HIERARCHICAL ELEMENTS, LOCAL MAPPINGS AND THE h–p VERSION OF THE FINITE ELEMENT METHOD (I)*

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Abstract

This is the first half of an article which develops a theory of the hierarchical elements for the h–p version of the finite element method in the two-dimentional case. The approximation properties of the hierarchical elements are discussed. The second part will address the convergence rate when geometric meshes are used.

§ 0. Introduction

In the finite element method, when conforming elements are used, the rate of convergence for an elliptic problem is determined by the approximability of the finite element space to the solution (Céa's Lemma). As a consequence, the select ion of a good finite element space is the key for achieving maximal rate of convergence.

There are three basic versions in the finite element method. In the h version, which is the traditional one, the mesh size h goes to zero while the degrees of elements are fixed. In contrast, the p version increases the degrees and fixes the mesh. The h–p version combines the two and obtains convergence by both refining the mesh and increasing the degrees.

Since the h–p version considers both mesh and degree distribution, it is more advantageous in the approximability of the finite element space. It was proved in [1]—[3] that in one-dimension if the solution has a singularity of x^α-type, the best order of convergence for the h–p version is $q_0^{\sqrt{(\alpha-\frac{1}{2})N}}$, where N is the number of degrees of freedom of the finite element space and $q_0=(\sqrt{2}-1)^2\approx 0.1715$. To achieve this rate of convergence a geometric mesh with the ratio q_0 and a linear distribution of degrees of elements with the slope $2\alpha-1$ were used. In this setting, the elements closer to the singularity have smaller sizes and lower degrees. If the degrees of elements are uniformly distributed, they should be increased as a multiple of the number of the elements with a factor $2\alpha-1$, which gives a rate of convergence of $q_0^{\sqrt{(\alpha-\frac{1}{2})N/2}}$. In 2-dimensions, when on a cornered domain, r^α-type singularities of the solution will arise at the corners. It was shown in [4] that when a geometric mesh is used and the degrees of elements are either optimally,

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or uniformly distributed, an order of convergence $e^{-\gamma\cdot\sqrt{N}}$ ($\gamma>0$) can be reached. However, the rules for selecting the ratio of the geometric mesh and for the increase of degrees of elements were not clear.

An important property in one-dimension is that the ratio of the geometric mesh for achieving the best convergence order is independent of the strength of the ingularity, i.e. the value α. This has been called "the 0.15 rule" (theoretically 0.1715). It has also been guessed that this rule could be valid also for two-dimensions. In this paper we will show that this guess is true for optimal mesh-degree combinations.

The basic tools for implementing the h-p version are the hierarchical elements and local mappings. In contrast with Lagrange elements and Hermit elements, when the degrees of elements need to be incresed, one can simply add new basis functions to the old basis set. The concept of hierarchical basis can go back to the original Galerkin's method. It was introduced by [5] in 1971 and suggested by [6], [7], [8] for p-version in the early 80's. The hierarchical basis is closely related with spectral expansions. And it has been used in the p-version finite element analysis program PROBE. In this paper we will give a more detailed analysis for the approximation properties of the finite element space based on the hierarchical elements.

The use of local mappings is an old technique in the finite element calculation. To handle the curved boundary the means of isoparametric elements were successfully used in the h version. In this paper a discussion will be given to it in cooperation with the hierarchical elements.

This part contains section 1, C^0 hierarchical elements in one-dimension and section 2, the $Q^0_?$ hierarchical elements in two-dimensions (the square elemets). The other two sections (section 3, C^0 compatible local mappings and geometric meshes and section 4, the h-p method and its error analysis) are left to Part II.

§1. C^0 Hierarchical Elements in One Dimension

1.1. Preliminaries

In the following, H^1, H^k, L_2 are used in the usual sense for Sobolev spaces. $P_n(x)$ ($n=0, 1, 2, \cdots$) are the standard Legendre polynomials:

$$P_n(x)=\frac{1}{(2n)!!}\frac{d^n}{dx^n}[(x^2-1)^n]. \tag{1.1.1}$$

For the properties of Legendre polynomials we refer to [7], [8] and [9]. The most important properties for our discussion are listed below:

1° $\{P_n(x)\}$ is an orthogonal basis of $L_2(-1, 1)$. And we have

$$\int_{-1}^{1} P_n(x)P_m(x)dx=\frac{2}{2n+1}\delta_{n,m} \tag{1.1.2}$$

where
$$\delta_{n,m}=\begin{cases}0, & \text{if } n\neq m,\\ 1, & \text{if } n=m.\end{cases}$$

2° If $u\in L_1(-1, 1)$, then $u(x)$ has a Fourier-Legendre expansion (or simply the Legendre expansion):

$$u(x)\sim\sum_{n=0}^{\infty} u_n P_n(x), \tag{1.1.3}$$

where
$$u_n=\frac{2n+1}{2}\int_{-1}^{1}u(x)P_n(x)dx.$$

3° If $u\in L_2(-1, 1)$, then (1.1.3) converges in $L_2(-1, 1)$, and we have
$$\int_{-1}^{1}u(x)^2dx=\sum_{n=0}^{\infty}u_n^2\frac{2}{2n+1}. \tag{1.1.4}$$

4° Denote
$$Q_n(x)=\int_{-1}^{x}P_n(t)dt,\quad n\geq 1 \tag{1.1.5}$$
and
$$Q_0^{-1}(x)=\frac{1-x}{2},\quad Q_0^1(x)=\frac{1+x}{2}.$$
It is easy to see that
$$Q_0^{-1}(-1)=1,\ Q_0^1(-1)=0,\ Q_0^{-1}(1)=0,\ Q_0^1(1)=1,$$
and $Q_n(\pm 1)=0$.

5° Let $u\in L_2(-1, 1)$ be the derivative of function $v(x)$; thus $v(x)$ is absolutely continuous on $[-1, 1]$. Define
$$S(x)=v(-1)Q_0^{-1}(x)+v(1)Q_0^1(x)+\sum_{n=1}^{\infty}u_nQ_n(x) \tag{1.1.6}$$
which is obtained by term-by-term integration of (1.1.3).

