}(n)))}\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0)))}\\
&\times\exp\left((n-n_0)\left(\int_{Q_0}^{P}\omega^{(3)}_{P_{0_2},P_{\infty_2}}-\omega^{0_3}(P_{0_2},P_{\infty_2})\right)\right),
\end{aligned} \tag{4.12}$$

$$\phi_{13}(P,n)=N_1(n)\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_{0_3},P_{\infty_2}}\right), \tag{4.13}$$

$$\phi_{23}(P,n)=N_2(n)\frac{\theta(\underline{z}(P,\underline{\hat{\xi}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1},P_{0_1}}\right), \tag{4.14}$$

where

$$\begin{aligned}
N_1(n) &= -\frac{1}{v(n_0)} \frac{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\mu}}^+(n_0)))}{\theta(\underline{z}(P_{0_3}, \underline{\hat{\mu}}^+(n_0)))} \exp\big(-\omega^{0_2}(P_{0_3}, P_{\infty_2})\big) \\
&\quad \times \exp\big((n-n_0)\big(\omega^{0_3}\big(P_{0_2}, P_{\infty_2}\big) - \omega^{\infty_2}\big(P_{0_2}, P_{\infty_2}\big)\big)\big), \\
N_2(n) &= \frac{1}{v^-(n_0)} \frac{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{0_3}, \underline{\hat{\mu}}(n_0)))} \frac{\theta(\underline{z}(P_{0_3}, \underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\xi}}(n)))} \exp\big(-\omega^{\infty_2}(P_{\infty_1}, P_{0_1})\big) \\
&\quad \times \exp\big((n-n_0)\big(\omega^{0_3}(P_{0_2}, P_{\infty_2}) - \omega^{\infty_2}(P_{0_2}, P_{\infty_2})\big)\big).
\end{aligned} \tag{4.15}$$

Moreover, the stationary finite genus solutions $u(n), v(n), w(n)$ *have the form*

$$\begin{aligned}
u(n) &= \frac{1}{v(n_0)} \frac{\theta(\underline{z}(P_{\infty_1}, \underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P_{\infty_1}, \underline{\hat{\mu}}(n)))} \frac{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\mu}}^+(n_0)))}{\theta(\underline{z}(P_{0_3}, \underline{\hat{\mu}}^+(n_0)))} \\
&\quad \times \exp\big((n-n_0)\big(\omega^{0_3}(P_{0_2}, P_{\infty_2}) - \omega^{\infty_2}(P_{0_2}, P_{\infty_2})\big) \\
&\qquad + \omega^{\infty_1}(P_{0_3}, P_{\infty_2}) - \omega^{0_2}(P_{0_3}, P_{\infty_2})\big), \\
v^-(n) &= v^-(n_0) \frac{\theta(\underline{z}(P_{0_3}, \underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{0_3}, \underline{\hat{\mu}}(n)))} \frac{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\mu}}(n_0)))} \\
&\quad \times \exp\big((n-n_0)(\omega^{\infty_2}(P_{0_2}, P_{\infty_2}) - \omega^{0_3}(P_{0_2}, P_{\infty_2})\big), \\
w(n) &= \big(1 - u(n)v(n)\big) \frac{v^-(n_0)}{v(n)} \frac{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\nu}}(n_0)))}{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\mu}}(n_0)))} \frac{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\xi}}(n)))}{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\nu}}(n)))} \frac{\theta(\underline{z}(P_{0_2}, \underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{0_2}, \underline{\hat{\xi}}(n)))} \\
&\quad \times \exp\big((n-n_0)(\omega^{\infty_2}(P_{0_2}, P_{\infty_2}) - \omega^{0_3}(P_{0_2}, P_{\infty_2})) \\
&\qquad + \omega^{\infty_2}(P_{\infty_1}, P_{0_1}) - \omega^{0_2}(P_{\infty_1}, P_{0_1})\big).
\end{aligned} \tag{4.16}$$

Proof. Let $\Psi_3(P, n, n_0)$ be the right-hand side of (4.12). Our point of departure is to prove $\Psi_3(P, n, n_0) = \psi_3(P, n, n_0)$. It follows from (4.3) and (4.4) that

$$\begin{aligned}
&\exp\left(\int_{P_0}^{Q} \omega^{(3)}_{P_{0_2}, P_{\infty_2}} - \omega^{0_3}(P_{0_2}, P_{\infty_2})\right) \underset{\zeta \to 0}{=} \zeta^{-1} + \mathcal{O}(1) \quad \text{as} \quad P \to P_{\infty_2}, \\
&\exp\left(\int_{P_0}^{Q} \omega^{(3)}_{P_{0_2}, P_{\infty_2}} - \omega^{0_3}(P_{0_2}, P_{\infty_2})\right) \\
&\underset{\zeta \to 0}{=} \zeta \exp\big(\omega^{0_2}(P_{0_2}, P_{\infty_2}) - \omega^{0_3}(P_{0_2}, P_{\infty_2})\big) + \mathcal{O}(\zeta^2) \qquad \text{as} \quad P \to P_{0_2}, \\
&\exp\left(\int_{P_0}^{Q} \omega^{(3)}_{P_{0_2}, P_{\infty_2}} - \omega^{0_3}(P_{0_2}, P_{\infty_2})\right) \underset{\zeta \to 0}{=} \exp\big(\mathcal{O}(\zeta)\big) \quad \text{as} \quad P \to P_{0_3}.
\end{aligned} \tag{4.17}$$

Employing Riemann's vanishing theorem and the Riemann-Roch theorem, we deduce that Ψ_3 has the same simple poles and zeros as ψ_3, especially that the holomorphic function

$\Psi_3/\psi_3=\gamma$, where γ is a constant. Using (4.17) and Lemma 3.2, one computes

$$\frac{\Psi_3}{\psi_3}=\frac{(1+\mathcal{O}(\zeta))\exp(\mathcal{O}(\zeta))}{1+\mathcal{O}(\zeta)}=1+\mathcal{O}(\zeta)\quad\text{as}\quad P\to P_{0_3},$$

from which one concludes $\gamma=1$. This proves (4.12).

By (3.20), (3.21) and the Riemann-Roch theorem, $\phi_{13}(P,n)$ and $\phi_{23}(P,n)$ are of the type in (4.13) and (4.14), where $N_1(n)$ and $N_2(n)$ are independent of $P\in\mathcal{K}_{m-1}$. A comparison of (4.12) and the asymptotic relations (3.25) near P_{∞_2}, then yields the following expression for $v(n)$ in (4.16):

$$\begin{aligned}\frac{v^-(n)}{v^-(n_0)}=&\frac{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}(n_0)))}{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}(n)))}\frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(n_0)))}\\&\times\exp\big((n-n_0)\big(\omega^{\infty_2}(P_{0_2},P_{\infty_2})-\omega^{0_3}(P_{0_2},P_{\infty_2})\big)\big).\end{aligned}$$

Comparing the asymptotic expressions of ϕ_{13} for $P\to P_{0_2}$ and $P\to P_{\infty_1}$ in (3.24), (3.25) and (4.13), one obtains

$$\begin{aligned}-\frac{1}{v(n)}&=N_1(n)\frac{\theta(\underline{z}(P_{0_2},\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P_{0_2},\underline{\hat{\mu}}(n)))}\exp\big(\omega^{0_2}(P_{0_3},P_{\infty_2})\big)\qquad\text{as}\quad P\to P_{0_2},\\-u(n)&=N_1(n)\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\nu}}(n)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(n)))}\exp\big(\omega^{\infty_1}(P_{0_3},P_{\infty_2})\big)\quad\text{as}\quad P\to P_{\infty_1},\end{aligned}\tag{4.18}$$

which gives the representation of $N_1(n)$,

$$\begin{aligned}N_1(n)=&-\frac{1}{v(n_0)}\frac{\theta(\underline{z}(P_{0_2},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{0_2},\underline{\hat{\nu}}(n)))}\frac{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}^+(n_0)))}\frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}^+(n_0)))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}^+(n)))}\\&\times\exp\big(-\omega^{0_2}(P_{0_3},P_{\infty_2})+(n-n_0)\big(\omega^{0_3}\,(P_{0_2},P_{\infty_2})-\omega^{\infty_2}(P_{0_2},P_{\infty_2})\big)\big).\end{aligned}$$

