LOCAL ASYMPTOTIC EXPANSION AND EXTRAPOLATION FOR FINITE ELEMENTS*

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Recently, the elliptic Ritz projection with linear finite elements is shown to admit asymptotic error expansions on certain uniform meshes if the exact solution is globally smooth (see [1, 4, 5, 6, 7]). We will prove in this note that the asymptotic error expansion even holds true for locally smooth solution.

Consider the Ritz projection method applied to the Dirichlet problem

$$-\Delta u=f \quad \text{in } \Omega,\ u=b \text{ on } \partial\Omega$$

on a convex polygonal domain $\Omega\subset R^2$. For linear finite elements on a piecewise uniform triangulation there hold

$$(P_h u-u)(z)=h^2e(z,u)+o(h^2), \quad \text{if } u\in W^{3,2+\delta}(\Omega)\cap H_0^1(\Omega); \tag{1}$$

$$\overline{D}(P_h u-u)(z)=h^2e'(z,u)+o(h^2), \quad \text{if } u\in W^{4,2+\delta}(\Omega)\cap H_0^1(\Omega), \tag{2}$$

in nodal points z, where $P_h u$ is the Ritz projection of u in the linear finite element space S_h, $\overline{D}$ is the nodal point averaged gradient[2,3] and $\delta>0$.

In general, however, the solution u is only locally smooth:

$$u\in W^{k,2+\delta}(U_\rho)\cap H^2(\Omega)\cap H_0^1(\Omega); \quad k=3,4, \tag{3}$$

where

$$U_\rho=\{x\in\Omega: |x-z|<\rho\}. \tag{4}$$

In Blum-Lin-Rannacher[1] it has been pointed out that there holds an extended expansion in interior nodal points and that an expansion like (1) in interior nodal points even holds true for interior smooth solution. We now prove the following local expansion theorem by a functional analysis method.

Theorem 1. *There hold*

$$(P_h u-u)(z)=h^2e(\rho,u)+o(h^2), \tag{5}$$

$$\overline{D}(P_h u-u)(z)=h^2e'(\rho,u)+o(h^2), \tag{6}$$

for nodal point z, and u in (3) and (4) ($k=3$, 4, respectively).

Proof. Let us divide the locally smooth solution u in (3) and (4) into a global smooth part

$$u_1=\omega u\in W^{3,2+\delta}(\Omega)\cap H_0^1(\Omega)$$

and a nonsmooth part

$$u_2=(1-\omega)u\in H^2(\Omega)\cap H_0^1(\Omega),$$

where

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$$\omega=\begin{cases}1 & \text{in } U_{\rho'}(\rho'<\rho),\\ 0 & \text{in } \Omega\backslash U_\rho\end{cases}$$

and $\omega\in C_0^\infty(U_\rho)$. Then

$$u_1=0 \quad \text{in } \Omega\backslash U_\rho, \quad v_2=0 \text{ in } U_{\rho'}.$$

For the good part u_1, it has been proved (see (1)) that there holds

$$\frac{1}{h^2}(P_hu_1-u_1)(z)=e(u_1)+o(1). \tag{7}$$

And the bad function u (or $u_2=(1-\omega)u$) can be approximated in $H^2(\Omega)$ by good functions

$$\tilde{u}(\text{or } \tilde{u}_2=(1-\omega)\tilde{u})\in W^{3,2+\delta}(\Omega)\cap H_0^1(\Omega)$$

which admit the expansion of the form

$$\frac{1}{h^2}(P_h\tilde{u}_2-\tilde{u}_2)(z)=e(\tilde{u}_2)+o(1).$$

It remains to prove the expansion

$$\frac{1}{h^2}(P_hu_2-u_2)(z)=e(u_2)+o(1). \tag{8}$$

For this, we define, corresponding to a nodal point z, the following linear functional

$$F_h(u)=\frac{1}{h^2}(P_hu_2-u_2)(z), \quad u_2=(1-\omega)u.$$

We have

$$\begin{aligned}|F_h(u)|&=\frac{1}{h^2}|(P_hu_2-u_2)(z)|=\frac{1}{h^2}|(\nabla(P_hu_2-i_hu_2),\nabla g_h^z)|\\&=\frac{1}{h^2}|(\nabla(u_2-i_hu_2),\nabla(g_h^z-g^z))|=\frac{1}{h^2}\left|\int_{\Omega\backslash U_{\rho''}}\nabla(u_2-i_hu_2)\nabla(g_h^z-g^z)dx\right|\\&\leqslant\frac{1}{h^2}\|u_2-i_hu_2\|_{H^1(\Omega)}\|g_h^z-g^z\|_{H^1(\Omega\backslash U_{\rho''})}\leqslant c\|u\|_{H^2(\Omega)},\end{aligned}$$

where $\rho''<\rho'$, $i_hu_2\in S_h$ is the interpolant of u_2, and $g_h^z\in\mathring{S}_h$ is the discrete Green function (see [1]). Then, for h, $h'\ll1$,

$$\begin{aligned}|F_h(u)-F_{h'}(u)|&\leqslant|F_h(u)-F_h(\tilde{u})|+|F_{h'}(u)-F_{h'}(\tilde{u})|+|F_h(\tilde{u})-F_{h'}(\tilde{u})|\\&\leqslant C\|u-\tilde{u}\|_{H^2(\Omega)}+\frac{\varepsilon}{2}<\varepsilon.\end{aligned}$$

Hence

$$\lim_{h\to0}F_h(u)=e(u_2)$$

and (8) holds true. Combining (7) and (8), we obtain the expansion

$$\frac{1}{h^2}(P_hu-u)(z)=e(u_1)+e(u_2)+o(1)=(\rho,u)+o(1).$$

Hence (5) holds true. (6) follows similarly from (2).

Theorem 2. *For $u\in C^6(\bar{\Omega})$, there holds*

$$(P_hu-u)(z)=h^2e^{(1)}(u)+h^4e^{(2)}(\rho,u)+o(h^4) \tag{9}$$

in nodal point z, with ρ described in (10).

Proof. We consider here only the case where the triangulation is uniform