Its N-th partial sum is
$$S_N(x)=v(-1)Q_0^{-1}(x)+v(1)Q_0^1(x)+\sum_{n=1}^{N}u_nQ_n(x). \tag{1.1.7}$$

We have the following:

Theorem 1.1.1. *If $u\in L_2(-1, 1)$, then $S(x)$ converges uniformly to*
$$v(x)=v(-1)+\int_{-1}^{x}u(t)dt$$
on $[-1, 1]$.

Proof. Let $v(x)$ be expanded in Legendre series:
$$v(x)\sim\sum_{n=0}^{\infty}v_nP_n(x) \tag{1.1.8}$$
and its N-th partial sum is denoted by $\bar{S}_N(x)=\sum_{n=0}^{N}v_nP_n(x)$.

Using the identity
$$(2n+1)Q_n(x)=P_{n+1}(x)-P_{n-1}(x)$$
one can easily obtain
$$S_N(x)=\bar{S}_{N-1}(x)+\frac{u_{N-1}}{2N-1}P_N(x)+\frac{u_N}{2N+1}P_{N+1}(x) \tag{1.1.9}$$
where S_N is defined by (1.1.7).

(1.1.4) implies that
$$u_n/n^{\frac{1}{2}}\to 0\quad \text{as } n\to\infty.$$
Thus by the embedding $H^1\subsetneq C^0$ we obtain
$$\|v-S_N\|_{L_\infty}\leq C\{\|v-S_N\|_{L_2}^2+\|u-S_N'\|_{L_2}^2\}^{\frac{1}{2}}\to 0\quad \text{as } N\to\infty,$$
since

$$\|v-S_N\|_{L_2}\leqslant\|v-\bar{S}_{N-1}\|_{L_2}+\left[\left|\frac{u_{N-1}}{2N-1}\right|+\left|\frac{u_N}{2N+1}\right|\right]\Big/n^{\frac{1}{2}}$$

and S'_N is the partial sum of the Legendre series of u. ▌

This shows that S_N is very close to the optimal approximation to u in $H^1(-1,1)$.

1.2. The hierarchical elements

We will give a more general definition for the hierarchical elements. Let Ω be a given domain in $\mathbb{R}^n$, Δ be a mesh on Ω, and $e_i\in\Delta$ be mesh elements. Let D be the standard element. There are non-singular local mappings T_i such that $T_i(D)=e_i$.

A set of linearly independent functions $\{\omega_l(P)\}$ (l may be multiple indices) defined on D is called a C^0-hierarchcial basis, if the set of functions $\{\omega_l(T_i^{-1}(x))\}$ (each is defined on one mesh element) can be used to assemble the basis for a C^0 finite element space on the given mesh. The elements of such a finite element space are called C^0-hierarchical elements.

The hierarchical basis is complete, if for any function $U(P)$ in $C^\infty(\bar{D})$ there exists an expansion

$$U(P)=\sum_l U_l\omega_l(P).$$

The local mappings which can be used for constructing a basis of a C^0 finite element space will be said to be C^0-compatible, and the mesh for which the C^0-compatible local mappings exist will be called a C^0-mesh.

According to the above definition the following conclusion is evident:

Theorem 1.2.1. *In one dimension let the standard element D be $[-1, 1]$. Then the set $\{Q_0^{-1}(x), Q_0^1(x), Q_n(x); n=1, 2, \cdots\}$ is a complete C^0-hierarchical basis. For any function $u\in H^1(-1, 1)$ we have the following uniformly convergent expansion:*

$$u(x)=u(-1)Q_0^{-1}(x)+u(1)Q_0^1(x)+\sum_{n=1}^{\infty}u_nQ_n(x) \tag{1.2.1}$$

with

$$u_n=\frac{2n+1}{2}\int_{-1}^{1}u'(x)P_n(x)dx. \tag{1.2.2}$$

Moreover, let $\Omega=[a, b]$. Then any mesh

$$\Delta=\{a=x_0<x_1<\cdots<x_m=b\}$$

is a C^0-mesh, and the local mappings

$$T_i(x)=x_{i-1}Q_0^{-1}(x)+x_iQ_0^1(x)=h_ix+c_i \tag{1.2.3}$$

with

$$h_i=\frac{1}{2}(x_i-x_{i-1}),\quad c_i=\frac{1}{2}(x_i+x_{i-1})$$

are C^0-compatible.

1.3. Error analysis for 1-d C^0 hierarchical elements

Let $P_n^{(k)}(x)$ be the k-th derivative of Legendre polynomial $P_n(x)$. The following orthogonality holds:

$$\int_{-1}^{1}(1-x^2)^kP_n^{(k)}(x)P_m^{(k)}(x)dx=\begin{cases}0, & m\neq n,\\ \dfrac{(n+k)!}{(n-k)!}\,\dfrac{2}{2n+1}, & m=n,\end{cases} \tag{1.3.1}$$

We define a weighted Hilbert space $H_{\rho_k}\equiv H_{\rho_k}(-1, 1)$ which has the inner product

$$(u,v)_{\rho_k}=\int_{-1}^{1}(1-x^2)^k u(x)v(x)dx \tag{1.3.2}$$

and the norm

$$\|u\|_{\rho_k}=\sqrt{(u,u)_{\rho_k}}. \tag{1.3.3}$$

Then $\{P_n^{(k)}\}_{n=k}^{\infty}$ is an orthogonal basis of H_{ρ_k}.

If $u^{(k+1)}\in H_{\rho_k}$, then one can show that the series obtained from term-by-term differentiation of the Legendre expansion (1.1.3) of $u(x)$ converges in H_{ρ_k}. Hence we have

$$\|u^{(k+1)}\|^2_{\rho_k}=\sum_{n=k}^{\infty} u_n^2\frac{(n+k)!}{(n-k)!}\frac{2}{2n+1}. \tag{1.3.4}$$

Suppose that $u(x)$ is expanded in (1.2.1) and let $u_p(x)$ be its partial sum of degree p:

$$u_p(x)=u(-1)Q_0^{-1}(x)+u(1)Q_0^1(x)+\sum_{n=1}^{p-1}u_nQ_n(x). \tag{1.3.5}$$

Then for $p\geq k$

$$\begin{aligned}\|u-u_p\|^2_{L_2}&=\sum_{n=p}^{\infty}u_n^2\frac{2}{2n+1}\leq\frac{(p-k)!}{(p+k)!}\sum_{n=k}^{\infty}u_n^2\frac{(n+k)!}{(n-k)!}\frac{2}{2n+1}\\&=\frac{(p-k)!}{(p+k)!}\|u^{(k+1)}\|^2_{\rho_k}.\end{aligned} \tag{1.3.6}$$

Lemma 1.3.1. $\dfrac{(p-k)!}{(p+k)!}\leq\left[\dfrac{e}{2p}\right]^{2k}, \quad p\geq k.$

Proof. Let $\gamma^{-1}=\dfrac{(p+k)!}{(p-k)!}\dfrac{1}{p^{2k}}=\left(1+\dfrac{k}{p}\right)\prod_{j=1}^{k-1}\left(1-\dfrac{j^2}{p^2}\right).$

Then
$$\begin{aligned}\ln\gamma&<-\sum_{j=1}^{k-1}\ln\left(1-\frac{j^2}{p^2}\right)=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{k-1}\left[\frac{j}{p}\right]^{2n}\\&<k\sum_{n=1}^{\infty}\frac{1}{n(2n+1)}=2k(1-\ln 2).\end{aligned}$$

Thus $\gamma<\left[\dfrac{e}{2}\right]^{2k}$ and the lemma follows. ▮

Using the identity

$$Q_n(x)=-\frac{1}{n(n+1)}(1-x^2)P_n'(x) \tag{1.3.7}$$

we have for $m,n>1$

$$\int_{-1}^{1}(1-x^2)^{-1}Q_n(x)Q_m(x)\,dx=\begin{cases}0, & m\neq n,\\ \dfrac{2}{n(n+1)(2n+1)}, & m=n.\end{cases} \tag{1.3.8}$$

Following (1.3.6) and Lemmas 1.3.1, 1.3.2, we have

Theorem 1.3.1. *Let $u^{(k+1)}\in H_{\rho_k}$, $p\geq k$. Then*

$$\|u'-u_p'\|_{L_2}\leq\left[\frac{e}{2p}\right]^k\|u^{(k+1)}\|_{\rho_k}\leq\left[\frac{e}{2p}\right]^k|u|_{k+1}, \tag{1.3.9}$$

$$\|u-u_p\|_{L_2}\leq\frac{1}{p}\left[\frac{e}{2p}\right]^k\|u^{(k+1)}\|_{\rho_k}\leq\frac{1}{p}\left[\frac{e}{2p}\right]^k|u|_{k+1}, \tag{1.3.10}$$

where $|\cdot|_{k+1}$ is the usual seminorm of H^{k+1}.

By the local mappings (1.2.7) one can easily obtain

Theorem 1.3.2. *Let $\Omega=[a, b]$ and Δ be a given mesh. $\mathscr{S}^{\Delta,p}$ is the C^0 finite element space based on Δ such that the degrees of the elements are equal to p. Let h be the maximal size of the mesh elements. If $u\in H^{k+1}(a, b)$, then there is $u_s\in \mathscr{S}^{\Delta,p}$ such that*

$$\|u'-u_s'\|_{L_2(a,b)}\leqslant\left[\frac{eh}{4p}\right]^k |u|_{k+1,(a,b)}, \tag{1.3.11}$$

$$\|u-u_s\|_{L_2(a,b)}\leqslant\left[\frac{eh}{4p}\right]^{k+1} |u|_{k+1,(a,b)}. \tag{1.3.12}$$

Remark 1.3.1. As seen in (1.3.6), if $u\in C^\infty$, then for any $p>1$

$$\|u-u_p\|^2_{L_2}\leqslant\frac{1}{(2p)!}\|u^{(p+1)}\|^2_{p_p}. \tag{1.3.13}$$

This estimate, however, is not accurate enough for the analytic functions.

1.4. Approximation to analytic functions

The following result can be found in [10].

Lemma 1.4.1. *Let $u(z)$ be analytic in a domain of the complex plane, containing segment $[-1, 1]$. Let ρ be the sum of semi-axes of the maximal ellipse which has the foci ±1 and in which $u(z)$ is regular. If $E_n(u)$ is the error of the best approximation by polynomials with degree $\leqslant n$ (in L_∞ norm), then*

$$\lim_{n\to\infty}\sup[E_n(u)]^{\frac{1}{n}}=\rho^{-1}. \tag{1.4.1}$$

It follows readily that for any $s>0$ there exists a constant $C=C(u, s)$ such that

$$\|u'-u_p'\|_{L_2}(-1, 1)\leqslant C(\rho^{-1}+s)^p \tag{1.4.2}$$

where u_p is defined in (1.3.5).

For some class of analytic functions we can obtain more accurate results. In [1] we have proved that for the function $(x+\mu)^\alpha$ with $\mu>1$ one can remove the s in the estimate (1.4.2). However, there α was restricted to be greater than $\frac{1}{2}$. We now give another proof which does not require the restriction.

We need the following classical theorem:

Lemma 1.4.2 (*Laplace asymptotic integration theory*). *Let $\phi(x)$, $h(x)$ and $f(x)=e^{h(x)}$ be defined on $[a, b]$. Moreover,*

(1) $\phi(x)[f(x)]^n\in L_1(a, b)$ *for* $n=0, 1, 2, \cdots$;

(2) $h(x)$ *assumes its effective maximum at an interior point* $\xi\in(a, b)$;

(3) $h(x)$ *is twice continuously differentiable at a neighborhood of* ξ *with* $h''(\xi)<0$;

(4) $\phi(x)$ *is continuous at* ξ *with* $\phi(\xi)\neq0$.

Then, as $n\to\infty$ the following asymptotic formula holds:

$$\int_a^b\phi(x)[f(x)]^n dx\sim\phi(\xi)[f(\xi)]^{n+\frac{1}{2}}\sqrt{\frac{2\pi}{-nf''(\xi)}}. \tag{1.4.3}$$

The proof can be seen in [11]. The term effective maximum means that for any $s>0$,

$$\sup_{x<\xi-s} h(x)<h(\xi), \quad \sup_{x>\xi+s} h(x)<h(\xi).$$

We now consider the coefficients of the Legendre expansions for the function

$$u(x)=(x+\mu)^\alpha, \quad \mu>1. \tag{1.4.4}$$

Lemma 1.4.3. *Let* $u_n=\frac{2n+1}{2}\int_{-1}^{1}u'(x)P_n(x)dx$. *Then for* $1>s>0$ *we have*

$$|u_n|\leqslant C(\alpha,\ s)n^{-\alpha+\frac{1}{2}}r^n \tag{1.4.5}$$

with

$$r=\mu-\sqrt{\mu^2-1},\qquad s\leqslant r\leqslant 1-s. \tag{1.4.6}$$

Proof. Since

$$u^{(n+1)}(x)=\alpha(\alpha-1)\cdots(\alpha-n)(x+\mu)^{\alpha-n-1},$$

we have

$$u_n=\frac{2n+1}{2}\int_{-1}^{1}u'(x)\frac{1}{(2n)!!}[(x^2-1)^n]^{(n)}dx$$
$$=\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}\,\frac{2n+1}{2^{n+1}}\int_{-1}^{1}(x+\mu)^{\alpha-1}\left[\frac{1-x^2}{x+\mu}\right]^n dx.$$

It is easy to check that the conditions of Lemma 1.4.2 are satisfied, and the effective maximum for

$$h(x)=\ln\left[\frac{1-x^2}{x+\mu}\right]$$

is at

$$x=-\mu+\sqrt{\mu^2-1}\equiv -r,\quad r\in(0,\ 1)$$

and by Lemma 1.4.2 it follows that

$$\int_{-1}^{1}(x+\mu)^{\alpha-1}\left[\frac{1-x^2}{x+\mu}\right]^n dx\sim\left[\frac{1-r^2}{2r}\right]^{\alpha-1}(2r)^{n+\frac{1}{2}}\sqrt{\frac{(1-r^2)\pi}{n}}. \tag{1.4.7}$$

Note that as $n\to\infty$, when α is not an integer we have

$$\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}=(-1)^n\frac{\Gamma(n-\alpha+1)}{\Gamma(\alpha)\Gamma(n+1)}\sim(-1)^n\Gamma(\alpha)^{-1}n^{-\alpha}. \tag{1.4.8}$$

Thus (1.4.5) holds for $s\leqslant r\leqslant 1-s$ $(s>0)$.

If α is a positive integer, then (1.4.5) is trivially true. If α is a negative integer, say $\alpha=-m$, then (1.4.8) becomes

$$\frac{(-m)(-m-1)\cdots(-m-n)}{n!}\sim(-1)^{n+1}C(m)\cdot n^m.$$

Thus the result still holds. ∎

Remark 1.4.1. It is easy to check that

$$r^{-1}=\frac{1}{\mu-\sqrt{\mu^2-1}}=\mu+\sqrt{\mu^2-1}$$

is the sum of semi-axes of the ellipse with foci ± 1 and passing through the point $z=-\mu$.

Therefore we obtain:

Theorem 1.4.1. *Let* α *be real and* $u(x)=(x+\mu)^\alpha$, $s^{-1}\geqslant\mu\geqslant 1+s$, $s>0$. *Then we have*

$$\|u'-u_p'\|_{L_2(-1,1)}\leqslant C(\alpha,\ s)p^{-\alpha}r^p \tag{1.4.9}$$

with r *defined by* (1.4.6). ∎

Remark 1.4.2. For a general interval $I=[a,\ b]$ and the singularity ξ which is real and outside I one has (see [2])

$$r=\frac{\sqrt{b-\xi}-\sqrt{a-\xi}}{\sqrt{b-\xi}+\sqrt{a-\xi}}.$$

This result can be extended to the function class A_α which is defined as follows: $u\in A_\alpha$ implies that u is analytic on the complex plane except at one point $\xi\in\mathbf{R}\backslash[a, b]$ and for $|z-\xi|<R$ there is $K=K(R)$ such that $|u(z)-u(\xi)|<K|z-\xi|^\alpha$.

§ 2. C^0 Hierarchical Elements in Two Dimensions (square elements)

2.1. The tensor product space

Let $H_i(i=1, 2)$ be Hilbert spaces with inner products $(\cdot, \cdot)_i$ and norms $\|\cdot\|_i$ respectively.

The (algebraic) tensor product space $\mathscr{H}_a$ is a linear space generated by the elements of the form

$$a^1\otimes a^2, \quad a^i\in H_i, \quad i=1, 2, \tag{2.1.1}$$

and satisfies the following properties:

(i) $a^1\otimes(a^2+b^2)=a^1\otimes a^2+a^1\otimes b^2, \quad (a^1+b^1)\otimes a^2=a^1\otimes a^2+b^1\otimes a^2$;

(ii) $a^1\otimes(\alpha a^2)=(\alpha a^1)\otimes a^2=\alpha(a^1\otimes a^2)$

where a^i, $b^i\in H_i(i=1, 2)$, and α is a scalar.

The inner product on $\mathscr{H}_a$ is defined to be

$$(a^1\otimes a^2, b^1\otimes b^2)=(a^1, b^1)_1\cdot(a^2, b^2)_2. \tag{2.1.2}$$

Thus the norm derived from the inner product satisfies

$$\|a^1\otimes a^2\|=\|a^1\|_1\cdot\|a^2\|_2. \tag{2.1.3}$$

The general elements of $\mathscr{H}_a$ are of the form

$$a=\sum_{n=1}^{k}\alpha_n\cdot(a_n^1\otimes a_n^2), \quad k\in Z^+ \tag{2.1.4}$$

The tensor product space $\mathscr{H}$ is then defined as the completion of $\mathscr{H}_a$ under the above norm. And we denote

$$\mathscr{H}=H_1\otimes H_2. \tag{2.1.5}$$

A basic fact for tensor product is that if $\{\omega_n^i\}_{n=1}^\infty$ are bases of H_i $(i=1, 2)$ respectively, then

$$\omega_l=\omega_{n_1}^1\otimes\omega_{n_2}^2, \quad l=(n_1, n_2) \tag{2.1.6}$$

form a basis of the product space $\mathscr{H}$. Especially, if $\{\omega_n^1\}_{n=1}^\infty$ and $\{\omega_n^2\}_{n=1}^\infty$ are orthonormal bases of H_1 and H_2 respectively, then $\{\omega_n^1\otimes\omega_m^2\}_{n,m=1}^\infty$ is an orthonormal basis of $\mathscr{H}$. And, if u, $v\in\mathscr{H}$ are given by

$$u=\sum_{n,m=1}^\infty u_{n,m}\omega_n^1\otimes\omega_m^2, \tag{2.1.7}$$

$$v=\sum_{n,m=1}^\infty v_{n,m}\omega_n^1\otimes\omega_m^2, \tag{2.1.8}$$

then

$$(u, v)=\sum_{n,m=1}^\infty u_{n,m}v_{n,m}, \tag{2.1.9}$$

$$\|u\|^2=\sum_{n,m=1}^\infty u_{n,m}^2. \tag{2.1.10}$$

Let us consider a few examples.

1° $\mathscr{H}^{0,0}=L_2(-1, 1)\otimes L_2(-1, 1)$.

Since the Legendre polynomials form an orthogonal basis of $L_2(-1, 1)$, every $u\in\mathscr{H}^{0,0}$ is representable by

$$u(x, y)=\sum_{n,m=1}^{\infty} u_{n,m}P_n(x)P_m(y) \tag{2.1.11}$$

with

$$u_{n,m}=\frac{2n+1}{2}\,\frac{2m+1}{2}\int_{-1}^{1}\int_{-1}^{1} u(x, y)P_n(x)P_m(y)dx\,dy. \tag{2.1.12}$$

And the norm on $\mathscr{H}^{0,0}$ is

$$\|u\|^2=\sum_{n,m=1}^{\infty} u_{n,m}^2\,\frac{2}{2n+1}\,\frac{2}{2m+1}=\int_{-1}^{1}\int_{-1}^{1} u(x, y)^2dx\,dy. \tag{2.1.13}$$

It is clear that $\mathscr{H}^{0,0}=L_2(D)$, where $D=[-1, 1]\times[-1, 1]$.

2° $\mathscr{H}^{1,1}=H^1(-1, 1)\otimes H^1(-1, 1)$.

In this case one can show that $\mathscr{H}^{1,1}\subsetneq H^1(D)$ with continuous embedding. However, they are no longer equal. In fact, the functions $u(x, y)$ in $\mathscr{H}^{1,1}$ have the characteristics that for any fixed y as a function of x, $u(\cdot, y)\in H^1(-1, 1)$, and, for fixed x, $u(x, \cdot)\in H^1(-1, 1)$. It can be shown that

$$\|u\|^2_{\mathscr{H}^{1,1}}=\|u\|^2_{L_2(D)}+\left\|\frac{\partial u}{\partial x}\right\|^2_{L_2(D)}+\left\|\frac{\partial u}{\partial y}\right\|^2_{L_2(D)}+\left\|\frac{\partial^2 u}{\partial x\,\partial y}\right\|^2_{L_2(D)}.$$

If we let $\mathscr{H}^{0,1}=L_2(-1, 1)\otimes H^1(-1, 1)$, and similarly, $\mathscr{H}^{1,0}=H^1(-1, 1)\otimes L_2(-1, 1)$, then for $u(x, y)\in\mathscr{H}^{1,1}$ we have

$$\frac{\partial u}{\partial x}(x, y)\in\mathscr{H}^{0,1},\quad \frac{\partial u}{\partial y}(x, y)\in\mathscr{H}^{1,0}, \tag{2.1.14}$$

and

$$\frac{\partial^2 u}{\partial x\,\partial y}(x, y)\in\mathscr{H}^{0,0}=L_2(D). \tag{2.1.15}$$

Recall that $\{Q_0^{-1}(x), Q_0^1(x), Q_n(x); n=1, 2, \cdots\}$ is a basis of $H^1(-1, 1)$ (see the previous section), it is clear that the set

$$\begin{aligned}&\{Q_0^{-1}(x)\cdot Q_0^{-1}(y),\ Q_0^1(x)\cdot Q_0^{-1}(y),\ Q_0^1(x)\cdot Q_0^1(y),\ Q_0^{-1}(x)\cdot Q_0^1(y),\\ &Q_n(x)\cdot Q_0^{-1}(y),\ Q_n(x)\cdot Q_0^{-1}(y),\ Q_0^{-1}(x)\cdot Q_n(y),\ Q_0^1(x)\cdot Q_n(y),\\ &Q_n(x)\cdot Q_m(y); n, m=1, 2, \cdots\}\end{aligned} \tag{2.1.16}$$

is a basis of $\mathscr{H}^{1,1}$.

2.2. C^0 hierarchical basis on standard square

We will use D to denote the standard element. In two dimensions we choose $D=[-1, 1]\times[-1, 1]$.

We discuss first the properties of the expansions of the functions in $\mathscr{H}^{1,1}$ on the basis (2.1.16).

Theorem 2.2.1. *If $u(x, y)\in\mathscr{H}^{1,1}$, then*

$$\begin{aligned}u(x, y)=&\,u(-1, -1)Q_0^{-1}(x)Q_0^{-1}(y)+u(1, -1)Q_0^1(x)Q_0^{-1}(y)\\ &+u(1, 1)Q_0^1(x)Q_0^1(y)+u(-1, 1)Q_0^{-1}(x)Q_0^1(y)\\ &+\sum_{n=1}^{\infty} b_nQ_n(x)Q_0^{-1}(y)+\sum_{n=1}^{\infty} b_n'Q_n(x)Q_0^{-1}(y)\end{aligned}$$

$$+\sum_{n=1}^{\infty} c_n Q_0^{-1}(x) Q_n(y) + \sum_{n=1}^{\infty} c'_n Q_0^1(x) Q_n(y)$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{n,m} Q_n(x) Q_m(y) \tag{2.2.1}$$

in which

$$b_n = \frac{2n+1}{2}\int_{-1}^{1}\frac{\partial u}{\partial x}(x,\ -1)P_n(x)dx, \tag{2.2.2}$$

$$b'_n = \frac{2n+1}{2}\int_{-1}^{1}\frac{\partial u}{\partial x}(x,\ 1)P_n(x)dx, \tag{2.2.3}$$

$$c_n = \frac{2n+1}{2}\int_{-1}^{1}\frac{\partial u}{\partial y}(-1,\ y)P_n(y)dy, \tag{2.2.4}$$

$$c'_n = \frac{2n+1}{2}\int_{-1}^{1}\frac{\partial u}{\partial y}(1,y)P_n(y)dy, \tag{2.2.5}$$

$$d_{n,m} = \frac{2n+1}{2}\,\frac{2m+1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^2 u}{\partial x\,\partial y}(x,\ y)P_n(x)P_m(y)dx\,dy. \tag{2.2.6}$$

Proof. If $u(x,\ y)\in\mathscr{H}^{1,1}$, then the following identity holds:

$$u(x,\ y) = u(-1,\ -1) + \int_{-1}^{x}\frac{\partial u}{\partial x}(\xi,\ -1)d\xi + \int_{-1}^{y}\frac{\partial u}{\partial y}(-1,\ \eta)d\eta$$

$$+\int_{-1}^{x}\int_{-1}^{y}\frac{\partial^2 u}{\partial x\,\partial y}(\xi,\ \eta)d\xi\ d\eta. \tag{2.2.7}$$

Since

$$\frac{\partial u}{\partial x}(x,\ -1) = \sum_{n=0}^{\infty} b_n P_n(x),\quad \frac{\partial u}{\partial y}(-1,\ y) = \sum_{n=0}^{\infty} c_n P_n(y),$$

$$\frac{\partial^2 u}{\partial x\,\partial y}(x,\ y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} d_{n,m} P_n(x)\cdot P_m(y)$$

substitute these expressions in the identity (2.2.7) and notice the following

$$b_0 = \frac{1}{2}(u(1,\ -1) - u(-1,\ -1)),$$

$$d_0 = \frac{1}{2}(u(-1,\ 1) - u(-1,\ -1)),$$

$$d_{0,0} = \frac{1}{2}(u(1,\ 1) - u(1,\ -1) - u(-1,\ 1) + u(-1,\ -1)),$$

$$d_{n,0} = \frac{1}{2}(b'_n - b_n),$$

$$d_{0,n} = \frac{1}{2}(c'_n - c_n).$$

Then (2.2.1) follows easily. ▮

Remark 2.2.1. Since $Q_n(x)Q_m(y)$ $(n,\ m\geq 1)$ equals zero at the boundary of D, and b_n, b'_n, c_n, c'_n are only determined by the tangential derivatives along the corresponding boundary of D, this makes the basis (2.1.16) able to act as the C^0 hierarchical basis on the square D.

In fact, suppose that a mesh is given, the elements of which are curvilinear quadrilaterals. Let its edges be represented by parametrized equations (the sides are numbered as shown in Fig. 2.2.1):

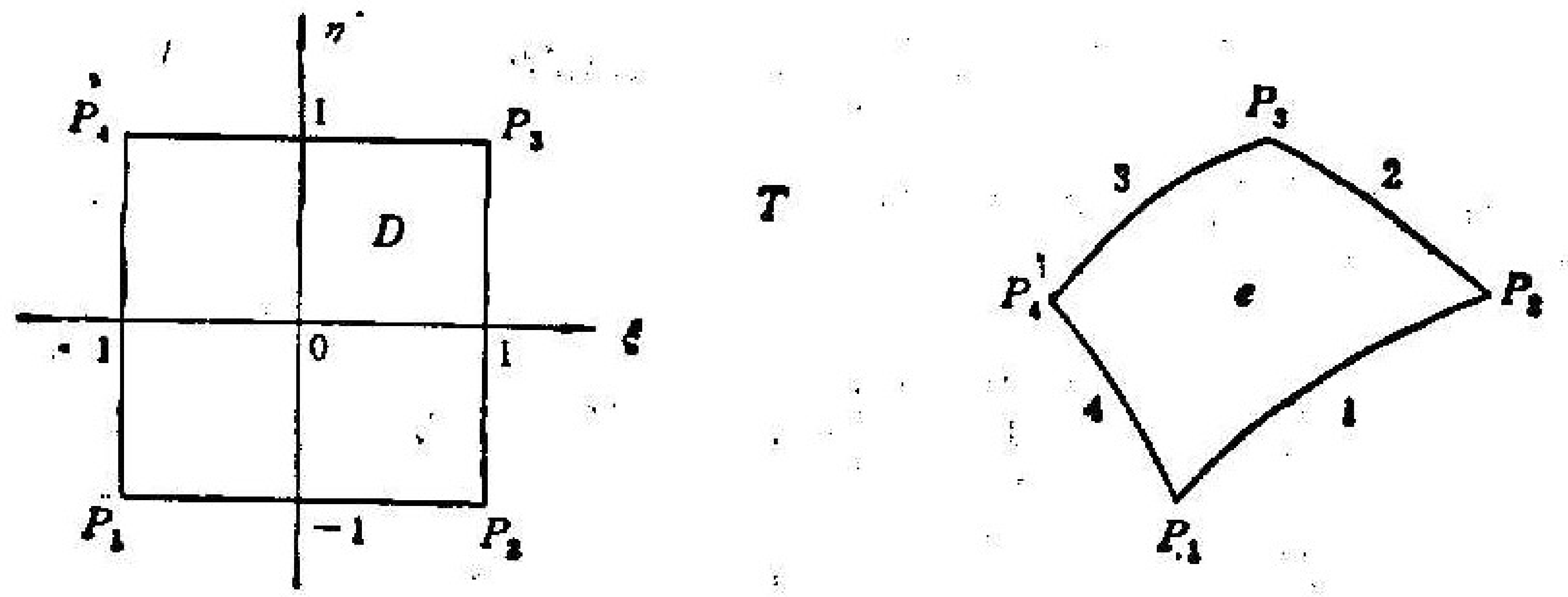

Fig. 2.2.1

$$\begin{cases} x = x_i(s), \\ y = y_i(s), \quad -1 \leqslant s \leqslant 1, \ i = 1, 2, 3, 4. \end{cases} \tag{2.2.8}$$

At the vertices, we must have

$$\begin{aligned} &x_1(-1) = x_4(-1),\ x_1(1) = x_2(-1),\ x_2(1) = x_3(1),\ x_3(-1) = x_4(1), \\ &y_1(-1) = y_4(-1),\ y_1(1) = y_2(-1),\ y_2(1) = y_3(1),\ y_3(-1) = y_4(1). \end{aligned} \tag{2.2.9}$$

A local mapping T

$$\begin{cases} x = x(\xi, \eta), \\ y = y(\xi, \eta) \end{cases} \tag{2.2.10}$$

is C^0 compatible, if on the edges of D equations (2.2.10) are reduced to the corresponding parametrized equations (2.2.8). Becauce of the properties mentioned in Remark 2.2.1 the partial sum of (2.2.2) under the local mapping T will be continuous across the boundary of the adjacent curvilinear quadrilateral elements of the given mesh, provided the same polynomial degrees on the corresponding edges of D are selected. Such a mapping was given in [4] on $[0, 1] \times [0, 1]$. For $D = [-1, 1] \times [-1, 1]$ it should be written as

$$\begin{cases} x = x(\xi, \eta) = x_1(\xi)Q_0^{-1}(\eta) + x_2(\eta)Q_0^1(\xi) + x_3(\xi)Q_0^1(\eta) \\ \quad + x_4(\eta)Q_0^{-1}(\xi) - x_1(-1)Q_0^{-1}(\xi)Q_0^{-1}(\eta) \\ \quad - x_1(1)Q_0^1(\xi)Q_0^{-1}(\eta) - x_3(1)Q_0^1(\xi)Q_0^1(\eta) \\ \quad - x_3(-1)Q_0^{-1}(\xi)Q_0^1(\eta), \\ y = y(\xi, \eta) = y_1(\xi)Q_0^{-1}(\eta) + y_2(\eta)Q_0^1(\xi) + y_3(\xi)Q_0^1(\eta) \\ \quad + y_4(\eta)Q_0^{-1}(\xi) - y_1(-1)Q_1^{-1}(\xi)Q_0^{-1}(\eta) \\ \quad - y_1(1)Q_0^1(\xi)Q_0^{-1}(\eta) - y_3(1)Q_0^1(\xi)Q_0^1(\eta) - y_3(-1)Q_0^{-1}(\xi)Q_0^1(\eta). \end{cases} \tag{2.2.11}$$

Of course one has to verify that the Jacobian of the mapping is not zero. In section 3 we will consider two local mappings which will be used in the h–p method.

We summarize the above result:

Theorem 2.2.2. *In two dimensions if the standard element is the square $D = [-1, 1] \times [-1, 1]$, then the set (2.1.16) forms a C^0 hierarchical basis.*

Remark 2.2.2. The hierarchical basis can be classified in three groups:

(1) the vertex modes: $Q_0^{-1}(x) \cdot Q_0^{-1}(y)$, $Q_0^1(x) \cdot Q_0^{-1}(y)$, $Q_0^1(x) \cdot Q_0^1(y)$, $Q_0^{-1}(x) \cdot Q_0^1(y)$, which are bilenear with value 1 at one vertex and zero at the others; (2) the edge modes: $Q_n(x) \cdot Q_0^{-1}(y)$, $Q_n(x) \cdot Q_0^{-1}(y)$, $Q_0^{-1}(x) \cdot Q_n(y)$, $Q_0^1(x) \cdot Q_n(y)$, $n = 1, 2, \cdots$, which are polynomials on one edge and zero on the others; (3) the internal modes

$Q_n(x)\cdot Q_m(y)$, n, $m=1, 2, \cdots$, which are zero on all edges of the square.

2.3. Error analysis for 2-D square C^0 hierarchical elements

Similarly to the one dimensional case, for $u(x, y)\in \mathscr{H}^{1,1}$ we consider the partial sum

$$\begin{aligned} u_{p,q}(x, y) = & u(-1, -1)Q_0^{-1}(x)Q_0^{-1}(y)+u(1, -1)Q_0^1(x)Q_0^{-1}(y) \\ & +u(1, 1)Q_0^1(x)Q_0^1(y)+u(-1, 1)Q_0^{-1}(x)Q_0^1(y) \\ & +\sum_{n=1}^{p-1} b_nQ_n(x)Q_0^{-1}(y)+\sum_{n=1}^{p-1} b'_nQ_n(x)Q_0^{-1}(y) \qquad (2.3.1) \\ & +\sum_{n=1}^{q-1} c_nQ_0^{-1}(x)Q_n(y)+\sum_{n=1}^{q-1} c'_nQ_0^1(x)Q_n(y) \\ & +\sum_{n=1}^{p-1}\sum_{m=1}^{q-1} d_{n,m}Q_n(x)Q_m(y), \end{aligned}$$

where b_n, b'_n, c_n, c'_n, $d_{n,m}$ are defined as in (2.2.2)—(2.2.6). Then it is a polynomial of degree p in x and degree q in y.

We have the following result:

Theorem 2.3.1. *If $u\in H^{k+1}(D)$, $p\geqslant k\geqslant 1$, and $e_{p,p}=u-u_{p,p}$ then*

$$\|Du-Du_{p,p}\|_{L_2(D)\times L_2(D)}\equiv\left\{\left\|\frac{\partial e_{p,p}}{\partial x}\right\|^2_{L_2(D)}+\left\|\frac{\partial e_{p,p}}{\partial y}\right\|^2_{L_2(D)}\right\}^{\frac{1}{2}}$$
$$\leqslant\sqrt{2}\left[\frac{e}{2p}\right]^k|u|_{k+1,D}, \qquad (2.3.2)$$

$$\|u-u_{p,p}\|_{L_2(D)}\equiv\|e_{p,p}\|_{L_2(D)}\leqslant\frac{1}{p}\left[\frac{e}{2p}\right]^k|u|_{k+1,D}. \qquad (2.3.3)$$

The proof is similar to the 1-d case and so is omitted.

Remark 2.3.1. One could choose $p\neq q$. However, the above estimates will then be controlled by the smaller one between p and q, unless the anisotropic property of the given function is known.

We now consider the estimates under local mappings.

Lemma 2.3.1. *Let e be a given mesh element, and $D=[-1, 1]\times[-1, 1]$. Let*

$$\begin{cases} x=x(\xi, \eta), \\ y=y(\xi, \eta) \end{cases} \qquad (2.3.4)$$

be a given C^0-compatible local mapping, and

$$U(\xi, \eta)=u(x(\xi, \eta), y(\xi, \eta)). \qquad (2.3.5)$$

Suppose that

$$M=\max_{(\xi,\eta)\in D}\left\{\sqrt{\left[\frac{\partial x}{\partial\xi}\right]^2+\left[\frac{\partial y}{\partial\xi}\right]^2},\ \sqrt{\left[\frac{\partial x}{\partial\eta}\right]^2+\left[\frac{\partial y}{\partial\eta}\right]^2}\right\}, \qquad (2.3.6)$$

$$0<\delta<\left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right|^{\frac{1}{2}}\leqslant C. \qquad (2.3.7)$$

Then

$$\|Du\|_{L_2(e)\times L_2(e)}\leqslant\frac{\sqrt{2}M}{\delta}\|DU\|_{L_2(D)\times L_2(D)}, \qquad (2.3.8)$$

$$\|u\|_{L_2(e)}\leqslant C\|U\|_{L_2(D)}. \qquad (2.3.9)$$

Proof. One has

$$\iint_e \left[\frac{\partial u}{\partial x}\right]^2 dx\,dy = \iint_D \left[\frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x}\right]^2 \left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right| d\xi\,d\eta$$

$$= \iint_D \left[\frac{\partial U}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial U}{\partial \eta}\frac{\partial y}{\partial \xi}\right]^2 \left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right|^{-1} d\xi\,d\eta$$

and

$$\iint_e \left[\frac{\partial u}{\partial y}\right]^2 dx\,dy = \iint_D \left[-\frac{\partial U}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial U}{\partial \eta}\frac{\partial x}{\partial \xi}\right]^2 \left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right|^{-1} d\xi\,d\eta.$$

Hence

$$\iint_e \left\{\left[\frac{\partial u}{\partial x}\right]^2 + \left[\frac{\partial u}{\partial y}\right]^2\right\} dx\,dy$$

$$= \iint_D \left\{\left[\frac{\partial U}{\partial \xi}\right]^2 \left[\left[\frac{\partial x}{\partial \eta}\right]^2 + \left[\frac{\partial y}{\partial \eta}\right]^2\right] - 2\frac{\partial U}{\partial \xi}\frac{\partial U}{\partial \eta}\left[\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta}\right]\right.$$

$$\left. + \left[\frac{\partial U}{\partial \eta}\right]^2 \left[\left[\frac{\partial x}{\partial \xi}\right]^2 + \left[\frac{\partial y}{\partial \xi}\right]^2\right]\right\} \left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right|^{-1} d\xi\,d\eta$$

$$< \frac{2M^2}{\delta^2} \iint_D \left\{\left[\frac{\partial U}{\partial \xi}\right]^2 + \left[\frac{\partial U}{\partial \eta}\right]^2\right\} d\xi\,d\eta,$$

$$\iint_e u(x, y)^2 dx\,dy = \iint_D U(\xi, \eta)^2 \left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right| d\xi\,d\eta \leqslant C^2 \iint U(\xi, \eta)^2 d\xi\,d\eta.$$

The results follow readily. ∎.

A direct application of the above is the error estimates by C^0 piecewise polynomials on a domain with a rectangular mesh.

Theorem 2.3.2. *Let Ω be a domain which can be partitioned with a rectangular mesh Δ. Let h be the maximal length of the edges of the mesh elements. Suppose that the mesh is regular, i.e. there is a constant $\gamma>0$ such that $h/h_{\min}<\gamma$ where $h_{\min}$ is the minimal edge length of the elements. If $u\in H^{k+1}(\Omega)$ (or $H^{k+1}(\Omega)\cap \mathring{H}^1(\Omega)$), then there is a C^0 piecewise polynomial $u_{p,p}(x, y)$ based on Δ with degree p in both x and y (if $u(x, y)$ vanishes on the boundary $\partial\Omega$ then so is $u_{p,p}(x, y)$) such that*

$$\|Du - Du_{p,p}\|_{L_2(\Omega)\times L_2(\Omega)} \leqslant C\left[\frac{eh}{4p}\right]^k |u|_{k+1,\Omega} \tag{2.3.10}$$

$$\|u - u_{p,p}\|_{L_2(\Omega)} \leqslant C\left[\frac{eh}{4p}\right]^{k+1} |u|_{k+1,\Omega} \tag{2.3.11}$$

where C is a constant which depends only on γ.

Proof. Let the mesh be constructed by lines parallel to the coordinate axes. A typical element of the mesh is a rectangle with vertices

$$P_1(x_1, y_1),\ P_2(x_2, y_1),\ P_3(x_2, y_2),\ P_4(x_1, y_2).$$

Then the local mapping is given by

$$\begin{cases} x = \frac{1}{2}(x_2 - x_1)\,\xi + \frac{1}{2}(x_2 + x_1), \\ y = \frac{1}{2}(y_2 - y_1)\,\eta + \frac{1}{2}(y_2 + y_1). \end{cases} \tag{2.3.12}$$

Thus the Jacobian $\dfrac{\partial(x, y)}{\partial(\xi, \eta)} = \dfrac{(x_2 - x_1)(y_2 - y_1)}{4} \geqslant \dfrac{h^2}{4\gamma}.$

And the theorem follows easily. ▮

Remark 2.3.1. A similar technique can be used for the approximation by polynomials of degree p (total degree in x and y) on squares. And it is easy to obtain similar results in the 3-dimensional case.

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