Moreover, we obtain from (4.7)-(4.11) and (4.6) that

$$\begin{aligned}&\theta\big(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}^+(n))\big)=\theta\big(\underline{z}(P_{0_2},\underline{\hat{\mu}}(n))\big),\\&\theta\big(\underline{z}(P_{0_3},\underline{\hat{\mu}}^+(n))\big)=\theta\big(\underline{z}(P_{\infty_2},\underline{\hat{\nu}}^+(n))\big)=\theta\big(\underline{z}(P_{0_2},\underline{\hat{\nu}}(n))\big).\end{aligned}$$

The representation of $N_1(n)$ can be simplified as shown in (4.15) by using the above relations. Consequently, the Riemann theta function representation of $u(n)$ in (4.16) is given by (4.18). Meanwhile, comparing the asymptotic expressions of ϕ_{23} for $P\to P_{\infty_2}$ and $P\to P_{0_2}$ in (3.18), (3.19) and (4.14), one obtains

$$\begin{aligned}\frac{1}{v^-(n)}&=N_2(n)\frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\xi}}(n)))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\nu}}(n)))}\exp\big(\omega^{\infty_2}(P_{\infty_1},P_{0_1})\big),\\\frac{1-u(n)v(n)}{v(n)w(n)}&=N_2(n)\frac{\theta(\underline{z}(P_{0_2},\underline{\hat{\xi}}(n)))}{\theta(\underline{z}(P_{0_2},\underline{\hat{\nu}}(n)))}\exp\big(\omega^{0_2}(P_{\infty_1},P_{0_1})\big),\end{aligned}$$

from which we obtain $N_2(n)$ and $w(n)$. This completes the proof. □

5. Finite Genus Solutions of Discrete Integrable Hierarchy

In this section, we extend the stationary Baker-Akhiezer function to the time-dependent case, from which all those results obtained in Sections 3 and 4 are generalized to the time-dependent cases. In particular, we obtain Riemann theta function representations for the time-dependent Baker-Akhiezer function, the time-dependent meromorphic functions, and the general time-dependent finite genus solutions of the discrete integrable hierarchy.

Similar to Eqs. (3.3), we introduce the time-dependent Baker-Akhiezer function

$$\begin{aligned}
&E\psi(P,n,n_0,t_r,t_{0,r}) = U\big(u(n,t_r),v(n,t_r),w(n,t_r);\lambda(P)\big)\psi(P,n,n_0,t_r,t_{0,r}),\\
&\psi_{t_r}(P,n,n_0,t_r,t_{0,r}) = \widetilde{V}^{(r)}\big(u(n,t_r),v(n,t_r),w(n,t_r);\lambda(P)\big)\psi(P,n,n_0,t_r,t_{0,r}),\\
&V^{(q)}\big(u(n,t_r),v(n,t_r),w(n,t_r);\lambda(P)\big)\psi(P,n,n_0,t_r,t_{0,r}) = y(P)\psi(P,n,n_0,t_r,t_{0,r}),\\
&\psi_3(P,n_0,n_0,t_{0,r},t_{0,r}) = 1, \quad P\in\mathscr{K}_{m-1}\backslash\{P_{\infty_1},P_{\infty_2}\}, \quad (n,t_r)\in\mathbb{Z}\times\mathbb{R}.
\end{aligned} \tag{5.1}$$

The compatibility conditions of the first three members in (5.1) yield

$$U_{t_r} - (E\widetilde{V}^{(r)})U + U\widetilde{V}^{(r)} = 0, \tag{5.2}$$

$$(EV^{(q)})U - UV^{(q)} = 0, \tag{5.3}$$

$$V^{(q)}_{t_r} - [\widetilde{V}^{(r)}, V^{(q)}] = 0. \tag{5.4}$$

A direct calculation shows that $yI - V^{(q)}$ satisfies (5.3) and (5.4), which implies that $\det(yI - V^{(q)})$ is independent of n and t_r. Define the associated time-dependent meromorphic functions

$$\begin{aligned}
\phi_{13}(P,n,t_r) &= \frac{\psi_1(P,n,n_0,t_r,t_{0,r})}{\psi_3(P,n,n_0,t_r,t_{0,r})}, \quad P\in\mathscr{K}_{m-1},\\
\phi_{23}(P,n,t_r) &= \frac{\psi_2(P,n,n_0,t_r,t_{0,r})}{\psi_3(P,n,n_0,t_r,t_{0,r})}, \quad P\in\mathscr{K}_{m-1},
\end{aligned} \tag{5.5}$$

which implies by (5.1) that

$$\begin{aligned}
\phi_{13}(P,n,t_r) &= \frac{yV^{(q)}_{12}(\lambda,n,t_r) + C_m(\lambda,n,t_r)}{yV^{(q)}_{32}(\lambda,n,t_r) + A_m(\lambda,n,t_r)}\\
&= \frac{\lambda F_{m-1}(\lambda,n,t_r)}{y^2V^{(q)}_{12}(\lambda,n,t_r) - y[C_m(\lambda,n,t_r) + V^{(q)}_{12}(\lambda,n,t_r)R_m(\lambda)] + D_m(\lambda,n,t_r)}\\
&= \frac{y^2V^{(q)}_{32}(\lambda,n,t_r) - y[A_m(\lambda,n,t_r) + V^{(q)}_{32}(\lambda,n,t_r)R_m(\lambda)] + B_m(\lambda,n,t_r)}{E_{m-1}(\lambda,n,t_r)},
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
\phi_{23}(P,n,t_r) &= \frac{yV^{(q)}_{21}(\lambda,n,t_r) + \mathscr{C}_m(\lambda,n,t_r)}{yV^{(q)}_{31}(\lambda,n,t_r) + \mathscr{A}_m(\lambda,n,t_r)}\\
&= \frac{H_{m-1}(\lambda,n,t_r)}{y^2V^{(q)}_{21}(\lambda,n,t_r) - y[\mathscr{C}_m(\lambda,n,t_r) + V^{(q)}_{21}(\lambda,n,t_r)R_m(\lambda)] + \mathscr{D}_m(\lambda,n,t_r)}
\end{aligned} \tag{5.7}$$

$$= \frac{y^2 V_{31}^{(q)}(\lambda,n,t_r) - y[\mathscr{A}_m(\lambda,n,t_r) + V_{31}^{(q)}(\lambda,n,t_r) R_m(\lambda)] + \mathscr{B}_m(\lambda,n,t_r)}{-\lambda E_{m-1}(\lambda,n,t_r)},$$

where the polynomials $A_m(\lambda,n,t_r), B_m(\lambda,n,t_r), C_m(\lambda,n,t_r), D_m(\lambda,n,t_r), \mathscr{A}_m(\lambda,n,t_r), \mathscr{B}_m(\lambda,n,t_r), \mathscr{C}_m(\lambda,n,t_r), \mathscr{D}_m(\lambda,n,t_r), E_{m-1}(\lambda,n,t_r), F_{m-1}(\lambda,n,t_r)$ and $H_{m-1}(\lambda,n,t_r)$ are defined as (3.9)-(3.11). Therefore, in the present context the relations (3.12), (3.13) also hold. Similarly, we have

$$\begin{aligned} &E_{m-1}(\lambda,n,t_r) = \alpha_0\beta_0 v^-(n,t_r)\prod_{j=1}^{m-1}\big(\lambda-\mu_j(n,t_r)\big),\\ &F_{m-1}(\lambda,n,t_r) = -\alpha_0^2\beta_0\big(u(n,t_r)\big)^2\prod_{j=1}^{m-1}\big(\lambda-\nu_j(n,t_r)\big),\\ &H_{m-1}(\lambda,n,t_r) = -\alpha_0^2\beta_0 u^+(n,t_r)\big(1-u(n,t_r)v(n,t_r)\big)\prod_{j=1}^{m-1}\big(\lambda-\xi_j(n,t_r)\big), \end{aligned} \tag{5.8}$$

where $\{\mu_j(n,t_r)\}_{j=1}^{m-1}$, and $\{\nu_j(n,t_r)\}_{j=1}^{m-1}$, $\{\zeta_j(n,t_r)\}_{j=1}^{m-1}$ are respectively the zeros of the polynomials $E_{m-1}(\lambda,n,t_r)$, $F_{m-1}(\lambda,n,t_r)$, and $H_{m-1}(\lambda,n,t_r)$. Since

$$\begin{aligned} E_{m-1}(\lambda,n,t_r)|_{\lambda=\mu_j(n,t_r)} &= \left(\lambda V_{21}^{(q)}\left(V_{32}^{(q)}\right)^2 + V_{31}^{(q)}V_{32}^{(q)}\left(V_{11}^{(q)}-V_{22}^{(q)}\right) - V_{12}^{(q)}\left(V_{31}^{(q)}\right)^2\right)\Big|_{\lambda=\mu_j(n,t_r)}\\ &= \left(V_{32}^{(q)}\mathscr{A}_m - V_{31}^{(q)}A_m\right)\Big|_{\lambda=\mu_j(n,t_r)} = 0, \end{aligned}$$

we have

$$\frac{\mathscr{A}_m(\mu_j(n,t_r),n,t_r)}{V_{31}^{(q)}(\mu_j(n,t_r),n,t_r)} = \frac{A_m(\mu_j(n,t_r),n,t_r)}{V_{32}^{(q)}(\mu_j(n,t_r),n,t_r)}.$$

Define

$$\begin{aligned} \hat{\mu}_j(n,t_r) &= \big(\mu_j(n,t_r), y(\hat{\mu}_j(n,t_r))\big)\\ &= \left(\mu_j(n,t_r), -\frac{A_m(\mu_j(n,t_r),n,t_r)}{V_{32}^{(q)}(\mu_j(n,t_r),n,t_r)}\right)\\ &= \left(\mu_j(n,t_r), -\frac{\mathscr{A}_m(\mu_j(n,t_r),n,t_r)}{V_{31}^{(q)}(\mu_j(n,t_r),n,t_r)}\right) \in \mathscr{K}_{m-1}, \quad j=1,\dots,m-1,\\ \hat{\nu}_j(n,t_r) &= \big(\nu_j(n,t_r), y(\hat{\nu}_j(n,t_r))\big)\\ &= \left(\nu_j(n,t_r), -\frac{C_m(\nu_j(n,t_r),n,t_r)}{V_{12}^{(q)}(\nu_j(n,t_r),n,t_r)}\right) \in \mathscr{K}_{m-1}, \quad j=1,\dots,m-1,\\ \hat{\xi}_j(n,t_r) &= \big(\xi_j(n,t_r), y(\hat{\xi}_j(n,t_r))\big)\\ &= \left(\xi_j(n,t_r), -\frac{\mathscr{C}_m(\xi_j(n,t_r),n,t_r)}{V_{21}^{(q)}(\xi_j(n,t_r),n,t_r)}\right) \in \mathscr{K}_{m-1}, \quad j=1,\dots,m-1. \end{aligned} \tag{5.9}$$

Now we can infer the divisors of $\phi_{13}(P,n,t_r)$ and $\phi_{23}(P,n,t_r)$ are given by

$$\big(\phi_{13}(P,n,t_r)\big)=\mathscr{D}_{P_{0_3},\hat{\nu}_1(n,t_r),\ldots,\hat{\nu}_{m-1}(n,t_r)}(P)-\mathscr{D}_{P_{\infty_2},\hat{\mu}_1(n,t_r),\ldots,\hat{\mu}_{m-1}(n,t_r)}(P), \tag{5.10}$$

$$\big(\phi_{23}(P,n,t_r)\big)=\mathscr{D}_{P_{\infty_1},\hat{\xi}_1(n,t_r),\ldots,\hat{\xi}_{m-1}(n,t_r)}(P)-\mathscr{D}_{P_{0_1},\hat{\mu}_1(n,t_r),\ldots,\hat{\mu}_{m-1}(n,t_r)}(P). \tag{5.11}$$

Similarly, $\phi_{13}(P,n,t_r)$ and $\phi_{23}(P,n,t_r)$ satisfy the Riccati-type equations

$$\begin{aligned}&\phi_{13}^{+}(P,n,t_r)\big(1+v(n,t_r)\phi_{13}(P,n,t_r)\big)=\lambda\phi_{23}(P,n,t_r),\\ &\phi_{23}^{+}(P,n,t_r)\big(1+v(n,t_r)\phi_{13}(P,n,t_r)\big)=\phi_{13}(P,n,t_r)+w(n,t_r)\phi_{23}(P,n,t_r)+u(n,t_r).\end{aligned}$$

The dynamics of the zeros μ_j of E_{m-1} concerning the variable t_r can be described by a first-order system of nonlinear differential equations in terms of Dubrovin-type equations.

Lemma 5.1. *Let the zeros $\{\mu_j(n,t_r)\}_{j=1,\ldots,m-1}$ of E_{m-1} remain distinct for $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$. Then $\{\mu_j(n,t_r)\}_{j=1,\ldots,m-1}$ satisfy the system of equations,*

$$\begin{aligned}\mu_{j,t_r}(n,t_r)=&\Big(\widetilde{V}_{32}^{(r)}(\mu_j(n,t_r),n,t_r)V_{31}^{(q)}(\mu_j(n,t_r),n,t_r)\\ &-\widetilde{V}_{31}^{(r)}(\mu_j(n,t_r),n,t_r)V_{32}^{(q)}(\mu_j(n,t_r),n,t_r)\Big)\\ &\times\frac{3y^2(\hat{\mu}_j(n,t_r))-2y(\hat{\mu}_j(n,t_r))R_m(\mu_j(n,t_r))+S_m(\mu_j(n,t_r))}{\alpha_0\beta_0 v^{-}(n,t_r)\prod_{k=1,k\neq j}^{m-1}(\mu_j(n,t_r)-\mu_k(n,t_r))},\quad 1\le j\le m-1.\end{aligned}\tag{5.12}$$

Proof. The relations (5.4) and (3.9)-(3.11) yield

$$\begin{aligned}E_{m-1,t_r}(n,t_r)=&\left(\lambda V_{21}^{(q)}\left(V_{32}^{(q)}\right)^2+V_{31}^{(q)}V_{32}^{(q)}\left(V_{11}^{(q)}-V_{22}^{(q)}\right)-V_{12}^{(q)}\left(V_{31}^{(q)}\right)^2\right)_{t_r}\\ =&\left(3\lambda\widetilde{V}_{33}^{(r)}-\widetilde{R}^{(r)}\right)\left(\lambda V_{21}^{(q)}\left(V_{32}^{(q)}\right)^2+V_{31}^{(q)}V_{32}^{(q)}\left(V_{11}^{(q)}-V_{22}^{(q)}\right)-V_{12}^{(q)}\left(V_{31}^{(q)}\right)^2\right)\\ &-\widetilde{V}_{31}^{(r)}\left(2V_{32}^{(q)}S_m-3B_m+A_mR_m\right)+\widetilde{V}_{32}^{(r)}\left(2V_{31}^{(q)}S_m-3\mathscr{B}_m+\mathscr{A}_mR_m\right),\end{aligned}$$

where $\widetilde{R}^{(r)}=\widetilde{V}_{11}^{(r)}+\widetilde{V}_{22}^{(r)}+\lambda\widetilde{V}_{33}^{(r)}$. Noticing (5.9), we have

$$\left.\frac{A_m}{V_{32}^{(q)}}\right|_{\lambda=\mu_j(n,t_r)}=\left.\frac{\mathscr{A}_m}{V_{31}^{(q)}}\right|_{\lambda=\mu_j(n,t_r)}=-y\left(\hat{\mu}_j(n,t_r)\right).$$

Recalling (3.12)-(3.15), we get

$$\begin{aligned}&\left.\left(2V_{32}^{(q)}S_m-3B_m+A_mR_m\right)\right|_{\lambda=\mu_j(n,t_r)}=V_{32}^{(q)}\left(-3y^2\left(\hat{\mu}_j(n,t_r)\right)+2R_my\left(\hat{\mu}_j(n,t_r)\right)-S_m\right),\\ &\left.\left(2V_{31}^{(q)}S_m-3\mathscr{B}_m+\mathscr{A}_mR_m\right)\right|_{\lambda=\mu_j(n,t_r)}=V_{31}^{(q)}\left(-3y^2\left(\hat{\mu}_j(n,t_r)\right)+2R_my\left(\hat{\mu}_j(n,t_r)\right)-S_m\right),\end{aligned}$$

hence

$$E_{m-1,t_r}(n,t_r)\big|_{\lambda=\mu_j(n,t_r)}=\left(\widetilde{V}_{31}^{(r)}V_{32}^{(q)}-\widetilde{V}_{32}^{(r)}V_{31}^{(q)}\right)$$

$$\times\left(3y^2\left(\hat{\mu}_j(n,t_r)\right)-2R_m y\left(\hat{\mu}_j(n,t_r)\right)+S_m\right). \tag{5.13}$$

Moreover, the derivative of $E_{m-1}(\lambda,n,t_r)$ in (5.8) with respect to t_r evaluated at $\lambda=\mu_j$ is

$$E_{m-1,t_r}(\lambda,n,t_r)\big|_{\lambda=\mu_j(n,t_r)}=-\alpha_0\beta_0 v^-(n,t_r)\mu_{j,t_r}(n,t_r)\prod_{\substack{k=1\\k\neq j}}^{m-1}\left(\mu_j(n,t_r)-\mu_k(n,t_r)\right),$$

which together with (5.13) gives rise to (5.12). □

Next, let us turn to the analytic properties of the Baker-Akhiezer function ψ_3. From the first two expressions of (5.1), we have

$$\begin{aligned}
&\psi_3(P,n,n_0,t_r,t_{0,r})\\
&=\exp\left(\int_{t_{0,r}}^{t_r}\left(\widetilde{V}_{31}(\lambda,n_0,t')\phi_{13}(P,n_0,t')+\lambda\widetilde{V}_{32}(\lambda,n_0,t')\phi_{23}(P,n_0,t')+\lambda\widetilde{V}_{33}(\lambda,n_0,t')\right)dt'\right)\\
&\quad\times\begin{cases}\prod_{n'=n_0}^{n-1}\left(v(n',t_r)\phi_2(P,n',t_r)+1\right) & \text{as}\quad n\geq n_0+1,\\ 1 & \text{as}\quad n=n_0,\\ \prod_{n'=n}^{n_0-1}\left(v(n',t_r)\phi_2(P,n',t_r)+1\right)^{-1} & \text{as}\quad n\leq n_0-1.\end{cases}
\end{aligned} \tag{5.14}$$

By inspection, we verify that

$$\psi_3(P,n,n_0,t_r,t_{0,r})=\psi_3(P,n_0,n_0,t_r,t_{0,r})\psi_3(P,n,n_0,t_r,t_r). \tag{5.15}$$

Considering the integrand in (5.14), let us define

$$\begin{aligned}
I_r(P,n,t_r)&=\widetilde{V}_{31}^{(r)}(\lambda,n,t_r)\phi_{13}(P,n,t_r)+\lambda\widetilde{V}_{32}^{(r)}(\lambda,n,t_r)\phi_{23}(P,n,t_r)+\lambda\widetilde{V}_{33}^{(r)}(\lambda,n,t_r)\\
&=\left(v^-\tilde{a}^{(r)-}+\tilde{c}^{(r)-}-w^-\tilde{c}^{(r)}\right)\phi_{13}+\lambda\tilde{c}^{(r)}\phi_{23}+\lambda\tilde{e}^{(r)}.
\end{aligned} \tag{5.16}$$

We have used (2.10) to obtain the second equation in (5.16). The homogeneous case of $\hat{I}_r(P,n,t_r)$ is defined by

$$\hat{I}_r(P,n,t_r)=\left(v^-\hat{a}^{(r)-}+\hat{c}^{(r)-}-w^-\hat{c}^{(r)}\right)\phi_{13}+\lambda\hat{c}^{(r)}\phi_{23}+\lambda\hat{e}^{(r)},$$

where

$$\begin{aligned}
\hat{a}^{(r)}&=\tilde{a}^{(r)}|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=0},\\
\hat{c}^{(r)}&=\tilde{c}^{(r)}|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=0},\\
\hat{e}^{(r)}&=\tilde{e}^{(r)}|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=0}.
\end{aligned}$$

It can be verified that

$$I_r(P,n,t_r)=\sum_{l=0}^{r}\tilde{\alpha}_{r-l}\hat{I}_l(P,n,t_r). \tag{5.17}$$

Lemma 5.2. *Suppose that* $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$. *Then*

$$\hat{I}_r(P,n,t_r) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{-r-1}+\mathcal{O}(1) & \text{as} \quad P\to P_{\infty_1}, \\ -\hat{e}_{r+1}-\dfrac{\hat{c}_{r+1}}{v^-}+\mathcal{O}(\zeta) & \text{as} \quad P\to P_{\infty_2}, \end{cases} \tag{5.18}$$

$$\hat{I}_r(P,n,t_r) \underset{\zeta\to 0}{=} \begin{cases} \mathcal{O}(1) & \text{as} \quad P\to P_{0_1}, \\ \mathcal{O}(1) & \text{as} \quad P\to P_{0_2}, \\ \mathcal{O}(\zeta) & \text{as} \quad P\to P_{0_3}. \end{cases} \tag{5.19}$$

Proof. We use the inductive method to prove (5.18). For $r=0$,

$$\begin{aligned} \hat{I}_0 &\underset{\zeta\to 0}{=} \left(v^-\hat{a}_0^- + \hat{c}_0^- - w^-\hat{c}_0\right)\phi_{13} + \zeta^{-1}\hat{c}_0\phi_{23} + \zeta^{-1}\hat{e}_0 \\ &\underset{\zeta\to 0}{=} -\zeta^{-1} - \hat{e}_1 + \mathcal{O}(\zeta) \quad \text{as} \quad P\to P_{\infty_1}, \\ \hat{I}_0 &\underset{\zeta\to 0}{=} \left(v^-\hat{a}_0^- + \hat{c}_0^- - w^-\hat{c}_0\right)\phi_{13} + \xi^{-2}\hat{c}_0\phi_{23} + \xi^{-2}\hat{e}_0 \\ &\underset{\zeta\to 0}{=} -\frac{\hat{c}_1}{v^-} - \hat{e}_1 + \mathcal{O}(\zeta) \quad \text{as} \quad P\to P_{\infty_2}. \end{aligned}$$

Suppose that

$$\begin{aligned} \hat{I}_r(P,n,t_r) &= -\zeta^{-r-1} + \sum_{j=0}^{\infty}\sigma_j(n,t_r)\zeta^j \quad \text{as} \quad P\to P_{\infty_1}, \\ \hat{I}_r(P,n,t_r) &= \sum_{j=0}^{\infty}\delta_j(n,t_r)\zeta^j \qquad\qquad \text{as} \quad P\to P_{\infty_2} \end{aligned} \tag{5.20}$$

for some coefficients $\{\sigma_j(n,t_r)\}_{j\in\mathbb{N}}, \{\delta_j(n,t_r)\}_{j\in\mathbb{N}}$ to be determined. Differentiating the first equation in (5.5) with respect to t_r and using (5.1) yields

$$\begin{aligned} (v\phi_{13}+1)_{t_r} &= \left(\frac{\psi_3^+}{\psi_3}\right)_{t_r} = \frac{\psi_{3,t_r}^+\psi_3 - \psi_3^+\psi_{3,t_r}}{\psi_3^2} = \frac{\psi_3^+}{\psi_3}\left(\frac{\psi_{3,t_r}^+}{\psi_3^+} - \frac{\psi_{3,t_r}}{\psi_3}\right) \\ &= (v\phi_{13}+1)\Delta\frac{\psi_{3,t_r}}{\psi_3} = (v\phi_{13}+1)\Delta I_r. \end{aligned} \tag{5.21}$$

In particular, we have

$$(v\phi_{13}+1)_{t_r}|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=0} = (v\phi_{13}+1)\Delta\hat{I}_r. \tag{5.22}$$

Inserting (3.22) and (5.20) into (5.22) as $P\to P_{\infty_i}, i=1,2$. Comparing the same powers of ζ in (5.21), we can obtain

$$\begin{aligned} (v\kappa_{1,j})_{t_r}|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=0} &= v\sum_{l=0}^{j}\kappa_{1,l}\Delta\sigma_{j-l} + \Delta\sigma_j, \quad j\geq 0, \\ (v\kappa_{2,-1})_{t_r}|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=0} &= v\kappa_{2,-1}\Delta\delta_0, \\ (v\kappa_{2,j})_{t_r}|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=0} &= v\sum_{l=-1}^{j}\kappa_{2,l}\Delta\delta_{j-l} + \Delta\delta_j, \quad j\geq 0. \end{aligned}$$

Using (5.2) and Lemma 3.1, we have

$$\begin{aligned}
&\Delta\sigma_0 = -\Delta\hat{e}_{r+1},\\
&\Delta\delta_0 = -\Delta\left(\hat{e}_{r+1}+\frac{\hat{c}_{r+1}}{v^-}\right),\\
&\Delta\delta_1 = -\Delta\hat{a}^-_{r+1}+\Delta\frac{v^{--}\hat{c}_{r+1}-v^-\hat{c}^-_{r+1}-u^-v^{--}v^-\hat{c}_{r+1}}{(v^-)^2},\\
&(E^2-1)\delta_2 = -(E^2-1)\left[\left(v^-\hat{a}^-_{r+1}-w^-\hat{c}_{r+1}+\hat{c}^-_{r+1}\right)\kappa_{2,0}+\hat{c}_{r+1}\chi_{2,2}+\hat{e}_{r+2}+\frac{\hat{c}_{r+2}}{v^-}\right].
\end{aligned}$$

Note that there are no arbitrary constants in the expansions of ϕ_{13} near P_{∞_i}, $i=1,2$, nor in the homogeneous coefficients $\hat{a}_j,\hat{b}_j,\hat{d}_j$ with the condition $\Delta\Delta^{-1}=\Delta^{-1}\Delta=1$, hence

$$\begin{aligned}
&\sigma_0 = -\hat{e}_{r+1},\\
&\delta_0 = -\hat{e}_{r+1}-\frac{\hat{c}_{r+1}}{v^-},\\
&\delta_1 = -\hat{a}^-_{r+1}+\frac{v^{--}\hat{c}_{r+1}-v^-\hat{c}^-_{r+1}-u^-v^{--}v^-\hat{c}_{r+1}}{(v^-)^2},\\
&\delta_2 = -(v^-\hat{a}^-_{r+1}-w^-\hat{c}_{r+1}+\hat{c}^-_{r+1})\kappa_{2,0}-\hat{c}_{r+1}\chi_{2,2}-\hat{e}_{r+2}-\frac{\hat{c}_{r+2}}{v^-}.
\end{aligned}$$

Then

$$\begin{aligned}
\hat{I}_{r+1}(P,n,t_r) &= \left(v^-(\hat{a}^{(r+1)})^- - w^-\hat{c}^{(r+1)}+(\hat{c}^{(r+1)})^-\right)\phi_{13}+\lambda\hat{c}^{(r+1)}\phi_{23}+\lambda\hat{e}^{(r+1)}\\
&\underset{\zeta\to 0}{=} \zeta^{-1}\hat{I}_r+\left(v^-\hat{a}^-_{r+1}+\hat{c}^-_{r+1}-w^-\hat{c}_{r+1}\right)\phi_{13}+\zeta^{-1}\hat{c}_{r+1}\phi_{23}+\zeta^{-1}\hat{e}_{r+1}\\
&\underset{\zeta\to 0}{=} -\zeta^{-r-1}+(\sigma_0+\hat{e}_{r+1})\zeta^{-1}+\mathcal{O}(1)\\
&\underset{\zeta\to 0}{=} \mathcal{O}(1) \quad \text{as} \quad P\to P_{\infty_1},
\end{aligned}$$

$$\begin{aligned}
\hat{I}_{r+1} &\underset{\zeta\to 0}{=} \zeta^{-2}\hat{I}_r+\left(v^-\hat{a}^-_{r+1}-w^-\hat{c}_{r+1}+\hat{c}^-_{r+1}\right)\phi_{13}+\zeta^{-2}\hat{c}_{r+1}\phi_{23}+\zeta^{-2}\hat{e}_{r+1}\\
&\underset{\zeta\to 0}{=} \zeta^{-2}\left(\delta_0+\frac{\hat{c}_{r+1}}{v^-}+\hat{e}_{r+1}\right)+\zeta^{-1}\left(\delta_1+\hat{a}^-_{r+1}+\frac{\hat{c}^-_{r+1}-w^-\hat{c}_{r+1}}{v^-}+\hat{c}_{r+1}\chi_{2,1}\right)\\
&\qquad +\delta_2+\left(v^-\hat{a}^-_{r+1}-w^-\hat{c}_{r+1}+\hat{c}^-_{r+1}\right)\kappa_{2,0}+\hat{c}_{r+1}\chi_{2,2}+\mathcal{O}(\zeta)\\
&\underset{\zeta\to 0}{=} -\hat{e}_{r+2}-\frac{\hat{c}_{r+2}}{v^-}+\mathcal{O}(\zeta) \quad \text{as} \quad P\to P_{\infty_2}.
\end{aligned}$$

In summary, we complete the proof of (5.18). Similarly, we can prove (5.19). □

Using Lemma 5.2 and (5.17), we arrive at

$$I_r(P,n,t_r) \underset{\zeta\to 0}{=} \begin{cases} -\sum_{l=0}^{r}\tilde{\alpha}_{r-l}\zeta^{-l-1}+\mathcal{O}(1) & \text{as} \quad P\to P_{\infty_1},\\ \dfrac{v^-_{t_r}(n,t_r)}{v^-(n,t_r)}+\mathcal{O}(\zeta) & \text{as} \quad P\to P_{\infty_2}, \end{cases} \tag{5.23}$$

$$I_r(P,n,t_r) \underset{\zeta\to 0}{=} \begin{cases} \mathcal{O}(1) & \text{as} \quad P\to P_{0_1}, \\ \mathcal{O}(1) & \text{as} \quad P\to P_{0_2}, \\ \mathcal{O}(\zeta) & \text{as} \quad P\to P_{0_3}. \end{cases} \tag{5.24}$$

Let $\omega^{(2)}_{P_{\infty_1},j}$ be the normalized differential holomorphic of second kind on $\mathcal{K}_{m-1}\backslash\{P_{\infty_1}\}$ with a pole of order $j\geq 2$ at P_{∞_1},

$$\omega^{(2)}_{P_{\infty_1},j} \underset{\zeta\to 0}{=} \big(\zeta^{-j}+\mathcal{O}(1)\big)d\zeta \quad \text{as} \quad P\to P_{\infty_1}, \quad \lambda=\zeta^{-1}$$

with vanishing a-periods

$$\int_{a_k}\omega^{(2)}_{P_{\infty_1},j}=0, \quad k=1,\dots,m-1.$$

From (5.23), we introduce the Abelian differential

$$\tilde{\Omega}^{(2)}_r(P)=\sum_{l=0}^{r}\tilde{\alpha}_{r-l}(l+1)\omega^{(2)}_{P_{\infty_1},l+2}(P). \tag{5.25}$$

Integrating (5.25) yields

$$\begin{aligned} &\int_{Q_0}^{P}\tilde{\Omega}^{(2)}_r(P) \underset{\zeta\to 0}{=} \begin{cases} -\sum_{l=0}^{r}\tilde{\alpha}_{r-l}\zeta^{-l-1}+\tilde{\Omega}^{\infty_1}_r(Q_0)+\mathcal{O}(\zeta), & P\to P_{\infty_1}, \\ \tilde{\Omega}^{\infty_2}_r(Q_0)+\mathcal{O}(\zeta), & P\to P_{\infty_2}, \end{cases} \\ &\int_{Q_0}^{P}\tilde{\Omega}^{(2)}_r(P) \underset{\zeta\to 0}{=} \begin{cases} \tilde{\Omega}^{0_1}_r(Q_0)+\mathcal{O}(\zeta), & P\to P_{0_1}, \\ \tilde{\Omega}^{0_2}_r(Q_0)+\mathcal{O}(\zeta), & P\to P_{0_2}, \\ \tilde{\Omega}^{0_3}_r(Q_0)+\mathcal{O}(\zeta), & P\to P_{0_3}, \end{cases} \end{aligned} \tag{5.26}$$

where $\tilde{\Omega}^{\infty_1}_r(Q_0)$, $\tilde{\Omega}^{\infty_2}_r(Q_0)$, $\tilde{\Omega}^{0_1}_r(Q_0)$, $\tilde{\Omega}^{0_2}_r(Q_0)$ and $\tilde{\Omega}^{0_3}_r(Q_0)$ are integration constants depending on the base point $Q_0\in\mathcal{K}_{m-1}\backslash\{P_{\infty_1},P_{\infty_2}\}$.

Lemma 5.3. *Let the curve $\mathcal{K}_{m-1}$ be nonsingular and irreducible. Let $P=(\lambda,y)\in\mathcal{K}_{m-1}\backslash\{P_{\infty_1},P_{\infty_2},P_{0_1},P_{0_2},P_{0_3}\}$ and $(n,n_0,t_r,t_{0,r})\in\mathbb{Z}^2\times\mathbb{R}^2$. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(n,t_r)}$ and $\mathcal{D}_{\underline{\hat{\xi}}(n,t_r)}$ is nonspecial for each $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, then we have*

$$\phi_{13}(P,n,t_r)=N_1(n,t_r)\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(n,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n,t_r)))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_{0_3},P_{\infty_2}}\right), \tag{5.27}$$

$$\phi_{23}(P,n,t_r)=N_2(n,t_r)\frac{\theta(\underline{z}(P,\underline{\hat{\xi}}(n,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n,t_r)))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1},P_{0_1}}\right), \tag{5.28}$$

$$
\begin{aligned}
&\psi_3(P,n,n_0,t_r,t_{0,r}) \\
&= \frac{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}(n_0,t_{0,r})))}{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}(n,t_r)))} \frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0,t_{0,r})))} \\
&\quad \times \exp\left((n-n_0)\left(\int_{Q_0}^{P} \omega^{(3)}_{P_{0_2},P_{\infty_2}} - \omega^{0_3}(P_{0_2},P_{\infty_2})\right) + (t_r - t_{0,r})\left(\int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_r - \widetilde{\Omega}^{0_3}_r\right)\right),
\end{aligned} \tag{5.29}
$$

where

$$
\begin{aligned}
N_1(n,t_r) = &-\frac{1}{\nu(n_0,t_{0,r})} \frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}^{+}(n_0,t_{0,r})))}{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}^{+}(n_0,t_{0,r})))} \\
&\times \exp\Big((t_r - t_{0,r})\big(\widetilde{\Omega}^{0_3}_r - \widetilde{\Omega}^{\infty_2}_r\big) + (n-n_0) \\
&\qquad \times \big(\omega^{0_3}(P_{0_2},P_{\infty_2}) - \omega^{\infty_2}(P_{0_2},P_{\infty_2})\big) - \omega^{0_2}(P_{\infty_1},P_{0_1})\Big),
\end{aligned} \tag{5.30}
$$

$$
\begin{aligned}
N_2(n,t_r) = &\frac{1}{\nu^{-}(n_0,t_{0,r})} \frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(n_0,t_{0,r})))}{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}(n_0,t_{0,r})))} \frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\nu}}(n,t_r)))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\xi}}(n,t_r)))} \\
&\times \exp\big(-\omega^{\infty_2}(P_{\infty_1},P_{0_1})\big) \exp\big((n-n_0)\big(\omega^{0_3}(P_{0_2},P_{\infty_2}) - \omega^{\infty_2}(P_{0_2},P_{\infty_2})\big) \\
&\qquad + (t_r - t_{0,r})\big(\widetilde{\Omega}^{0_3}_r - \widetilde{\Omega}^{\infty_2}_r\big)\big).
\end{aligned} \tag{5.31}
$$

Proof. Denote the right-hand side of (5.29) by $\Psi_3(P,n,n_0,t_r,t_{0,r})$. Our goal is to show that

$$\Psi_3(P,n,n_0,t_r,t_{0,r}) = \psi_3(P,n,n_0,t_r,t_{0,r}).$$

In fact,

$$\Psi_3(P,n,n_0,t_r,t_{0,r}) = \Psi_3(P,n,n_0,t_r,t_r)\Psi_3(P,n_0,n_0,t_r,t_{0,r}). \tag{5.32}$$

It is easily seen from (5.14), and (5.15), (5.27) that

$$
\begin{aligned}
\psi_3(P,n,n_0,t_r,t_r) &= \begin{cases}
\displaystyle\prod_{n'=n_0}^{n-1} \big(\nu(n',t_r)\phi_{13}(P,n',t_r) + 1\big) & \text{as} \quad n \geq n_0 + 1, \\
1 & \text{as} \quad n = n_0, \\
\displaystyle\prod_{n'=n}^{n_0-1} \big(\nu(n',t_r)\phi_{13}(P,n',t_r) + 1\big)^{-1} & \text{as} \quad n \leq n_0 - 1
\end{cases} \\
&= \Psi_3(P,n,n_0,t_r,t_r).
\end{aligned}
$$

It is necessary to prove

$$\psi_3(P,n_0,n_0,t_r,t_{0,r}) = \Psi_3(P,n_0,n_0,t_r,t_{0,r}). \tag{5.33}$$

Now Lemma 5.1 and (5.6), (5.7), (5.16), give

$$I_r(P,n,t_r) = \widetilde{V}^{(r)}_{31}(\lambda,n,t_r)\phi_{13}(P,n,t_r) + \lambda\widetilde{V}^{(r)}_{32}(\lambda,n,t_r)\phi_{23}(P,n,t_r) + \lambda\widetilde{V}^{(r)}_{33}(\lambda,n,t_r)$$

$$
\begin{aligned}
&= \widetilde{V}_{31}^{(r)} \frac{y^2 V_{32}^{(q)} - y(A_m + V_{32}^{(q)} R_m) + B_m}{E_{m-1}} \\
&\quad - \lambda \widetilde{V}_{32}^{(r)} \frac{y^2 V_{31}^{(q)} - y(\mathscr{A}_m + V_{31}^{(q)} R_m) + \mathscr{B}_m}{E_{m-1}} + \lambda \widetilde{V}_{33}^{(r)} \\
&= \widetilde{V}_{31}^{(r)} \frac{V_{32}^{(q)}(3y^2 - 2R_m y + S_m)}{E_{m-1}} - \widetilde{V}_{32}^{(r)} \frac{V_{31}^{(q)}(3y^2 - 2R_m y + S_m)}{E_{m-1}} + \lambda \widetilde{V}_{33}^{(r)} \\
&= \frac{(\widetilde{V}_{31}^{(r)} V_{32}^{(q)} - \widetilde{V}_{32}^{(r)} V_{31}^{(q)})(3y^2 - 2R_m y + S_m)}{E_{m-1}} + \lambda \widetilde{V}_{33}^{(r)} \\
&= -\frac{\mu_{j,t_r}(n,t_r)}{\lambda - \mu_j(n,t_r)} + \mathscr{O}(1) \\
&= \partial_{t_r} \ln\big(\lambda - \mu_j(n,t_r)\big) + \mathscr{O}(1) \quad \text{as} \quad P \to \hat{\mu}_j(n,t_r), \quad \lambda \to \mu_j(n,t_r),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\psi_3(P, n_0, n_0, t_r, t_{0,r}) \\
&= \exp\left(\int_{t_{0,r}}^{t_r} I_{t_r}(P, n_0, t') dt' \right) = \frac{\lambda - \mu_j(n_0, t_r)}{\lambda - \mu_j(n_0, t_{0,r})} \mathscr{O}(1) \\
&= \begin{cases} \big(\lambda - \mu_j(n_0, t_r)\big)\mathscr{O}(1) & \text{for} \quad P \quad \text{near} \quad \hat{\mu}_j(n,t_r) \neq \hat{\mu}_j(n, t_{0,r}), \\ \mathscr{O}(1) & \text{for} \quad P \quad \text{near} \quad \hat{\mu}_j(n,t_r) = \hat{\mu}_j(n, t_{0,r}), \\ \big(\lambda - \mu_j(n_0, t_r)\big)^{-1}\mathscr{O}(1) & \text{for} \quad P \quad \text{near} \quad \hat{\mu}_j(n,t_{0,r}) \neq \hat{\mu}_j(n, t_r), \end{cases}
\end{aligned}
$$

where $\mathscr{O}(1) \neq 0$. Hence, all zeros and poles of $\psi_3(P, n_0, n_0, t_r, t_{0,r})$ and $\Psi_3(P, n_0, n_0, t_r, t_{0,r})$ on $\mathscr{K}_{m-1} \backslash \{P_{\infty_1}, P_{\infty_2}, P_{0_1}, P_{0_2}, P_{0_3}\}$ are simple and coincide. Taking into account (5.23) and (5.26), we note that the singularities of $\psi_3(P, n_0, n_0, t_r, t_{0,r})$ and $\Psi_3(P, n_0, n_0, t_r, t_{0,r})$ at $P_{\infty_1}, P_{\infty_2}, P_{0_1}, P_{0_2}$ and P_{0_3} are coincide. The Riemann-Roch uniqueness result is that

$$
\frac{\Psi_3(P, n_0, n_0, t_r, t_{0,r})}{\psi_3(P, n_0, n_0, t_r, t_{0,r})} = \gamma
$$

for a constant $\gamma \in \mathbb{C}$. It is not difficult to verify that $\gamma = 1$ from the asymptotic expansions near P_{0_3}, that is

$$
\frac{\Psi_3(P, n_0, n_0, t_r, t_{0,r})}{\psi_3(P, n_0, n_0, t_r, t_{0,r})} \underset{\zeta \to 0}{=} \frac{(1 + \mathscr{O}(\zeta)) \exp(\mathscr{O}(\zeta))}{\exp(\mathscr{O}(\zeta))} \underset{\zeta \to 0}{=} 1 + \mathscr{O}(\zeta) \quad \text{as} \quad P \to P_{0_3}.
$$

Hence, using (5.15), (5.32) and (5.33), we have demonstrated (5.29).

Similar to Lemma 4.2, we conclude that ϕ_{13} and ϕ_{23} are of the form (5.27) and (5.28), respectively. As $P \to P_{\infty_2}$, the asymptotic expansions of ψ_3 shows

$$
\frac{v^-(n,t_r)}{v^-(n_0,t_{0,r})} = \frac{\theta(\underline{z}(P_{0_3}, \underline{\hat{\mu}}(n_0, t_{0,r})))}{\theta(\underline{z}(P_{0_3}, \underline{\hat{\mu}}(n, t_r)))} \frac{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\mu}}(n, t_r)))}{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\mu}}(n_0, t_{0,r})))}
$$

$$\times \exp\left((t_r - t_{0,r})\left(\widetilde{\Omega}_r^{\infty_2} - \widetilde{\Omega}_r^{0_3}\right) + (n-n_0)(\omega^{\infty_2}(P_{0_2}, P_{\infty_2}) - \omega^{0_3}(P_{0_2}, P_{\infty_2}))\right), \qquad (5.34)$$

which yields (5.39) for $v^-(n,t_r)$. Comparing the asymptotic expressions of ϕ_{13} for $P \to P_{0_2}$ in (3.19) and (5.27), one obtains

$$-\frac{1}{v(n,t_r)} = N_1(n,t_r)\frac{\theta(\underline{z}(P_{0_2}, \underline{\hat{v}}(n,t_r)))}{\theta(\underline{z}(P_{0_2}, \underline{\hat{\mu}}(n,t_r)))}\exp\left(\omega^{0_2}(P_{0_3}, P_{\infty_2})\right) \quad \text{as} \quad P \to P_{0_2}.$$

Similar to the stationary case, $N_1(n,t_r)$ can be simplified as (5.30). Meanwhile, comparing the asymptotic expressions of ϕ_{23} for $P \to P_{\infty_2}$ in (3.18) and (5.28), one obtains

$$\frac{1}{v^-(n,t_r)} = N_2(n,t_r)\frac{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\xi}}(n,t_r)))}{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{v}}(n,t_r)))}\exp\left(\omega^{\infty_2}(P_{\infty_1}, P_{0_1})\right) \quad \text{as} \quad P \to P_{\infty_2},$$

which yields (5.31) for $N_2(n,t_r)$. This completes the proof. □

The b-period of the differential $\widetilde{\Omega}_r^{(2)}$ is denoted by

$$\underline{\widetilde{U}}_r^{(2)} = \left(\widetilde{U}_{r,1}^{(2)}, \dots, \widetilde{U}_{r,m-1}^{(2)}\right), \quad \widetilde{U}_{r,k}^{(2)} = \frac{1}{2\pi \mathrm{i}}\int_{b_k} \Omega_r^{(2)}, \quad k = 1, \dots, m-1.$$

Lemma 5.4. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular and let (n,t_r), $(n_0,t_{0,r}) \in \mathbb{Z}\times\mathbb{R}$. Then*

$$\underline{\rho}^{(1)}(n,t_r) = \underline{\rho}^{(1)}(n_0,t_{0,r}) - \underline{U}^{(3)}(n-n_0) - \underline{\widetilde{U}}_r^{(2)}(t_r - t_{0,r}) \pmod{\mathcal{T}_{m-1}}, \qquad (5.35)$$

$$\underline{\rho}^{(2)}(n,t_r) = \underline{\rho}^{(2)}(n_0,t_{0,r}) - \underline{U}^{(3)}(n-n_0) - \underline{\widetilde{U}}_r^{(2)}(t_r - t_{0,r}) \pmod{\mathcal{T}_{m-1}}, \qquad (5.36)$$

$$\underline{\rho}^{(3)}(n,t_r) = \underline{\rho}^{(3)}(n_0,t_{0,r}) - \underline{U}^{(3)}(n-n_0) - \underline{\widetilde{U}}_r^{(2)}(t_r - t_{0,r}) \pmod{\mathcal{T}_{m-1}}. \qquad (5.37)$$

Proof. We introduce the meromorphic differential on $\mathcal{K}_{m-1}$ by

$$\Omega(n,n_0,t_r,t_{0,r}) = \frac{\partial}{\partial\lambda}\ln\left(\psi_3(n,n_0,t_r,t_{0,r})\right)d\lambda.$$

From the representation of (5.29) we have

$$\Omega(n,n_0,t_r,t_{0,r}) = (n-n_0)\omega^{(3)}_{P_{0_2},P_{\infty_2}} + (t_r - t_{0,r})\widetilde{\Omega}_r^{(2)} + \sum_{j=1}^{m-1}\omega^{(3)}_{\hat{\mu}_j(n,t_r),\hat{\mu}_j(n_0,t_{0,r})} + \bar{\omega},$$

where $\bar{\omega}$ denotes a holomorphic differential on $\mathcal{K}_{m-1}$, that is $\bar{\omega} = \sum_{j=1}^{m-1}\bar{e}_j\omega_j$ for some $\bar{e}_j \in \mathbb{C}, j = 1,\dots,m-1$. Since $\psi_3(P,n,n_0,t_r,t_{0,r})$ is single-valued on $\mathcal{K}_{m-1}$, all a-periods and b-periods of Ω are integer multiples of $2\pi\mathrm{i}$ and hence

$$2\pi\mathrm{i}M_k = \int_{a_k}\Omega(n,n_0,t_r,t_{0,r}) = \int_{a_k}\bar{\omega} = \bar{e}_k, \quad k = q,\dots,m-1,$$

form some $M_k \in \mathbb{Z}$. Similarly, for some $N_k \in \mathbb{Z}$,

$$\begin{aligned}
2\pi \mathrm{i} N_k =& \int_{b_k} \Omega(n, n_0, t_r, t_{0,r}) \\
=& (n-n_0)\int_{b_k} \omega^{(3)}_{P_{0_2},P_{\infty_2}} + (t_r - t_{0,r})\int_{b_k} \widetilde{\Omega}^{(2)}_r + \sum_{j=1}^{m-1}\int_{b_k} \omega^{(3)}_{\hat{\mu}_j(n,t_r),\hat{\mu}_j(n_0,t_{0,r})} + \int_{b_k}\bar{\omega} \\
=& 2\pi \mathrm{i}(n-n_0)U^{(3)}_k + 2\pi \mathrm{i}(t_r - t_{0,r})\widetilde{U}^{(2)}_{r,k} + 2\pi \mathrm{i}\sum_{j=1}^{m-1}\int_{\hat{\mu}_j(n_0,t_{0,r})}^{\hat{\mu}_j(n,t_r)} \omega_k + 2\pi \mathrm{i}\sum_{j=1}^{m-1} M_j \int_{b_k}\omega_j \\
=& 2\pi \mathrm{i}(n-n_0)U^{(3)}_k + 2\pi \mathrm{i}(t_r - t_{0,r})\widetilde{U}^{(2)}_{r,k} + 2\pi \mathrm{i}\left(\sum_{j=1}^{m-1}\int_{Q_0}^{\hat{\mu}_j(n,t_r)} \omega_k - \sum_{j=1}^{m-1}\int_{Q_0}^{\hat{\mu}_j(n_0,t_{0,r})} \omega_k\right) \\
&+ 2\pi \mathrm{i}\sum_{j=1}^{m-1} M_j \tau_{jk}, \quad k = 1,\dots,m-1.
\end{aligned}$$

Therefore, we have

$$\underline{N} = (n-n_0)\underline{U}^{(3)} + (t_r - t_{0,r})\widetilde{U}^{(2)}_r + \sum_{j=1}^{m-1}\int_{Q_0}^{\hat{\mu}_j(n,t_r)} \underline{\omega} - \sum_{j=1}^{m-1}\int_{Q_0}^{\hat{\mu}_j(n_0,t_{0,r})} \underline{\omega} + \underline{M}\tau,$$

where $\underline{N} = (N_1,\dots,N_{m-1})$, $\underline{M} = (M_1,\dots,M_{m-1}) \in \mathbb{Z}^{m-1}$, and hence (5.35) holds. Applying Abel's Theorem to (5.10) and (5.11), we have

$$\begin{aligned}
\underline{\rho}^{(2)}(n,t_r) &= \underline{\rho}^{(1)}(n,t_r) + \underline{\mathscr{A}}(P_{\infty_2}) - \underline{\mathscr{A}}(P_{0_3}), \\
\underline{\rho}^{(3)}(n,t_r) &= \underline{\rho}^{(1)}(n,t_r) + \underline{\mathscr{A}}(P_{0_1}) - \underline{\mathscr{A}}(P_{\infty_1}),
\end{aligned}$$

then we can prove (5.36) and (5.37). □

Recalling (4.6), we extend the function $\underline{z}(P,\underline{\hat{\mu}}(n))$, $\underline{z}(P,\underline{\hat{\nu}}(n))$ and $\underline{z}(P,\underline{\hat{\xi}}(n))$ into time-dependent cases

$$\begin{aligned}
\underline{z}\big(P,\underline{\hat{\mu}}(n,t_r)\big) &= \underline{\Lambda} - \underline{\mathscr{A}}(P) + \underline{\rho}^{(1)}(n,t_r), \quad P \in \mathscr{K}_{m-1}, \\
\underline{z}\big(P,\underline{\hat{\nu}}(n,t_r)\big) &= \underline{\Lambda} - \underline{\mathscr{A}}(P) + \underline{\rho}^{(2)}(n,t_r), \quad P \in \mathscr{K}_{m-1}, \\
\underline{z}\big(P,\underline{\hat{\xi}}(n,t_r)\big) &= \underline{\Lambda} - \underline{\mathscr{A}}(P) + \underline{\rho}^{(3)}(n,t_r), \quad P \in \mathscr{K}_{m-1}.
\end{aligned}$$

