A FINITE ELEMENT METHOD OF SEMI-DISCRETIZATION WITH MOVING GRID*

LIANG GUO-PING (梁国平)
(*Institute of Mathematics, Academia Sinica, Beijing, China*)

§ 1. Introduction

In solving a parabolic equation, a finite element method with variable mesh is more efficient than one with fixed mesh, if the space domain to be solved changes with time, such as the moving boundary problem, or if the peak value on the curved surface of the solution in the space domain moves with time, such as the speading of flame. In spite of the existence of this kind of methods[1-6], however, there is lack of its theoretical analysis; especially, there is hardly any proof of its optimal order accuracy.

Jamet has proved[7] that a method proposed by himself and Bonnerot, where the finite element is adopted in both space and time, has the optimal order accuracy. But his proof was made under the special condition of one dimension and uniform meshes and as a generalization of the Crank-Nicolson difference scheme, and is difficult to be extended to finite elements of more general form. Jamet also proved the convergence of their discontinuous finite element method and applied it to complex one-dimensional Stefan problem with many phases. Li[6] wrote the Stefan problem in enthalpy form so as to make his treatment of the moving boundary condition more natural when using Jamet's method. His method has strong adaptability and is fit for complex problems, but it requires many times the amount of calculation and storage than the continuous finite element method.

The purpose of this paper is to present a semi-discretization finite element method with grid moving continuously with time and to prove its optimal order accuracy. A stable difference scheme with second-order accuracy is given for the solution of an ordinary differential equation system derived from our method.

§ 2. The Semi-Discretization Finite Elements with Moving Grid

Consider solving the initial boundary value problem of second-order parabolic equation:

$$(\mathrm{P})\qquad \begin{cases} \dfrac{\partial u}{\partial t}+Lu=f, \quad (x,t)\in\mathscr{D}, & (1)\\ u|_{t=0}=u_0(x), \quad x\in D_0, & (2)\\ u|_{\partial D_t}=0, \quad x\in\partial D_t,\ 0\leqslant t\leqslant T, & (3)\end{cases}$$

where $\mathscr{D}=\{(x,t)\,|\,x\in D_t,\ 0\leqslant t\leqslant T\}$ is a bounded simply connected domain in $r+1$

* Received June 3, 1985.

dimensional space, D_t is a bounded simply connected domain in r-dimensional space dependent on time, ∂D_t is the contour of D_t in r-dimensional space, L is a self-adjoint differential operator of second order.

To solve problem (P) using moving finite elements, suppose at any moment $t\in[0, T]$ there is a one to one mapping

$$x=X(y, t) \tag{4}$$

to map the simply connected bounded domain Ω in y-space into a simply connected bounded domain D_t in space x, so that the cylinder domain $\Omega\times[0, T]$ in (y, t)-space can be mapped into the bounded domain $\mathscr{D}$ in (x, t)-space.

Then subdivide Ω into finite elements, construct finite element basis functions $\psi_1(y)$, $\psi_2(y)$, $\cdots$, $\psi_m(y)$, and denote by $S_h(\Omega)$ the m-dimensional linear space expanded from these basis functions. At any moment $t\in[0, T]$, the finite element subdivision of Ω, the basis functions $\{\psi_i(y)\}_{i=1}^{i=m}$ and m-dimensional linear space $S_h(\Omega)$ are transformed into the finite element subdivision of D_t, the basis functions $\{\varphi_i(x, t)\}_{i=1}^{i=m}$ and the m-dimensional linear space $S_h(D_t)$ respectively, where the basis functions in the two spaces satisfy the following relation:

$$\varphi_i(X(y, t), t)=\psi_i(y), \quad i=1, 2, \cdots, m. \tag{5}$$

Differentiate this equation by t

$$\frac{\partial\varphi_i}{\partial x}\frac{\partial X}{\partial t}+\frac{\partial\varphi_i}{\partial t}=\frac{d\psi_i}{dt},$$

where $\frac{d}{dt}$ denotes differentiation for t under (y, t)-space coordinate and $\frac{\partial}{\partial t}$ denotes differentiation for t under the (x, t)-space coordinate,

$$\frac{\partial\varphi_i}{\partial x}=\left(\frac{\partial\varphi_i}{\partial x_1}, \frac{\partial\varphi_i}{\partial x_2}, \cdots, \frac{\partial\varphi_i}{\partial x_r}\right),$$

$$\frac{\partial X}{\partial t}=\left(\frac{\partial X_1}{\partial t}, \frac{\partial X_2}{\partial t}, \cdots, \frac{\partial X_r}{\partial t}\right)^\tau,$$

where τ denotes the transpose of a vector or a matrix. Because $\psi_i(y)$ (in (y, t)-space coordinate) is independent of t, so

$$\frac{\partial\varphi_i}{\partial t}+\frac{\partial\varphi_i}{\partial x}\frac{\partial X}{\partial t}=0, \quad i=1, 2, \cdots, m, \tag{6)$'$}$$

or, in matrix form,

$$\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial X}\cdot\frac{\partial X}{\partial t}=0, \tag{6)$''$}$$

where $\varphi=(\varphi_1, \varphi_2, \cdots, \varphi_m)^\tau$,

$$\frac{\partial\varphi}{\partial x}=\begin{pmatrix}\frac{\partial\varphi_1}{\partial x_1}, & \frac{\partial\varphi_1}{\partial x_2}, & \cdots, & \frac{\partial\varphi_1}{\partial x_r}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial\varphi_m}{\partial x_1}, & \frac{\partial\varphi_m}{\partial x_2}, & \cdots, & \frac{\partial\varphi_m}{\partial x_r}\end{pmatrix}. \tag{7}$$

Take the finite element approximate solution

$$u^h(x, t)=\varphi^\tau(x, t)U(t)=\sum_{i=1}^{m}U_i\varphi_i(x, t). \tag{8}$$