ON THE APPROXIMATION OF LINEAR HAMILTONIAN SYSTEMS*1)

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Abstract

When we study the oscillation of a physical system near its equilibrium and ignore dissipative effects, we may assume it is a linear Hamiltonian system (H-system), which possesses a special symplectic structure. Thus there arises a question: how to take this structure into account in the approximation of the H-system? This question was first answered by Feng Kang for finite dimensional H-systems[1-4].

We will in this paper discuss the symplectic difference schemes preserving the symplectic structure and its related properties, with emphasis on the infinite dimensional H-systems.

In the first section we propose the notion of symmetry of a difference scheme, and obtain the equivalence between symmetries and the conservation of first integrals. In the second section we discuss hyperbolic equations with constant coefficients in one space variable. This kind of H-system possesses not only a symplectic structure. but also a unitary structure. Our result is that a difference scheme is symplectic iff its amplification factors are of modulus one. In the third section we discuss symmetric hyperbolic equations with constant coefficients in several space variables. Although the antisymmtric operator of the symplectic structure is not invertible in that case, we obtain a similar conclusion. In the fourth section, we propose the notion of multiple-level symplectic difference schemes Finally, we derive the generating function for K-symplectic transformation, and construct a SDS for a hyperbolic equation with variable coefficients using the generating functions.

§1. Symmetries of Difference Schemes

a) Consider a linear Hamiltonian system with quadratic Hamiltonian $H(z)=\frac{1}{2}z^TAz$:

$$\frac{dz}{dt}=J^{-1}Az, \tag{0}$$

where $J=\begin{bmatrix}0 & I_n\\ -I_n & 0\end{bmatrix}$ and A is a $2n\times 2n$ symmetric matrix, and a difference scheme

$$z^{m+1}=\phi(J^{-1}A)z^m. \tag{1}$$

Definition. *We say* (1) *is a symplectic difference scheme if the matrix* $\phi=\phi(J^{-1}A)$ *is a symplectic matrix.*

Now we perform a canonical coordinate transformation $z\to w:z=Pw$, and the H-system written in the new coordinate w is

$$\frac{dw}{dt}=J^{-1}P^TAPw \tag{2}$$

and the scheme (1) is

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$$w^{m+1}=P^{-1}\phi(J^{-1}A)Pw^m. \tag{3}$$

Now we construct the difference scheme for the H-system (2)

$$w^{m+1}=\phi(J^{-1}P^TAP)w^m. \tag{4}$$

Then there arises a problem: Is scheme (3) equivalent to scheme (4)? The answer leads to the notion of symmetry.

Definition. *We say a scheme (1) is invariant under a group G of linear symplectic transformations if*

$$P^{-1}\phi(J^{-1}A)P=\phi(J^{-1}P^TAP)$$

for all A symmetric and all $P\in G$.

Another question of interest is: If a quadratic form $f(z)=\frac{1}{2}z^TBz$ is a first integral of the H-system, is it conserved by scheme (1), i.e.,

$$f(z^{m+1})=f(z^m)$$

for all $z^{m+1}=\phi(J^{-1}A)z^m$?

Example 1. Euler's mid-point scheme

$$z^{m+1}-z^m=\tau J^{-1}A((z^{m+1}+z^m)/2), \tag{5}$$

where τ is the time step, is symplectic and is invariant under the full symplectic group $S_p(2n)$. All the quadratic first integrals are conserved by the scheme (5)[2].

Theorem 1. *Let the quadratic form $f(z)=\frac{1}{2}z^TBz$ is a first integral of the H-system (0), $G^t=\exp(tJ^{-1}B)$ is the phase flow of the H-system with Hamiltonian $f(z)$, the scheme (1) is symplectic. Then f is conserved by (1) iff (1) is invariant under the phase flow G^t.*

Proof. By definition,

$$(G^t)^{-1}\phi(J^{-1}A)G^t=\phi((G^t)^{-1}J^{-1}AG^t)=\phi(J^{-1}(G^t)^TAx^t), \tag{6}$$

f is a first integral of the H-system. By the Noether theorem, $(G^t)^TAG^t=A$. Thus it follows from (6) that

$$(G^t)^{-1}\phi(J^{-1}A)G^t=\phi(J^{-1}A), \text{ or } \phi(J^{-1}A)G^t=G^t\phi(J^{-1}A).$$

Taking derivative at $t=0$ and making use of $\left.\frac{d}{dt}\right|_{t=0}G^t=J^{-1}B$, we obtain

$$(\phi(J^{-1}A))^TB\phi(J^{-1}A)=(\phi(J^{-1}A))^TJ\phi(J^{-1}A)J^{-1}B=JJ^{-1}B=B.$$

Thus the symmetries imply conservations.

Now we assume conservations, i.e, $(\phi(J^{-1}A))^TB\phi(J^{-1}A)=B$. Then

$$J^{-1}B(\phi(J^{-1}A))=\phi(J^{-1}A)J^{-1}B,$$

and

$$\phi(J^{-1}A)\exp(J^{-1}Bt)=\exp(J^{-1}Bt)\phi(J^{-1}A).$$

Example 2. The generalized Euler scheme

$z^{m+1}=\phi(\tau J^{-1}A)z^m$, where $\phi_p(\lambda)$ is the p-th diagonal Pade approximant of $\exp\lambda$, is invariant under $S_p(2n)$. Hence all the first integrals of quadratic form are conserved by the difference scheme[1,2,4].

b) We will generalize the above notion to the nonlinear Hamiltonian system: