NUMERICAL SOLUTION OF NON-STEADY STATE POROUS FLOW FREE BOUNDARY PROBLEMS*

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Abstract

The aim of this paper is the study of the convergence of a finite element approximation for a variational inequality related to free boundary problems in non-steady fluid flow through porous media. There have been many results in the stationary case, for example, the steady dam problems ([3, 1]), the steady flow well problems[6], etc. In this paper we shall deal with the axisymmetric non-steady porous flow well problem. It is well known that by means of Torelli's transform this problem, similar to the non-steady rectangular dam problem, can be reduced to a variational inequality, and the existence, uniqueness and regularity of the solution can be obtained ([12, 7]). Now we study the numerical solution of this variational inequality.

The main results are as follows:

1. We establish new regularity properties for the solution W of the variational inequality. We prove that $W\in L^\infty(0, T; H^2(D))$, $\gamma_0 W\in L^\infty(0, T; H^2(\Gamma_n))$ and $D_t\gamma_0 W\in L^2(0, T; H^1(\Gamma_n))$ (see Theorem 2.5). Friedman and Torelli[7] obtained $W\in L^2(0, T; H^2(D))$. Our new regularity properties will be used for error estimation.

2. We prove that the error estimate for the finite element solution of the variational inequality is

$$\{\sum_{i=1}^{K}\|W^i-W_h^i\|^2_{H^1(D)}\Delta t\}^{1/2}=O(h+\Delta t^{1/2})$$

(see Theorem 3.4). In the stationary case the error estimate is $\|W-W_h\|_{H^1(D)}=O(h)$ ([3, 6]).

3. We give a numerical example and compare the result with the corresponding result in the stationary case.

The results of this paper are valid for the non-steady rectangular dam problem with stationary or quasi-stationary initial data (see [7], p. 534).

§ 1. Introduction

In this section we state the non-steady porous flow well problem and the related results.

1.1. Statement of the problem.

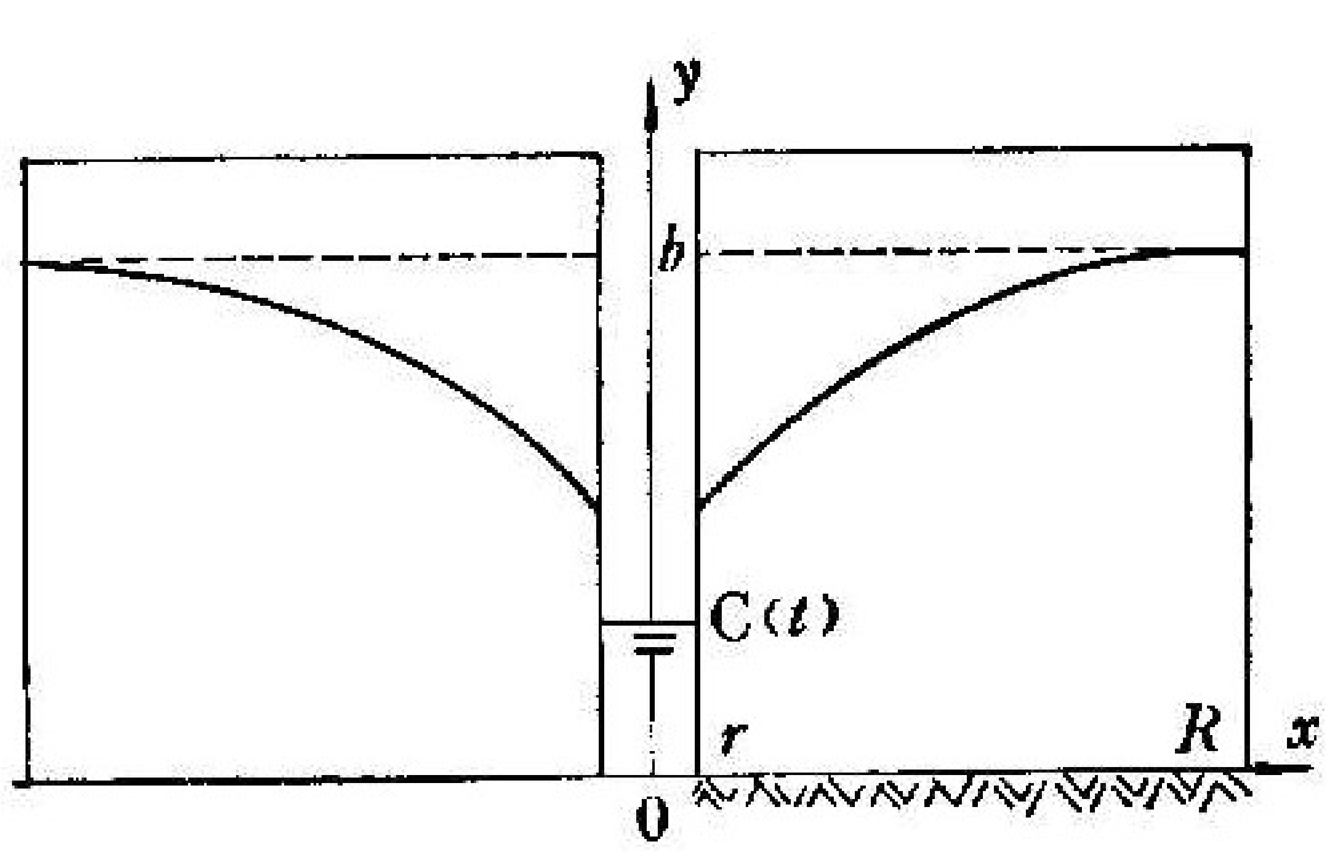

Fig. 1

The non-steady state problem to be considered is shown in Figure 1. An axisymmetric well of radius r is sunk into a soil aquifer of depth b and radius R. The bottom of the aquifer is impervious. The outer boundary of the aquifer adjoins a catchment area and the hydraulic head $u(x, t)$ is equal to the constant b_0 along this boundary. $[0, T]$, with $T>0$, is the time interval during which the filtration process is studied. $C(t)$ is the water level in the well. We assume

* Received June 18, 1984.

that $C(0)=b_0$ and $0<C(t)\leq b_0$, $\forall t\in[0, T]$. The water-air interface is a free boundary. $\varphi(x, t)$ represents the height of the free boundary. We suppose that $u(0, t)=b_0$, $\forall t\in[0, T]$. Finally we assume that the water is incompressible and the porou medium is homogeneous.

The mathematical problem can now be formulated as follows (see [2]):

Problem 1.1. We look for a triplet $\{\varphi, \Omega, u\}$ such that:

i) φ is a *regular* function defined in $[r, R]\times[0, T]$, satisfying

$$\begin{cases} 0<\varphi(x, t)\leq b_0, & \forall(x, t)\in[r, R]\times[0, T], \\ \varphi(r, t)\geq C(t), \ \varphi(R, t)=b_0, & \forall t\in[0, T], \\ \varphi(x, 0)=b_0, & \forall x\in[r, R]; \end{cases} \tag{1.1}$$

ii) Ω is defined by the relation:

$$\Omega=\{(x, y, t);\ r<x<R,\ 0<t<T,\ 0<y<\varphi(x, t)\}; \tag{1.2}$$

iii) u is a *regular* function defined in $\bar{\Omega}$ such that:

$$Eu\equiv(xu_x)_x+xu_{yy}=0, \quad \text{in } \Omega, \tag{1.3}$$

$$\begin{cases} u(r, y, t)=C(t) & \text{if } 0\leq y\leq C(t),\ 0<t\leq T, \\ u(r, y, t)=y & \text{if } C(t)<y\leq\varphi(r, t),\ 0<t\leq T, \\ u(R, y, t)=b_0 & \text{if } 0\leq y\leq b_0,\ 0<t\leq T, \\ u_y(x, 0, t)=0 & \text{if } r<x<R,\ 0<t\leq T. \end{cases} \tag{1.4}$$

On the free boundary

$$\Sigma=\{(x, y, t);\ r<x<R,\ 0<t<T,\ y=\varphi(x, t)\},$$

u satisfies the relations

$$\begin{cases} u(x, y, t)=y, \\ u_x^2+u_y^2-u_y=u_t. \end{cases} \tag{1.5}$$

This problem corresponds to the non-steady rectangular dam problem with stationary initial data. Furthermore we suppose that

$$C(t)\in C^1(0, T), \qquad C'(t)>-1. \tag{1.6}$$

1.2. Formulation as a variational inequality.

In this section we reformulate Problem 1.1 as a variational inequality.

