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Increasing Stability of Determining Both the Potential and Source for the Biharmonic Wave Equation

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Abstract. This paper is concerned with the inverse scattering problems of simultaneously determining the unknown potential and unknown source for the biharmonic wave equation. We first derive an increasing stability estimate for the inverse potential scattering problem without a priori knowledge of the source function by multi-frequency active boundary measurements. The stability estimate consists of the Lipschitz type data discrepancy and the high frequency tail of the potential function, where the latter decreases as the upper bound of the frequency increases. The key ingredients in the analysis are employing scattering theory to derive an analytic domain and resolvent estimates and an application of the quantitative analytic continuation principle. Utilizing the derived stability for the inverse potential scattering, we further prove an increasing stability estimate for the inverse source problem. The main novelty of this paper is that both the source and potential functions are unknown.

AMS subject classifications: 35R30, 78A46

Key words: Inverse scattering problem, increasing stability, biharmonic wave equation.

1. Introduction

Let $e^{\mathrm{i}\kappa x\cdot d}$ be the incident plane wave where $d\in\mathbb{S}^2$ and let $\kappa>0$ be the wavenumber. We consider the scattering problem modeled by the biharmonic wave equation in three dimensions

$$\Delta^2 u(x,\kappa)-\kappa^4 u(x,\kappa)+V(x)u(x,\kappa)=f(x),\quad x\in\mathbb{R}^3, \tag{1.1}$$

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where $u = e^{\mathrm{i}\kappa x\cdot d} + u^{sc}$ is the total field, u^{sc} the scattered field, $V(x)$ the potential, and $f(x)$ the external source. We assume that both f and V are real-valued and have compact supports contained in B_R. Here

$$B_R = \left\{x \in \mathbb{R}^3 : |x| \le R\right\}$$

with $R > 0$ being a constant. Denote the boundary of B_R by ∂B_R. The Sommerfeld radiation conditions

$$\lim_{r\to\infty} r\,(\partial_r u^{sc} - \mathrm{i}\kappa u^{sc}) = 0, \quad \lim_{r\to\infty} r\,[\partial_r(\Delta u^{sc}) - \mathrm{i}\kappa(\Delta u^{sc})] = 0 \tag{1.2}$$

are imposed on u^{sc} and Δu^{sc} uniformly in all directions $\hat{x} = x/|x|$ with $r = |x|$, which implies that the solution is outgoing and guarantees the uniqueness of the direct problem. This paper is concerned with the inverse problem of determining both the unknown potential V and the unknown source f from the boundary measurements $u(x,\kappa), \Delta u(x,\kappa), \partial_\nu u(x,\kappa)$, $\partial_\nu \Delta u(x,\kappa)$ on ∂B_R with the wavenumber κ being given in a finite interval. Here ν denotes the unit outward normal vector to ∂B_R.

Biharmonic wave equations are encountered in the theory of plate bending and thin plate elasticity, where the variable u represents the deflection of the plate in the vertical direction, while Δu is closely related to the bending moments of the plate [2], which can be practically measured. Then the terms $\partial_\nu u$ and $\partial_\nu \Delta u$ in the measurement data can be computed from u and Δu on ∂B_R through a decomposition method and the transparent boundary conditions. For details we refer the reader to [10].

Compared with the well-developed inverse scattering theory of acoustic, elastic and electromagnetic waves, the inverse scattering problems of the biharmonic wave equations are much less studied. Recently, motivated by their important applications in a wide range of scientific areas including offshore runway design, seismic cloaks, and platonic crystal [7], the inverse scattering problems of the biharmonic wave equations have received much attention [6–9, 14]. For the inverse problem, a fundamental issue is the stability. In general, the inverse problem at a fixed frequency is lack of stability — i.e. a small variation of the data might lead to a huge error in the reconstruction. On the other hand, it has been realized that the use of multi-frequency data is an effective method to overcome the instability encountered at a single frequency [1]. In this paper, we investigate the stability issue for the inverse scattering problem of the biharmonic wave equation by using multi-frequency data. We exhibit the intrinsic nature of the inverse scattering problem that the stability improves when the frequency of the data increases in a more complicated situation where both the source and potential are unknown.

Given a known potential, the stability estimate for the inverse source problem of the biharmonic wave equation was studied in [8, 9] by passive measurements. By passive measurements it means that no incident plane waves are used and the scattering is only generated by the source function. On the other hand, uniqueness and stability on the inverse potential scattering problems were studied in [7, 14, 15] without the source function. The major novelty of this work is that we derive the increasing stability estimates of determining simultaneously the potential and source of the biharmonic wave equation. To this end,