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Global Solvability of Two-Dimensional Stochastic Chemotaxis-Navier-Stokes System

Fan Xu[1,2] and Bin Liu[1,2,*]

[1] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, P.R. China.*
[2] *Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, P.R. China.*

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Abstract. This paper considers a chemotaxis model coupled a stochastic Navier-Stokes equation in two-dimensional case. In non-convex bounded domain, it is proved that the stochastic chemotaxis-Navier-Stokes system possesses at least one global martingale weak solution when the chemotactic sensitivity function χ is nonsingular. In convex bounded domain, under the conditions of χ and per capita oxygen consumption rate h are appropriately relaxed (where χ allows singularity), it is proved that the system admits a unique global mild solution. Our results generalize previously known ones.

AMS subject classifications: 35K55, 35K20, 35R60, 60H15

Key words: Stochastic, chemotaxis-Navier-Stokes system, martingale solution, mild solution.

1. Introduction

Chemotaxis serves as a pivotal mechanism employed by cells to effectively navigate their environment in search of nutrients, while simultaneously avoiding potential hazards and seeking out other cells for essential communication or mating purposes [17]. In a study conducted by Tuval *et al*. [27], the intricate dynamics of bacteria in suspension were meticulously observed, leading to the proposal of the chemotaxis-Navier-Stokes (CNS) system

$$\begin{aligned}
&dn+u\cdot\nabla n\,dt=\Delta n\,dt-\chi\nabla\cdot(n\nabla c)\,dt && \text{in } \mathbb{R}^+\times\mathscr{O},\\
&dc+u\cdot\nabla c\,dt=\Delta c dt-cn\,dt && \text{in } \mathbb{R}^+\times\mathscr{O},\\
&du+(u\cdot\nabla)u\,dt+\nabla P\,dt=\Delta u\,dt+n\nabla\phi\,dt && \text{in } \mathbb{R}^+\times\mathscr{O},\\
&\nabla\cdot u=0 && \text{in } \mathbb{R}^+\times\mathscr{O},
\end{aligned} \tag{1.1}$$

where the unknown functions $n(t,x)$, $c(t,x)$ and $u(t,x)$ denote the bacterial density, the chemical concentration, and the fluid velocity field, respectively. The scalar function $P(t,x)$

*Corresponding author. *Email addresses:* 1754149729@qq.com (F. Xu), binliu@mail.hust.edu.cn (B. Liu)

stands for the pressure, and $\phi(x)$ represents the gravitational potential. The CNS system (1.1) provides a quantitative framework comprehensively characterizing the intricate interrelationships among bacterial density, chemical concentration, and fluid transport. This sophisticated system not only delves into the nuanced impact of fluid transport on the density of bacteria and the concentration of chemical substances but also discerns the reciprocal influence of bacterial gravity on fluid velocity. To simulate a more realistic environment, several authors considered the CNS system being influenced by some random external forces [35,36]. Specially, Zhai and Zhang [35] first did some work about a stochastic version of the CNS system. They discussed the global solvability of the following system:

$$\begin{aligned}
&dn + u\cdot\nabla n\,dt = \Delta n\,dt - \chi\nabla\cdot(n\nabla c)\,dt && \text{in } \mathbb{R}^+\times\mathcal{O},\\
&dc + u\cdot\nabla c\,dt = \Delta c\,dt - cn\,dt && \text{in } \mathbb{R}^+\times\mathcal{O},\\
&du + (u\cdot\nabla)u\,dt + \nabla P\,dt = \Delta u\,dt + n\nabla\phi\,dt + B(u)\,dW(t) && \text{in } \mathbb{R}^+\times\mathcal{O},\\
&\nabla\cdot u = 0 && \text{in } \mathbb{R}^+\times\mathcal{O}
\end{aligned}$$

with the initial and boundary conditions

$$\begin{aligned}
&n(0,x) = n_0(x), \quad c(0,x) = c_0(x), \quad u(0,x) = u_0(x) && \text{in } \mathcal{O},\\
&\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, \quad u = 0 && \text{in } \mathbb{R}^+\times\partial\mathcal{O},
\end{aligned}$$

where $\mathcal{O}\subset\mathbb{R}^2$ is a convex bounded domain with a smooth boundary and ν denotes the outward normal vector on the boundary $\partial\mathcal{O}$. The term $B(u)dW(t)$ denotes a random force that influences the fluid transport continuously in time.

Numerous scholars have undertaken an exhaustive investigation into the profound influence of viscous fluid on chemotaxis phenomena, examining it through the perspective of deterministic partial differential equations. The outcomes of these studies predominantly revolve around such aspects as global well-posedness [1,4,8,18,21,28,29,31–33], boundedness [14,20,25,26], and the asymptotic behavior [7,11,16,30] of solutions. The overall goal is to provide a comprehensive insight into the complex dynamics involved in the chemotaxis phenomenon in fluids. However, in comparison with the deterministic case, the available references concerning the stochastic CNS system are rather limited and we refer to [13,35,36]. The relevant results in the literature predominantly focus on the global well-posedness of the stochastic system. To be specific, Zhai and Zhang [35] established the existence of a unique global mild and weak solutions to the 2D stochastic CNS system in a bounded and convex domain. Then in 2D unbounded cases, Zhang and Liu [36] established the existence and uniqueness of global pathwise weak solutions to the stochastic CNS system.

This paper is devoted to appropriately relaxing the structural assumptions mentioned in [35] to obtain the global mild solution and pathwise weak solution for the following 2D stochastic chemotaxis-Navier-Stokes system:

$$dn + u\cdot\nabla n\,dt = \Delta n\,dt - \nabla\cdot\big(\chi(c)n\nabla c\big)\,dt \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \tag{1.2a}$$

$$dc + u\cdot\nabla c\,dt = \Delta c\,dt - h(c)n\,dt \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \tag{1.2b}$$

$$du + (u\cdot\nabla)u\,dt + \nabla P\,dt = \Delta u\,dt + n\nabla\phi\,dt + B(u)\,dW(t) \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \tag{1.2c}$$