Let

$$D=\{(x, y);\ r<x<R,\ 0<y<b\},$$

$$Q=D\times(0, T),$$

$$\Gamma_1=\{(x, y);\ x=r,\ 0<y<b\},$$

$$\Gamma_2=\{(x, y);\ x=R,\ 0<y<b\},$$

$$\Gamma_n=\{(x, y);\ r<x<R,\ y=0\}.$$

Set

$$\Gamma_d=\partial D\backslash\Gamma_n.$$

We introduce the functions

$$\begin{cases} g_1(y, t) = \int_0^t [C(t) - (y+t-\tau)]^+ d\tau + Z(y+t), \\ g_2(y, t) = \int_0^t [b_0 - (y+t-\tau)]^+ d\tau + Z(y+t), \\ G(x, y, t) = g_1 + \dfrac{x-r}{R-r}(g_2 - g_1), \end{cases} \tag{1.7}$$

where

$$Z(\lambda) = \frac{1}{2}[(b_0-\lambda)^+]^2, \quad \forall \lambda \in \mathbb{R}^1. \tag{1.8}$$

The condition (1.6) ensures that[12]

$$\begin{cases} g_1 \in W^{2,\infty}(\Gamma_1 \times (0, T)), \\ g_2 \in W^{2,\infty}(\Gamma_2 \times (0, T)), \\ G \in W^{2,\infty}(Q). \end{cases} \tag{1.9}$$

We now introduce the spaces, with the graph norms,

$$\begin{cases} W = \{v \in H^1(D): v = 0 \quad \text{on } \Gamma_d\}, \\ V = \{v \in W; \gamma_0 v \in H_0^1(\Gamma_n)\}, \end{cases} \tag{1.10}$$

where γ_0 is the trace operator on Γ_n.

Let

$$\begin{cases} a(u, v) = \int_D x(u_x v_y + u_x v_y)\, dx dy, \\ b(u, v) = \int_{\Gamma_n} x D_x \gamma_0 u \cdot D_x \gamma_0 v\, dx, \\ (u, v) = \int_D u \cdot v\, dx dy. \end{cases} \tag{1.11}$$

We finally introduce the closed convex set

$$K(t) = \{v \in H^1(D);\ v = G \quad \text{on } \Gamma_d\}.$$

We now introduce a variational inequality:

Problem 1.2. Find a function $W(t) \in K(t)$ such that

$$W \in L^\infty(0, T; H^{3/2}(D)),\ W_t \in L^2(0, T; H^1(D)); \tag{1.12}$$

$$\nabla(x \cdot \nabla W(t)) \in L^\infty(0, T; L^2(D)), \tag{1.13}$$

where the gradient is taken with respect to variables (x, y),

$$a(W(t), v - W(t)) + \langle x\gamma_1 W(t), \gamma_0(v - W(t)\rangle + (x, v^+ - W(t)^+) \geqslant 0, \tag{1.14}$$

$$\forall v \in K(t), \text{ a.e., } t \in [0, T],$$

where $\gamma_1 W$ is the trace of $\partial W/\partial y$ on Γ_n, and the brackets $\langle , \rangle$ indicate the duality between $(H_{0,0}^{1/2}(\Gamma_n))'$ and $H_{0,0}^{1/2}(\Gamma_n)$,

$$x D_t \gamma_1 W + D_x(x D_x)\gamma_0 W = 0 \tag{1.15}$$

in the sense of $\mathscr{D}'(0, T; (H_{0,0}^{3/2}(\Gamma_n)')$,

$$\gamma_1 W(0) = q, \tag{1.16}$$

where

$$q = Z'(0) = -b_0.$$

This problem corresponds to Problem W of [7]. We now state the main results about Problem 1.1 and Problem 1.2. Since these results are similar to the corresponding results of [7] and [12], we omit their proof.

Let

$$Q_1=\{(x,\ y,\ t);\ r<x<R,\ 0<t<T,\ 0<y<b+T-t\}.$$

Let $\{\varphi,\ \Omega,\ u\}$ be a solution of Problem 1.1. We introduce the functions

$$\begin{cases} \tilde{u}=\begin{cases} u(x,\ y,\ t) & \text{in } \bar{\Omega}, \\ y & \text{in } \bar{Q}_1\setminus\bar{\Omega}, \end{cases} \\ W=\int_0^t \{\tilde{u}(x,\ y+t-\tau,\ \tau)-(y+t-\tau)\}d\tau+Z(y+t) \quad \text{in } \bar{Q}. \end{cases} \tag{1.17}$$

Theorem 1.3 (See [12] Theorem 2.1). *Let $\{\varphi,\ \Omega,\ u\}$ be a solution of Problem 1.1, and let W be defined by* (1.17); *then*

$$W>0 \quad \text{in } \Omega, \quad W=0 \quad \text{in } Q\setminus\Omega, \tag{1.18}$$

and W satisfies Problem 1.2.

Theorem 1.4 (See [7] Theorems 2.1 and 2.2). *There exists a unique solution of Problem* 1.2.

Theorem 1.3 and Theorem 1.4 show that Problem 1.2 may be considered a weak formulation of Problem 1.1. Moreover Theorem 1.4 gives a uniqueness theorem for Problem 1.1.

1.3. Some regularity properties for the solution W of Problem 1.2.

Theorem 1.5 (See [7] Theorem 4.5).

$$W\in L^2(0,\ T;\ H^2(D)); \qquad W\geqslant 0,\ W_y\leqslant 0 \quad \text{in } Q.$$

Let

$$\zeta=(D_t-D_y)W. \tag{1.9}$$

About the function ζ we have:

Theorem 1.6 (See [7] Theorem 3.3 and Lemma 4.1).

$$\zeta\in L^2(0,\ T;\ H^1(D)); \qquad \zeta\geqslant 0 \quad \text{in } Q.$$

Let

$$\varphi(x,\ t)=\sup\{y;\ W(x,\ y,\ t)>0\}, \qquad r<x<R, \quad 0\leqslant t\leqslant T. \tag{1.20}$$

We now consider two porous flow well problems with data b_0^i, $C^i(t)$ $(i=1,\ 2)$. Denote by $W^i(x,\ y,\ t)$ the solution of Problem 1.2 corresponding to the data b_0^i, $C^i(t)$ and denote by $y=\varphi^i(x,\ t)$ the free boundary corresponding to W^i.

Theorem 1.7 (See [7] Theorem 8.4).

If

$$b_0^1\geqslant b_0^2; \qquad C^1(t)\geqslant C^2(t), \quad 0\leqslant t\leqslant T.$$

then

$$W^1\geqslant W^2 \quad \text{in } \bar{Q},$$

$$D_yW^1\leqslant D_yW^2 \quad \text{in } \bar{Q},$$

$$\varphi^1(x,\ t)\geqslant\varphi^2(x,\ t), \qquad r<x<R,\ 0\leqslant t\leqslant T.$$

From Theorem 1.5 and (1.12), we have

$$W\in L^2(0,\ T;\ H^2(D)),\quad W_t\in L^2(0,\ T;\ H^1(D)).$$

Hence (see [11] Ch. 1, Theorem 3.1)

$$W\in C([0,\ T];\ \ H^{3/2}(D)),\ W\in C(\bar{Q}). \tag{1.21}$$

1.4. Another form of Problem 1.2.

Let

$$\tilde{K}(t)=\{v\in H^1(D);\ \ v=G \text{ on } \Gamma_d,\ \gamma_0 v\in H^1(\Gamma_n)\}.$$

Problem 1.8. Find a function $W(t)\in\tilde{K}(t)$ such that

$$W\in L^\infty(0,\ T;\ H^{3/2}(D)),\ W_t\in L^2(0,\ T;\ H^1(D)). \tag{1.22}$$

$$a(W(t),\ v-W(t))+b\Big(\int_0^t W(\tau)d\tau,\ v-W(t)\Big)+(x,\ v^+-W(t)^+)$$

$$\geqslant-\int_{\Gamma_n} x\cdot q\cdot\gamma_0(v-W(t))dx,\quad \forall v\in\tilde{K}(t),\ \text{a.e. } t\in[0,\ T]. \tag{1.23}$$

Theorem 1.9 (See [12] Theorem 3.2). *Problems* 1.2 *and* 1.8 *are equivalent.*

About Problem 1.8 we prove the following two theorems.

Theorem 1.10. *Let W be the solution of Problem* 1.8, *then* (1.23) *holds for any* $t\in[0,\ T]$.

Proof. From (1.21) we deduce

$$\gamma_0\int_0^t W(t)d\tau\in C([0,\ T];\ H^1(\Gamma_n)). \tag{1.24}$$

Let $v_0\in V$, By (1.9) we have $G(t)\in C([0,\ T];\ H^{3/2}(D))$, $\gamma_0 G\in C([0,\ T];H^1(\Gamma_n))$. Then

$$\begin{cases} v_0+G(t)\in C([0,\ T];\ H^1(D)),\\ \gamma_0(v_0+G(t))\in C([0,\ T];\ H^1(\Gamma_n)).\end{cases} \tag{1.25}$$

We choose $v=v_0+G(t)$ in (1.23). Since (1.23) holds for a.e. $t\in[0,\ T]$, by (1.21), (1.24) and (1.25) we deduce that (1.23) holds for any $t\in[0,\ T]$.

Theorem 1.11. *Let W be the solution of Problem* 1.8, *then*

$$W(0)=Z(y). \tag{1.26}$$

Proof. By Theorem 1.10, $W(0)$ satisfies the following variational inequality

$$\begin{cases} W(0)\in\tilde{K}(0),\\ a(W(0),\ v-W(0))+(x,\ v^+-W(0)^+)\\ \quad\geqslant-\int_{\Gamma_n} x\cdot q\cdot\gamma_0(v-W(0))dx,\quad \forall v\in\tilde{K}(0).\end{cases} \tag{1.27}$$

It is well known that (1.27) has a unique solution (see [10] Ch. 2, Theorem 8.5) on the one hand. On the other hand we can verify directly that the function $Z(y)$ is a solution of (1.27).

§ 2. Some New Regularity Properties for the Solution of Problem 1.8

The main object of this section is to establish some new regularity properties for the solution of Problem 1.8 (cf. Theorem 2.5). They will be used for error estimation. First we need the following lemmas.

Lemma 2.1. *Let W be the solution of Problem 1.8; then*

$$EW = x \cdot H(W) \quad a.e.\ in\ Q \tag{2.1}$$

where H is a Heaviside function.

Proof. Let

$$\Omega = \{(x, y, t) \in Q;\ W(x, y, t) > 0\}. \tag{2.2}$$

We take $\psi \in \mathscr{D}(\Omega)$. Since $W \in C(Q)$, there exists $\lambda_\psi > 0$ such that for each real λ with $|\lambda| \leqslant \lambda_\psi$ one has $W + \lambda_\psi \geqslant 0$ in Q, hence $(W + \lambda_\psi)^+ = W + \lambda_\psi$. We take $v = W + \lambda_\psi$ in (1.23). Then by integration on $[0, T]$ we obtain

$$\int_0^T a(W, \lambda_\psi)dt + \int_0^T (x, \lambda_\psi)dt \geqslant 0,$$

from which, as the sign of λ is arbitrary,

$$\int_0^T a(W, \psi)dt + \int_0^T (x, \psi)dt = 0.$$

By Green's formula we have

$$\int_0^T (-EW + x, \psi)\, dt = 0,$$

i.e.

$$EW = x \qquad \text{a.e. in } \Omega,$$

and we can deduce (2.1).

Lemma 2.2. *Let W be the solution of Problem 1.8, and*

$$C_0 = \inf_{0 \leqslant t \leqslant T} C(t), \tag{2.3}$$

then

$$\begin{cases} W > 0 & if\ y < C_0, \\ W = 0 & if\ y \geqslant b_0. \end{cases} \tag{2.4}$$

Proof. In Theorem 1.7 we take

$$b_0^1 = b_0,\ b_0^2 = C_0,$$

$$C^1(t) = b_0,\ C^2(t) = C_0.$$

It is easy to verify that

$$W^1 = Z(y) = \frac{1}{2}(b_0 - y)^{+2}, \qquad W^2 = Z_0(y) = \frac{1}{2}(C_0 - y)^{+2}.$$

Hence

$$Z_0(y) \leqslant W(t) \leqslant Z(y),$$

from which we can deduce (2.4).

From Lemma 2.2 we have

$$W_y(x, b, t) = 0. \tag{2.5}$$

Lemma 2.3. *Let W be the solution of Problem 1.8, then*

$$W \in L^\infty(0, T; H^2(D_\varepsilon))$$

where $\quad D_\varepsilon = \{(x, y) \in D;\ y > \varepsilon\}, \quad 0 < \varepsilon < b.$

Proof. By (1.21), (2.1), (1.9) and (2.4) we have

$$\begin{cases} W\in C(0,\ T;\ H^1(D)), \\ EW=xH(W)\in L^\infty(Q), \\ W|_{\Gamma_1}=g_1\in L^{2,\infty}(\Gamma_1\times(0,\ T)),\ W|_{\Gamma_2}=g_2\in L^{2,\infty}(\Gamma_2\times(0,\ T)), \\ W=0 \quad \text{if } b_0\leqslant y\leqslant b, \end{cases} \tag{2.6}$$

from which and from usual regularity results (see for instance [11] Ch. 2) we obtain

$$W\in L^\infty(0,\ T;\ H^2(D_s)).$$

Set

$$D_1=\{(x,\ y)\in D;\ y<C_1\}, \tag{2.7}$$

where C_1 is a constant with $0<C_1<C_0$, C_0 being defined by (2.3).

Lemma 2.4. *Let W be the solution of Problem 1.8, then ζ defined by*

$$\zeta=(D_t-D_y)W \tag{2.8}$$

satisfies

$$\zeta\in C^\infty(\overline{D}_1),\quad a.e.\ t\in[0,\ T], \tag{2.9}$$

$$\zeta\in L^2(0,\ T;\ H^K(D_1)) \tag{2.10}$$

for any integer $K\geqslant 2$.

Proof. From (2.1) we have

$$EW=x,\quad \text{a.e. in } \Omega.$$

Hence

$$E\zeta=0\quad \text{in the sense of } \mathscr{D}'(\Omega).$$

Since $\zeta\in L^2(0,\ T;\ H^1(D))$ (see Theorem 1.6) we deduce for a.e. $t\in[0,\ T]$

$$E\zeta=0\quad \text{in the sense of } \mathscr{D}'(\Omega(t)). \tag{2.11}$$

where

$$\Omega(t)=\{(x,\ y)\in D;\ W(x,\ y,\ t)>0\}. \tag{2.12}$$

Let γ_0^i be the trace operator on $\Gamma_i(i=1,\ 2)$. Since $W|_{\Gamma_i}=g_i$ where g_i is defined by (1.7), we deduce (cf. [7] Lemma 3.1) in the sense of $L^2(0,\ T;\ H^{1/2}(\Gamma_i))$

$$\gamma_0^i\zeta=\gamma_0^i(D_t-D_y)W=(D_t-D_y)\gamma_0^iW=(D_t-D_y)g_i.$$

Hence (see (1.7))

$$\begin{cases} \gamma_0^1\zeta=[C(t)-y]^+ \quad \text{in the sense of } L^2(0,\ T;\ H^{1/2}(\Gamma_1)), \\ \gamma_0^2\zeta=[b_0-y]^+ \quad \text{in the sense of } L^2(0,\ T;\ H^{1/2}(\Gamma_2)). \end{cases} \tag{2.13}$$

From (2.1) we have

$$EW=x\quad \text{a.e. in } D_1\times(0,\ T).$$

Hence

$$D_y\zeta=D_y(D_t-D_y)W=D_tD_yW+\frac{1}{x}D_x(xD_xW)-1\quad \text{a.e. in } D_1\times(0,\ T),$$

from which and (1.15) we conclude (cf. [7] Lemma 3.1)

$$\gamma_1\zeta=\gamma_0(D_y\zeta)=D_t\gamma_1 W+\frac{1}{x}D_x(xD_x\gamma_0 W)-1$$
$$=-1 \quad \text{in the sense of } L^2(0,\ T;\ (H^{1/2}_{00}(\Gamma_n)'). \tag{2.14}$$