We rewrite $\theta(\underline{z}(P_{0_j},\underline{\hat{\mu}}(n,t_r)))$, $\theta(\underline{z}(P_{0_j},\underline{\hat{\nu}}(n,t_r)))$, $\theta(\underline{z}(P_{0_j},\underline{\hat{\xi}}(n,t_r)))$, $\theta(\underline{z}(P_{\infty_l},\underline{\hat{\mu}}(n,t_r)))$, $\theta(\underline{z}(P_{\infty_l},\underline{\hat{\nu}}(n,t_r)))$ and $\theta(\underline{z}(P_{\infty_l},\underline{\hat{\xi}}(n,t_r)))$, respectively, as

$$\begin{aligned}
\theta\big(\underline{z}(P_{0_j},\underline{\hat{\mu}}(n,t_r))\big) &= \theta\left(\underline{\widetilde{M}}^{(1)}_j - \underline{U}^{(3)}n - \underline{\widetilde{U}}^{(2)}_r t_r\right), \\
\theta\big(\underline{z}(P_{\infty_l},\underline{\hat{\mu}}(n,t_r))\big) &= \theta\left(\underline{\widetilde{K}}^{(1)}_l - \underline{U}^{(3)}n - \underline{\widetilde{U}}^{(2)}_r t_r\right),
\end{aligned}$$

$$\theta(\underline{z}(P_{0_j},\underline{\hat{\nu}}(n,t_r))) = \theta\left(\underline{\widetilde{M}}_j^{(2)} - \underline{U}^{(3)}n - \underline{\widetilde{U}}_r^{(2)}t_r\right),$$
$$\theta\big(\underline{z}(P_{\infty_l},\underline{\hat{\nu}}(n,t_r))\big) = \theta\left(\underline{\widetilde{K}}_l^{(2)} - \underline{U}^{(3)}n - \underline{\widetilde{U}}_r^{(2)}t_r\right),$$
$$\theta\big(\underline{z}(P_{0_j},\underline{\hat{\xi}}(n,t_r))\big) = \theta\left(\underline{\widetilde{M}}_j^{(3)} - \underline{U}^{(3)}n - \underline{\widetilde{U}}_r^{(2)}t_r\right),$$
$$\theta\big(\underline{z}(P_{\infty_l},\underline{\hat{\xi}}(n,t_r))\big) = \theta\left(\underline{\widetilde{K}}_l^{(3)} - \underline{U}^{(3)}n - \underline{\widetilde{U}}_r^{(2)}t_r\right),$$

where

$$\underline{\widetilde{M}}_j^{(s)} = \Lambda - \underline{\mathscr{A}}(P_{0_j}) + \rho^{(s)}(n_0,t_{0,r}) + \underline{U}^{(3)}n_0 + \underline{\widetilde{U}}_r^{(2)}t_{0,r},$$
$$\underline{\widetilde{K}}_l^{(s)} = \Lambda - \underline{\mathscr{A}}(P_{\infty_k}) + \rho^{(s)}(n_0,t_{0,r}) + \underline{U}^{(3)}n_0 + \underline{\widetilde{U}}_r^{(2)}t_{0,r}, \quad j=1,2,3, \quad l=1,2, \quad s=1,2,3$$

are constants independent of the discrete variable n and the time variable t_r. Then we arrive at the following theorem.

Theorem 5.1. *Assume that the curve $\mathscr{K}_{m-1}$ is nonsingular and irreducible. Let $P=(\lambda,y)\in \mathscr{K}_{m-1}\backslash\{P_{\infty_1},P_{\infty_2},P_{0_1},P_{0_2},P_{0_3}\}$ and let $(n_0,t_{0,r}),(n,t_r)\in\mathbb{Z}\times\mathbb{R}$. Suppose that $\mathscr{D}_{\underline{\hat{\mu}}}(n,t_r)$ and $\mathscr{D}_{\underline{\hat{\nu}}}(n,t_r)$ and $\mathscr{D}_{\underline{\hat{\xi}}}(n,t_r)$ is nonspecial. Then, finite genus solutions of the discrete integrable hierarchy* (2.13) *read*

$$\begin{aligned}
u(n,t_r) = {} & \frac{1}{v(n_0,t_{0,r})}\frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}^+(n_0,t_{0,r})))}{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}^+(n_0,t_{0,r})))}\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\nu}}(n,t_r)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(n,t_r)))} \\
& \times \exp\Big((n-n_0)\big(\omega^{0_3}(P_{0_2},P_{\infty_2}) - \omega^{\infty_2}(P_{0_2},P_{\infty_2})\big) \\
& + (t_r - t_{0,r})\big(\widetilde{\Omega}_r^{0_3} - \widetilde{\Omega}_r^{\infty_2}\big) + \omega^{\infty_1}(P_{\infty_2},P_{0_3}) - \omega^{0_2}(P_{\infty_2},P_{0_3})\Big),
\end{aligned} \tag{5.38}$$

$$\begin{aligned}
v^-(n,t_r) = {} & v^-(n_0,t_{0,r})\frac{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}(n_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(n_0,t_{0,r})))}\frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P_{0_3},\underline{\hat{\mu}}(n,t_r)))} \\
& \times \exp\Big((t_r - t_{0,r})\big(\widetilde{\Omega}_r^{\infty_2} - \widetilde{\Omega}_r^{0_3}\big) \\
& + (n-n_0)\big(\omega^{\infty_2}(P_{\infty_2},P_{0_3}) - \omega^{0_3}(P_{\infty_2},P_{0_3})\big)\Big),
\end{aligned} \tag{5.39}$$

$$\begin{aligned}
w(n,t_r) = {} & \big(1 - u(n,t_r)v(n,t_r)\big)\frac{v^-(n_0,t_{0,r})}{v(n,t_r)}\frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\nu}}(n_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(n_0,t_{0,r})))} \\
& \times \frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\xi}}(n,t_r)))}{\theta(\underline{z}(P_{0_3},\underline{\hat{\nu}}(n,t_r)))}\frac{\theta(\underline{z}(P_{0_2},\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P_{0_2},\underline{\hat{\xi}}(n,t_r)))} \\
& \times \exp\Big(\omega^{\infty_2}(P_{0_1},P_{\infty_1}) - \omega^{0_2}(P_{0_1},P_{\infty_1}) + (t_r - t_{0,r})\big(\widetilde{\Omega}_r^{\infty_2} - \widetilde{\Omega}_r^{0_3}\big) \\
& + (n-n_0)\big(\omega^{\infty_2}(P_{\infty_2},P_{0_3}) - \omega^{0_3}(P_{\infty_2},P_{0_3})\big)\Big).
\end{aligned} \tag{5.40}$$

Proof. From (5.34), we can get the expressions of $v(n,t_r)$ in (5.39). By considering the asymptotic expansions of ϕ_{13} near P_{∞_1} and ϕ_{23} near P_{0_2}, we arrive at

$$-u(n,t_r) = N_1(n,t_r)\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\nu}}(n,t_r)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(n,t_r)))}\exp\big(\omega^{\infty_1}(P_{0_3},P_{\infty_2})\big) \quad \text{as} \quad P\to P_{\infty_1},$$

$$\frac{1-u(n,t_r)v(n,t_r)}{v(n,t_r)w(n,t_r)} = N_2(n,t_r)\frac{\theta(\underline{z}(P_{0_2},\underline{\hat{\xi}}(n,t_r)))}{\theta(\underline{z}(P_{0_2},\underline{\hat{\nu}}(n,t_r)))}\exp\left(\omega^{0_2}(P_{\infty_1},P_{0_1})\right) \quad \text{as} \quad P \to P_{0_2},$$

which together with (5.30) and (5.31) yields (5.38) and (5.40). □

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