From (2.11), (2.13) and (2.14) we obtain that for a.e. $t\in[0,\ T]$ the function ζ satisfies

$$\begin{cases} E\zeta=0 \quad \text{in the sense of } \mathscr{D}'(\Omega(t)), \\ \gamma_0^1\zeta=[C(t)-y]^+ \quad \text{in the sense of } H^{1/2}((0,\ b)), \\ \gamma_0^2\zeta=[b_0-y]^+ \quad \text{in the sense of } H^{1/2}((0,\ b)), \\ \gamma_1\zeta=-1 \quad \text{in the sense of } (H^{1/2}_{00}(\Gamma_n))'. \end{cases} \tag{2.15}$$

We now introduce the function

$$\xi=\zeta+y. \tag{2.16}$$

Then for a.e. $t\in[0,\ T]$ the function ξ satisfies

$$\begin{cases} \xi\in H^1(D_0), \\ E\xi=0 \quad \text{in the sense of } \mathscr{D}'(D_0), \\ \gamma_0^1\xi=C(t),\ \gamma_0^2\xi=b_0 \quad \text{in the sense of } H^{1/2}((0,\ C_0)), \\ \gamma_1\xi=0 \quad \text{in the sense of } (H^{1/2}_{00}(\Gamma_n))', \end{cases} \tag{2.17}$$

where

$$D_0=\{(x,\ y)\in D;\ y<C_0\}, \tag{2.18}$$

and C_0 is defined by (2.3).

We now extend ξ across $y=0$ by reflection and denote the extended function still by ξ. Then from usual regularity results (see e.g. [8] Sections 8.3 and 8.4) we can conclude that for a.e. $t\in[0,\ T]$

$$\begin{cases} \xi\in C(D_1), \\ \|\xi\|_{H^K(D_1)}\leqslant C_1\|\xi\|_{H^1(D_0)}+C_2 \end{cases} \tag{2.19}$$

for any integer $K\geqslant 2$, where C_1 and C_2 are constants independent of t. From this we have

$$\begin{cases} \xi\in C^\infty(D_1), \quad \text{a.e. } t\in[0,\ T], \\ \xi\in L^2(0,\ T;\ H^K(D_1)). \end{cases}$$

Hence

$$\begin{cases} \xi\in C^\infty(\overline{D}_1), \quad \text{a.e. } t\in[0,\ T], \\ \zeta\in L^2(0,\ T;\ H^K(D_1)). \end{cases}$$

Theorem 2.5. *Let W be the solution of Problem 1.8, then*

$$W\in L^\infty(0,\ T;\ H^2(D)), \tag{2.20}$$
$$\gamma_0 W\in L^\infty(0,\ T;\ H^2(\Gamma_n)),\ D_t\gamma_0 W\in L^2(0,\ T;\ H^1(\Gamma_n)). \tag{2.21}$$

Proof. i) Proof of (2.20). By Lemma 2.3 it is sufficient to prove

$$W\in L^\infty(0,\ T;\ H^2(D_2)), \tag{2.22}$$

where

$$D_2=\{(x,\ y)\in D;\ y<C_2\}, \tag{2.23}$$

and C_2 is a constant with $0<C_2<C_1/2$.

Let

$$\tilde{D}=\{(x, y);\ r<x<R,\ 0<y<b+T\},\quad \tilde{Q}=\tilde{D}\times(0, T).$$

We introduce the function

$$\tilde{\zeta}=\begin{cases}\zeta(x, y, t) & \text{if } (x, y, t)\in\bar{Q},\\ 0 & \text{if } (x, y, t)\in\bar{\tilde{Q}}\setminus\bar{Q},\end{cases}\tag{2.24}$$

where ζ is defined by (2.8). By (2.4) we have $\zeta=0$ if $b_0\leqslant y\leqslant b$. Using this and Theorem 1.6, we deduce that

$$\tilde{\zeta}\in L^2(0, T;\ H^1(\tilde{D})).\tag{2.25}$$

Let us now set

$$W^*=\int_0^t\tilde{\zeta}(x,\ y+t-\tau,\ \tau)d\tau+Z(y+t)\quad \text{in } \bar{Q}.$$

Since $(D_t-D_y)W^*=(D_t-D_y)W=\zeta$ in Q and since

$$W^*|_{t=0}=W|_{t=0}=Z(y),\quad W^*|_{y=b}=W|_{y=b}=0,$$

it is deduced that

$$W=W^*=\int_0^t\tilde{\zeta}(x,\ y+t-\tau,\ \tau)d\tau+Z(y+t)\quad \text{in } \bar{Q}.\tag{2.26}$$

If $(x, y)\in D_2$ and $t>C_2$, then

$$W=\int_0^{t-C_2}\tilde{\zeta}(x,\ y+t-\tau,\ \tau)d\tau+\int_{t-C_2}^t\tilde{\zeta}(x,\ y+t-\tau,\ \tau)d\tau+Z(y+t)$$

$$=W(x,\ y+C_2,\ t-C_2)+\int_{t-C_2}^t\tilde{\zeta}(x,\ y+t-\tau,\ \tau)d\tau.\tag{2.27}$$

Since $0\leqslant y\leqslant y+t-\tau\leqslant y+C_2<C_1$ if $(x, y)\in D_2$ and $t-C_2\leqslant\tau\leqslant t$, we have

$$\left\|\int_{t-C_2}^t\tilde{\zeta}(x,\ y+t-\tau,\ \tau)d\tau\right\|_{C^2(D_2)}\leqslant\int_{t-C_2}^t\|\zeta(x,\ y+t-\tau,\ \tau)\|_{C^2(D_1)}d\tau,\tag{2.28}$$

$$\int_{t-C_2}^t\|\zeta(x,\ y,\ \tau)\|_{C^2(D_1)}d\tau\leqslant C\|\zeta\|_{L^2(0,T;C^2(D_1))}.$$

Taking $K=4$ in (2.10) and using $H^{2+2}(D_1)\subsetneq C^2(\bar{D}_1)$, we obtain for a.e. $t\in(C_2,\ T)$

$$\left\|\int_{t-C_2}^t\tilde{\zeta}(x,\ y+t-\tau,\ \tau)d\tau\right\|_{C^2(D_2)}\leqslant C\cdot\|\zeta\|_{L^2(0,T;H^4(D_1))}.$$

Hence on the one hand, we have

$$\int_{t-C_2}^t\tilde{\zeta}(x,\ y+t-\tau,\ \tau)d\tau\in L^\infty(C_2,\ T;\ H^2(D_2)).\tag{2.29}$$

On the other hand by Lemma 2.3 we have

$$W(x,\ y+C_2,\ t-C_2)\in L^\infty(C_2,\ T;\ H^2(D_2)).\tag{2.30}$$

From (2.29), (2.30) and (2.27) we get

$$W\in L^\infty(C_2,\ T;\ H^2(D_2)).\tag{2.31}$$

If $(x, y)\in D_2$ and $t<C_2$ then

$$W=\int_0^t\tilde{\zeta}(x,\ y+t-\tau,\ \tau)d\tau+Z(y+t).$$

Analogously to (2.29) we have

$$\int_0^t \tilde{\zeta}(x,\ y+t-\tau,\ \tau)d\tau \in L^\infty(0,\ C_2;\ H^2(D_2)).$$

Hence

$$W \in L^\infty(0,\ C_2;\ H^2(D_2)). \tag{2.32}$$

From (2.31), (2.32) we have

$$W \in L^\infty(0,\ T;\ H^2(D_2)).$$

ii) Proof of (2.21). Let

$$\begin{cases} D_{03} = \{(x,\ y) \in D;\ C_3 < y < C_0\}, \\ D_{12} = \{(x,\ y) \in D;\ C_2 < y < C_1\}, \end{cases} \tag{2.33}$$

where C_3 is a constant with $0 < C_3 < C_2$ (cf. (2.3), (2.7) and (2.23)).

By (2.20), (2.1) and (1.9) we have

$$\begin{cases} W \in L^\infty(0,\ T;\ H^2(D_{03})), \\ EW = x \quad \text{a.e. in } D_{03} \times (0,\ T), \\ \gamma_0^1 W = g_1 \in L^\infty(0,\ T;\ H^2(\Gamma_1)), \\ \gamma_0^2 W = g_2 \in L^\infty(0,\ T;\ H^2(\Gamma_2)). \end{cases} \tag{2.34}$$

From usual regularity results we conclude (see e.g. [11] Theorem 5.1)

$$W \in H^{5/2}(D_{12}),\quad \text{a.e. } t \in [0,\ T],$$

and

$$\|W\|_{H^{5/2}(D_{12})} \leqslant C\{\|EW\|_{H^{1/2}(D_{03})} + \|\gamma_0^1 W\|_{H^2(\Gamma_1^*)} + \|\gamma_0^2 W\|_{H^2(\Gamma_2^*)} + \|W\|_{H^2(D_{03})}\},\quad \text{a.e. } t \in [0,\ T],$$

where C is a constant independent of t, $\Gamma_1^* = \Gamma_1 \cap \partial D_{03}$, $\Gamma_2^* = \Gamma_2 \cap \partial D_{03}$. Hence

$$W \in L^\infty(0,\ T;\ H^{5/2}(D_{12})). \tag{2.35}$$

For $t > C_2$ we set

$$W^* = \int_{y+t-C_2}^t \zeta(x,\ y+t-\tau,\ \tau)d\tau + W(x,\ C_2,\ y+t-C_2) \text{ if } (x,\ y) \in \overline{D}_2.$$

It is clear that $(D_t - D_y)W^* = \zeta$. Hence

$$(D_t - D_y)W = (D_t - D_y)W^* = \zeta \quad \text{if } (x,\ y) \in \overline{D}_2,\ t > C_2.$$

Since $W(x,\ C_2,\ t) = W^*(x,\ C_2,\ t)$ we deduce

$$W = W^* = \int_{y+t-C_2}^t \zeta(x,\ y+t-\tau,\ \tau)d\tau + W(x,\ C_2,\ y+t-C_2) \quad \text{if } (x,\ y) \in \overline{D}_2,\ t > C_2. \tag{2.36}$$

For $t < C_2$ we set

$$W^* = \int_0^t \zeta(x,\ y+t-\tau,\ \tau)d\tau + W(x,\ y+t,\ 0) \quad \text{if } (x,\ y) \in \overline{D}_2.$$

Analogously to (2.36) we have

$$W = W^* = \int_0^t \zeta(x,\ y+t-\tau,\ \tau)d\tau + W(x,\ y+t,\ 0) \text{ if } (x,\ y) \in \overline{D}_2,\ t < C_2. \tag{2.37}$$

Taking $y=0$ in (2.36), (2.37) we get

$$W(x,\ 0,\ t)=\begin{cases}\int_{t-C_2}^{t}\zeta(x,\ t-\tau,\ \tau)d\tau+W(x,\ C_2,\ t-C_2) & \text{if } C_2<t<T,\\ \int_0^t\zeta(x,\ t-\tau,\ \tau)d\tau+W(x,\ t,\ 0) & \text{if } 0<t<C_2.\end{cases} \tag{2.38}$$

Analogously to the proof of (2.29) we have

$$\int_{t-C_2}^{t}\zeta(x,\ t-\tau,\ \tau)d\tau\in L^\infty(C_2,\ T;\ H^2(r,\ R)). \tag{2.39}$$

By (2.35) we have

$$W(x,\ C_2,\ t-C_2)\in L^\infty(C_2,\ T;\ H^2(r,\ R)). \tag{2.40}$$

From (2.39), (2.40) we get

$$W(x,\ 0,\ t)\in L^\infty(C_2,\ T;\ H^2(\Gamma_n)).$$

Similarly we have

$$W(x,\ 0,\ t)\in L^\infty(0,\ C_2;\ H^2(\Gamma_n)).$$

Hence

$$W(x,\ 0,\ t)\in L^\infty(0,\ T;\ H^2(\Gamma_n)).$$

Similarly one can prove

$$D_tW(x,\ 0,\ t)\in L^2(0,\ T;\ H^1(\Gamma_n)),$$

and (2.21) is proved.

Theorem 2.6. *Let W be the solution of Problem* 1.8, *then for any $t\in[0,\ T]$ the function W satisfies*

$$EW=xH(W)\quad a.e.\ in\ D, \tag{2.41}$$

$$W=G\quad on\ \Gamma_d, \tag{2.42}$$

$$\int_{\Gamma_n}x\gamma_1W\cdot\gamma_0v\,dx=b\Big(\int_0^tW(\tau)d\tau,\ v\Big)+\int_{\Gamma_n}x\cdot q\cdot\gamma_0v\,dx,\quad \forall v\in V. \tag{2.43}$$

Proof. By Theorem 1.10 we have for any $t\in[0,\ T]$

$$a(W,\ v-W)+b\Big(\int_0^tWd\tau,\ v-W\Big)+(x,\ v^+-W^+)$$
$$\geqslant-\int_{\Gamma_n}x\cdot q\cdot\gamma_0(v-W)dx,\quad \forall v\in\tilde{K}(t). \tag{2.44}$$

Let

$$\Omega(t)=\{(x,\ y)\in D;\ W(x,\ y,\ t)>0\}. \tag{2.45}$$

We take $\psi\in\mathscr{D}(\Omega(t))$. Analogously to the proof of Lemma 2.1 we have

$$(-EW+x,\ \psi)=0.$$

Hence for any $t\in[0,\ T]$

$$EW=x\quad \text{in the sense of } \mathscr{D}'(\Omega(t)). \tag{2.46}$$

Since $W\in L^\infty(0,\ T;\ H^2(D))\cap C(Q)$ we can deduce (cf. Lemma 3.3)

$$W(t)\in H^2(D)\quad \text{for any } t\in[0,\ T]. \tag{2.47}$$

From (2.46), (2.47) we have for any $t\in[0,\ T]$

$$EW = x \quad \text{a.e. in } \Omega(t).$$

Hence for any $t \in [0, T]$

$$EW = xH(W) \quad \text{a.e. in } D.$$

We now prove the assertion (2.43).

By (1.21) and (2.41) we have

$$\begin{cases} W \in C([0, T]; H^{3/2}(D)), \\ EW \in L^\infty(D), \quad \forall t \in [0, T]. \end{cases}$$

Then it is deduced (by [11] Ch. 1, Theorem 7.3) that

$$\gamma_1 W \in C([0, T]; L^2(D)). \tag{2.48}$$

It is clear that on the one hand

$$\int_0^t \gamma_0 W(\tau) d\tau \in C([0, T]; H^1(\Gamma_n)). \tag{2.49}$$

On the other hand, from (1.15) and (1.16) we can deduce (cf. [12] (3.25)) that for a.e. $t \in [0, T]$

$$\int_{\Gamma_n} x \cdot \gamma_1 W \cdot \gamma_0 v \, dx = b\left(\int_0^t W(\tau) d\tau, \ v\right) + \int_{\Gamma_n} x \cdot q \cdot \gamma_0 v \, dx, \quad \forall v \in V. \tag{2.50}$$

The assertion (2.43) now follows from (2.50), (2.48) and (2.49).

Finally the assertion (2.42) follows from (1.21).

§ 3. Numerical Approximation

In this section a finite element method is used to obtain numerical approximation of Problem 1.8, and the error estimate is obtained.

Let $\{T_h\}$ be a regular family of triangulations of D, where $h = \max_{U \in T_h} \dim(U)$. Let $K = \frac{T}{N}$, where N is a positive integer. We denote by Π_h the linear-interpolation operator. For a continuous function $v(x, y, t)$ defined in Q, we set

$$\begin{cases} v^i = v(iK), \ v_I^i = \Pi_h v(iK) \quad \text{for } i = 0, 1, \cdots, N, \\ v_{I,K}(t) = v_I^i \quad \text{if } t \in ((i-1)K, iK), \ i = 0, 1, \cdots, N. \end{cases} \tag{3.1}$$

We put now

$$\begin{cases} X_h = \{v_h; \ v_h \in C(\overline{D}), \ v_h|_K \in P_1, \ \forall K \in T_h\}, \\ K_h^i = \{v_h^i \in X_h; \ v_h^i = G_I^i \ \text{ on } \Gamma_d, \ v_h^i \geqslant 0 \text{ in } \overline{D}\}, \end{cases} \tag{3.2}$$

where P_1 is the set of all polynomials in two variables of degree $\leqslant 1$ and $G_I^i = \Pi_h G(iK)$. We set

$$v_{h,K}(t) = v_h^i \quad \text{if } t \in ((i-1)K, iK), \ i = 0, 1, \cdots, N, \tag{3.3}$$

where $v_h^i \in K_h^i$ $(i = 0, 1, \cdots, N)$.

We now consider the following discrete problem:

Problem 3.1. Find a sequence $W_h^i \in K_h^i$ $(i = 0, 1, \cdots, N)$, such that

$$W_h^0 = \Pi_h Z(y), \tag{3.4}$$

$$a(W_h^i,\ v_h^i-W_h^i)+Kb(W_h^i,\ v_h^i-W_h^i)+(x,\ v_h^i-W_h^i)$$
$$\geqslant -Kb\Big(\sum_{j=1}^{i-1}W_h^j,\ v_h^i-W_h^i\Big)-\int_{\Gamma_n}x\cdot q\cdot\gamma_0(v_h^i-W_h^i)dx,\quad \forall v_h^i\in K_h^i \tag{3.5}$$

where $i=1,\ 2,\ \cdots,\ N$.

Theorem 3.2. *Problem* 3.1 *has a unique solution.*

Proof. Consider the space

$$\tilde{V}=\{v\in H^1(D);\ \gamma_0 v\in H^1(\Gamma_n)\}$$

with the graph norm. The convex set K_h^i is closed in $\tilde{V}$ and the bilinear form

$$a(W,\ u)+Kb(W,\ u)$$

is coercive in $\tilde{V}$. Hence by applying inductively (see e.g. [10] Ch. 2, Theorem 8.5) we find that there exists one and only one solution W_h^i of Problem 3.1.

We now derive an error estimate for $\Big\{\sum_{i=1}^{N}\|W^i-W_h^i\|^2_{H^1(D)}K\Big\}^{1/2}$. To do this we need the following lemma:

Lemma 3.3. *If* $v\in L^\infty(0,\ T;\ H^2(D))\cap C(\bar{Q})$, *then*

$$v(t)\in H^2(D),\qquad \forall t\in[0,\ T]. \tag{3.6}$$

Moreover

$$\|v^i-v_I^i\|_{H^l(D)}\leqslant C\cdot h^{2-l}\|v\|_{L(0,T;H^2(D))},\qquad \forall 0\leqslant i\leqslant N,\ l=0,\ 1. \tag{3.7}$$

Proof. Take $t\in[0,\ T]$. Since $v\in L^\infty(0,\ T;\ H^2(D))$ then there exists $t\in[0,T]$ such that $t_\alpha\to t$ and $v(t_\alpha)\in H^2(D)$.

Clearly we have

$$\|v(t_\alpha)\|_{H^2(D)}\leqslant\|v\|_{L(0,T;H^2(D))},\qquad \forall\alpha,$$

and so we can on the one hand extract a subsequence of α, still denoted by α, such that

$$v(t_\alpha)\to v^*\quad \text{in } H^2(D) \text{ weak.} \tag{3.8}$$

On the other hand, since $v\in C(\bar{Q})$ we have

$$v(t_\alpha)\to v(t)\quad \text{in } L^2(D). \tag{3.9}$$

Comparing (3.8) with (3.9) we find

$$v(t)=v^*\in H^2(D).$$

This establishes (3.6). The assertion (3.7) follows from (3.6) and Theorem 3.2.1 in [4].

We now prove the main theorem of this paper.

Theorem 3.4. *Let* W *be the solution of Problem* 1.8 *and* W_h^i $(i=0,\ 1,\ \cdots,\ N)$ *be the solution of Problem* 3.1; *then*

$$\Big\{\sum_{i=1}^{N}\|W^i-W_h^i\|_{H^1(D)}K\Big\}^{1/2}\leqslant C(h+K^{1/2}). \tag{3.10}$$

where C *is a constant independent of* $h,\ K$.

Proof. We consider the relation

$$a(W^i-W_h^i,\ W^i-W_h^i)=a(W^i-W_h^i,\ W^i-W_I^i)+a(W^i,\ W_I^i-W_h^i)$$
$$-a(W_h^i,\ W_I^i-W_h^i). \tag{3.11}$$

By Green's formula we have

$$a(W^i,\ W_I^i-W_h^i)=-\int_{\Gamma_n}x\gamma_1 W^i\gamma_0(W_I^i-W_h^i)dx+(-EW^i,\ W_I^i-W_h^i). \tag{3.12}$$

Take $v_h^i = W_I^i$ in (3.5). Then

$$-a(W_h^i, W_I^i - W_h^i) \leqslant (x, W_I^i - W_h^i) + Kb\left(\sum_{j=1}^{i} W_h^j, W_I^i - W_h^i\right) + \int_{\Gamma_n} x\cdot q\cdot\gamma_0(W_I^i - W_h^i)dx. \tag{3.13}$$

Substituting (3.12), (3.13) into (3.11), we obtain

$$\begin{aligned}a(W^i - W_h^i, W^i - W_h^i) \leqslant a(W^i - W_h^i, W^i - W_I^i) + (-EW^i + x, W_I^i - W_h^i)\\ + Kb\left(\sum_{j=1}^{i} W_h^j, W_I^i - W_h^i\right) - \int_{\Gamma_n} x\gamma_1 W^i \gamma_0(W_I^i - W_h^i)dx\\ + \int_{\Gamma_n} x\cdot q\cdot\gamma_0(W_I^i - W_h^i)dx, \quad i = 1, 2, \cdots, N.\end{aligned} \tag{3.14}$$

By (2.43) we have

$$-\int_{\Gamma_n} x\gamma_1 W^i\cdot\gamma_0(W_I^i - W_h^i)dx + \int_{\Gamma_n} x\cdot q\cdot\gamma_0(W_I^i - W_h^i)dx = -b\left(\int_0^{iK} W(\tau)d\tau, W_I^i - W_h^i\right). \tag{3.15}$$

By (2.41) we have $-EW^i + x \geqslant 0$, $(-EW^i + x, W^i) = 0$. From this we obtain

$$\begin{aligned}(-EW^i + x, W_I^i - W_h^i) = (-EW^i + x, W_I^i - W^i) - (-EW^i + x, W_h^i)\\ + (-EW^i + x, W^i) \leqslant (-EW^i + x, W_I^i - W^i).\end{aligned} \tag{3.16}$$

It is clear that

$$Kb\left(\sum_{j=1}^{i} W_h^j, W_I^i - W_h^i\right) = b\left(\int_0^{iK} W_{h,K}(\tau)d\tau, W_I^i - W_h^i\right), \tag{3.17}$$

Substituting (3.15), (3.16) and (3.17) into (3.14), we obtain

$$\begin{aligned}a(W^i - W_h^i, W^i - W_h^i) \leqslant a(W^i - W_h^i, W^i - W_I^i) + (-EW^i + x, W_I^i - W^i)\\ + b\left(\int_0^{iK} (W_{h,K} - W)d\tau, W_I^i - W_h^i\right), \quad i = 1, 2, \cdots, N.\end{aligned} \tag{3.18}$$

Multiplying (3.18) by K and summing over i, we obtain

$$\begin{aligned}&\sum_{i=1}^{l} a(W^i - W_h^i, W^i - W_h^i)K\\ &\leqslant \sum_{i=1}^{l} a(W^i - W_h^i, W^i - W_I^i)K + \sum_{i=1}^{l}(-EW^i + x, W_I^i - W_h^i)K\\ &\quad + \sum_{i=1}^{l} b\left(\int_0^{iK} (W_{h,K} - W)d\tau, W_I^i - W_h^i\right)K = S_1 + S_2 + S_3, \quad 1 \leqslant l \leqslant N.\end{aligned} \tag{3.19}$$

By (1.21), (2.20) we have $W \in L^\infty(0, T; H^2(D)) \cap C(Q)$ and then by Lemma 3.3 we obtain

$$\begin{cases}\|W^i - W_I^i\|_{L^2(D)} \leqslant Ch^2 \|W\|_{L^\infty(0,T;H^2(D))},\\ \|W^i - W_I^i\|_{H^1(D)} \leqslant Ch \|W\|_{L^\infty(0,T;H^2(D))}, \quad i = 1, 2, \cdots, N.\end{cases} \tag{3.20}$$

Using this we can deduce that

$$S_1 \leqslant \frac{1}{2}\sum_{i=1}^{l} a(W^i - W_h^i, W^i - W_h^i)K + Ch^2 \|W\|^2_{L^\infty(0,T;H^2(D))}, \tag{3.21}$$

$$S_2\leqslant Ch^2\|W\|_{L^\infty(0,T;H^2(D))}. \tag{3.22}$$

Substituting (3.21), (3.22) into (3.19) and then applying the Poincare inequality we obtain

$$\sum_{i=1}^{l}\|W^i-W_h^i\|^2_{H^1(D)}K\leqslant Ch^2+S_3, \tag{3.23}$$

where

$$\begin{aligned}S_3&=\sum_{i=1}^{l}b\left(\int_0^{iK}(W_{h,K}-W)d\tau,\ W_I^i-W_h^i\right)K\\&=\sum_{i=1}^{l}b\left(\int_0^{iK}(W-W_{h,K})d\tau,\int_{(i-1)K}^{iK}(W-W_{I,K})d\tau\right)\\&\quad-\sum_{i=1}^{l}b\left(\int_0^{iK}(W-W_{h,K})d\tau,\int_{(i-1)K}^{iK}(W-W_{h,K})d\tau\right)=S_3^1-S_3^2.\end{aligned}$$

After simple calculation we obtain

$$\begin{aligned}S_3^2&=b\left(\int_0^{lK}(W-W_{h,K})d\tau,\int_0^{lK}(W-W_{h,K})\,d\tau\right)\\&\quad-\sum_{i=2}^{l}b\left(\int_{(i-1)K}^{iK}(W-W_{h,K})d\tau,\int_0^{(i-1)K}(W-W_{h,K})d\tau\right)\\&\geqslant b\left(\int_0^{lK}(W-W_{h,K})d\tau,\int_0^{lK}(W-W_{h,K})d\tau\right)-S_3^2.\end{aligned}$$

Consequently,

$$\begin{aligned}S_3^2&\geqslant\frac{1}{2}b\left(\int_0^{lK}(W-W_{h,K})\,d\tau,\int_0^{lK}(W-W_{h,K})d\tau\right)\\&\geqslant\alpha\left\|\int_0^{lK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)},\end{aligned} \tag{3.24}$$

where α is a constant. On the other hand we have

$$\begin{aligned}S_3^1&\leqslant\frac{\alpha}{2T}\sum_{i=1}^{l}\left\|\int_0^{iK}\gamma_0(W-W_{h,K})d\tau\right\|_{H^1(\Gamma_n)}K\\&\quad+\frac{C}{K}\sum_{i=1}^{l}\left\|\int_{(i-1)K}^{iK}\gamma_0(W-W_{I,K})d\tau\right\|^2_{H^1(\Gamma_n)},\end{aligned} \tag{3.25}$$

where C is a constant. Substituting (3.24), (3.25) into (3.23) we obtain

$$\begin{aligned}&\sum_{i=1}^{l}\|W^i-W_h^i\|^2_{H^1(D)}K+\alpha\left\|\int_0^{lK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)}\\&\leqslant\frac{\alpha}{2T}\sum_{i=1}^{l}\left\|\int_0^{iK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)}K\\&\quad+\frac{C}{K}\sum_{i=1}^{l}\left\|\int_{(i-1)K}^{iK}\gamma_0(W-W_{I,K})\,d\tau\right\|^2_{H^1(\Gamma_n)}+Ch^2,\quad\forall 1\leqslant l\leqslant N.\end{aligned} \tag{3.26}$$

Hence

$$\begin{aligned}&\alpha\left\|\int_0^{lK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)}\\&\leqslant\frac{\alpha}{2T}\sum_{i=1}^{N}\left\|\int_0^{iK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)}K\\&\quad+\frac{C}{K}\sum_{i=1}^{N}\left\|\int_{(i-1)K}^{iK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)}+Ch^2,\quad\forall 1\leqslant l\leqslant N.\end{aligned} \tag{3.27}$$

Multiplying (3.27) by K and summing over l, we obtain

$$\alpha\sum_{i=1}^{N}\left\|\int_0^{iK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)}K$$
$$\leqslant\frac{\alpha}{2}\sum_{i=1}^{N}\left\|\int_0^{iK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)}K$$
$$+\frac{CT}{K}\sum_{i=1}^{N}\left\|\int_{(i-1)K}^{iK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)}+CTh^2.$$

Consequently,

$$\sum_{i=1}^{N}\left\|\int_0^{iK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)}K$$
$$\leqslant\frac{C}{K}\sum_{i=1}^{N}\left\|\int_{(i-1)K}^{iK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)}+Ch^2. \tag{3.28}$$

Taking $l=N$ in (3.26) and then substituting (3.28) into (3.26), we obtain

$$\sum_{i=1}^{N}\|W^i-W^i_h\|^2_{H^1(D)}K$$
$$\leqslant\frac{C}{K}\sum_{i=1}^{N}\left\|\int_{(i-1)K}^{iK}\gamma_0(W-W_{h,K})d\tau\right\|^2_{H^1(\Gamma_n)}+Ch^2. \tag{3.29}$$

It is clear that

$$\left\|\int_{(i-1)K}^{iK}\gamma_0(W-W_{h,K})d\tau\right\|_{H^1(\Gamma_n)}\leqslant\left\|\int_{(i-1)K}^{iK}\gamma_0(W-W^i)d\tau\right\|_{H^1(\Gamma_n)}$$
$$+\left\|\int_{(i-1)K}^{iK}\gamma_0(W-W_{h,K})d\tau\right\|_{H^1(\Gamma_n)}. \tag{3.30}$$

By (2.21) we have

$$\left\|\int_{(i-1)K}^{iK}\gamma_0(W-W^i)d\tau\right\|_{H^1(\Gamma_n)}\leqslant\int_{(i-1)K}^{iK}\int_{\tau}^{iK}\|D_t\gamma_0W\|_{H^1(\Gamma_n)}dt\,d\tau$$
$$\leqslant\int_{(i-1)K}^{iK}(iK-\tau)^{1/2}d\tau\|D_t\gamma_0W\|_{L^2(0,T;H^1(\Gamma_n))}$$
$$=K^{3/2}\|D_t\gamma_0W\|_{L^2(0,T;H^1(\Gamma_n))}. \tag{3.31}$$

On the other hand by (2.21) we obtain $\gamma_0W\in L^\infty(0,T;H^2(\Gamma_n))\cap C(\overline{\Gamma}_n\times[0,T])$. Analogously to (3.7) we have

$$\|\gamma_0(W^i-W^i_I)\|_{H^1(\Gamma_n)}\leqslant Ch\|\gamma_0W\|_{L^\infty(0,T;H^2(\Gamma_n))}. \tag{3.32}$$

Hence

$$\left\|\int_{(i-1)K}^{iK}\gamma_0(W^i-W^i_I)d\tau\right\|_{H^1(\Gamma_n)}\leqslant\int_{(i-1)K}^{iK}\|\gamma_0(W^i-W^i_I)\|_{H^1(\Gamma_n)}d\tau$$
$$\leqslant Ch\|\gamma_0W\|_{L^\infty(0,T;H^2(\Gamma_n))}. \tag{3.33}$$

Substituting (3.31), (3.33) into (3.30) we obtain

$$\left\|\int_{(i-1)K}^{iK}\gamma_0(W-W_{h,K})d\tau\right\|_{H^1(\Gamma_n)}\leqslant C(K^{3/2}+Kh),\quad\forall 1\leqslant i\leqslant N. \tag{3.34}$$

From (3.29), (3.34) we deduce

$$\sum_{i=1}^{N}\|W^i-W^i_h\|^2_{H^1(D)}K\leqslant C(h^2+K).$$

This completes the proof of the theorem.

§ 4. A Numerical Example

Let

$$J^i(v)=\frac{1}{2}(a(v,\ v)+Kb(v,\ v))+(x,\ v)+Kb\left(\sum_{j=1}^{i-1}W_h^j,\ v\right)$$
$$+\int_{\Gamma_n}x\cdot q\cdot\gamma_0 v\,dx,\quad i=1,\ 2,\ \cdots,\ N. \tag{4.1}$$

In view of Theorem 1.2 in [9], Problem 3.1 is equivalent to the following minimization problem:

Problem 4.1. Find a sequence $W_h^i\in K_h^i$ $(i=0,\ 1,\ \cdots,\ N)$, such that

$$W_h^0=\Pi_h Z(y), \tag{4.2}$$

$$J^i(W_h^i)=\min_{v\in K_h^i}J^i(v),\quad i=1,\ 2,\ \cdots,\ N. \tag{4.3}$$

(4.3) can be written in the following form:

$$J^i(W_h^i)=\min_{v\in K_h^i}\{V^TAV+2B^TV\}, \tag{4.4}$$

where A is a known $M\times M$ matrix, B is a known M-vector, M is the dimension of X_h, and V is an M-vector $(v_{p,q})$, $v_{p,q}$ denoting the values of v at the gridpoint $(x_p,\ y_q)$.

The solution W_h^i $(i=0,\ 1,\ \cdots,\ N)$ of Problem 4.1 was computed using S.O.R. with projection (see e.g. [5]).

As an example we consider the specific case $r=4.8$, $R=76.8$, $b_0=48$, because it was previously considered for the stationary case by several authors (see e.g. [6]). We note that in [6] the stationary water level in the well is equal to 12. We now set

$$C(t)=\begin{cases}-0.36t+48 & \text{if } 0\leqslant t\leqslant 50,\\ 18\times10^{-4}(t-150)^2+12 & \text{if } 50<t\leqslant 150,\\ 12 & \text{if } 150<t.\end{cases} \tag{4.5}$$

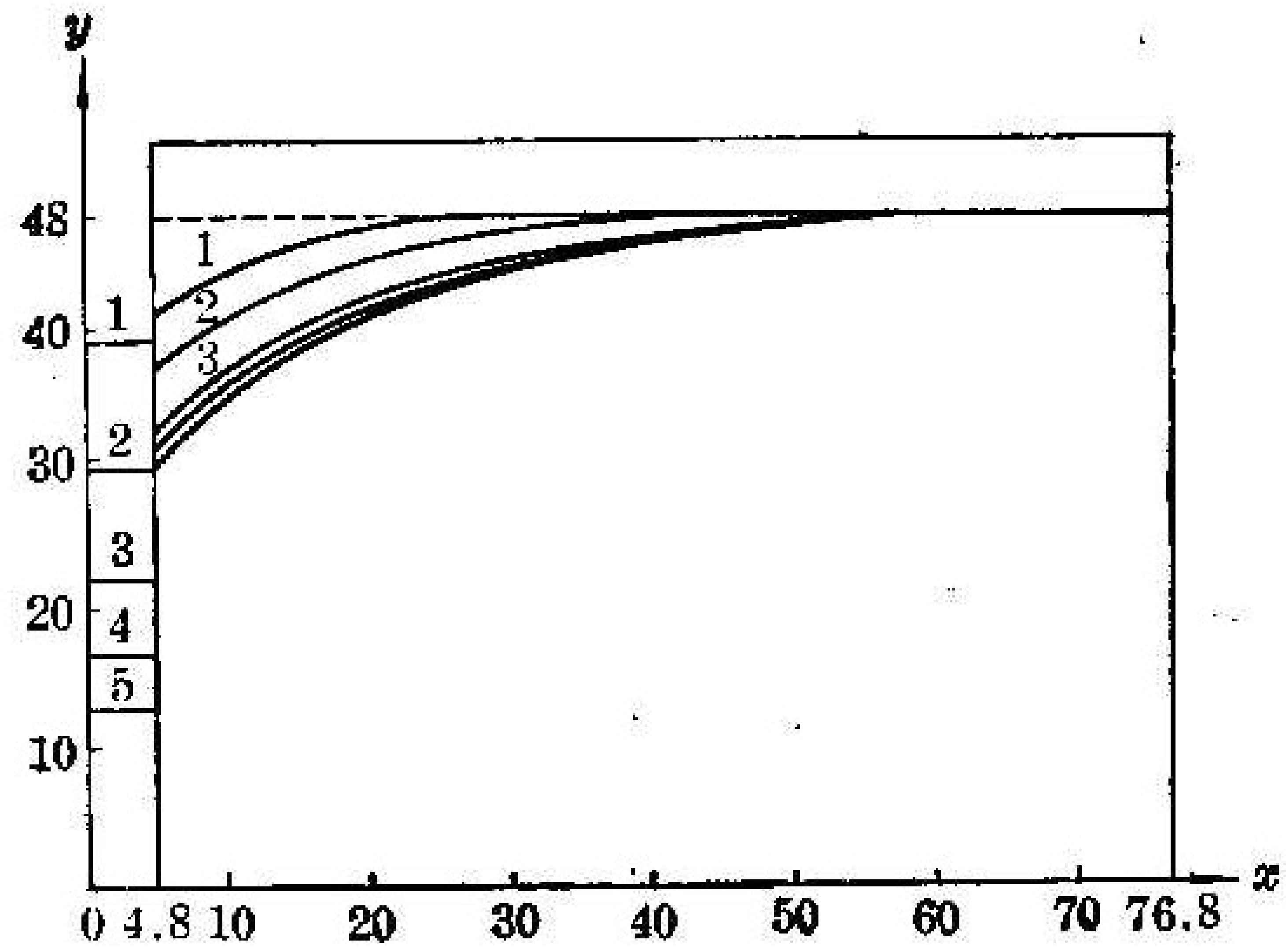

Fig. 2 Approximate free boundary at the different values of time for $m=16$, $n=24$, $K=1$.
1. $t=25$K; 2. $t=50$K; 3. $t=75$K; 4. $t=100$K; 5. $t=125$K.

Because the solution changes most rapidly near the well, the subdivisions were taken to be uniform in the y-direction and logarithmic in the x-direction. If n and m denote the number of subdivisions in the x- and y-directions, the coordinates of the gridpoints were given by

$$\begin{cases}x_i=r\exp[(i/n)\ln(R/r)], & 0\leqslant i\leqslant n,\\ y_j=j\cdot\dfrac{b_0}{m}, & 0\leqslant j\leqslant m.\end{cases} \tag{4.6}$$

For $m=4$, $n=6$, $K=1$, the solution W_h^{200} of Problem 4.1 is given in Table 2, where the position of the approximate free boundary is shown by the first zero term in each column. For comparison with the stationary case, in Table 1 we give the solution for the stationary case which was computed in [6]. In Figure 2 we plot the approximate free boundary at the different values of the time for $m=16$, $n=24$, $K=1$.

Finally we thank Prof. Xiao Shu-tie and Prof. Jiang Li-shang for the useful disoussions in preparing this paper.

Table 1 Solution in [6] for $m=4$, $n=6$

y \ x	4.80	7.61	12.09	19.20	30.47	48.38	76.80
48	0	0	0	0	0	0	0
36	0	0	0	0	3.03	31.59	72.00
24	0	17.55	43.61	81.33	133.09	204.06	288.00
12	0	88.58	182.02	283.14	394.37	517.10	648.00
0	72.00	252.00	432.00	612.00	792.00	972.00	1152.00

Table 2 Solution for $m=4$, $n=6$, $K=1$, $t=200K$

y \ x	4.80	7.61	12.09	19.20	30.47	40.38	76.80
48	0	0	0	0	0	0	0
36	0	0	0	0	5.64	34.43	72.00
24	0	19.25	47.00	86.39	139.90	210.54	288.00
12	0	92.08	188.92	293.11	406.76	528.86	648.00
0	72.00	258.15	443.85	627.96	812.00	991.95	1152.00

References

[1] H. W. Alt, Numerical solution of steady-state porous flow free boundary problems, *Numer. Math.*, **36** (1981), 73—98.

[2] J. Bear, Dynamics of Fluids in Porous Media, American Elsevier, New York, 1972.

[3] F. Brezzi, G. Sacchi, A finite element approximation of a variational inequality related to hydraulics, *Calcolo*, **13** (1976), 259—273.

[4] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[5] C. W. Cryer, The solution of a quadratic programming problem using systematic over-relaxation, *SIAM. J. Control.*, **9** (1971), 385—392.

[6] C. W. Cryer, H. Fetter, The numerical solution of axisymmetric free boundary porous flow well problems using variational inequalities, M. R. C. Univ. Wisconsin, 1761 (1977).

[7] A. Friedman, A. Torelli, A free boundary problem connected with non-steady filtration in porous media, *Nonlinear Analysis, Theory, Meth od & Applications*, **1**: 5 (1977), 503—545.

[8] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1977.

[9] J. L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer, Berlin, 1971.

[10] J. L. Lions, Quelque Méthodes de Résolutions des Problemes aux Limites non Lineares, Dunod, Paris, 1969.

[11] J. L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Grandlehre B. 181, Springer, 1972.

[12] A. Torelli, On a free boundary value problem connected with a non steady filtration phenomenon, *Ann. Scuola Norm. Sup. Pisa*, (I)4 (1977), 33—59.