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MULTIVARIATE PADÉ APPROXIMATION AND ITS APPLICATION IN SOLVING SYSTEMS OF NONLINEAR EQUATIONS*1)

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Abstract

For a certain kind of multivariate Padé approximation problems, we establish in this paper some results about the solvability and uniqueness of its solution. We give also the necessary and sufficient conditions for the continuity of Padé approximation operator. The application of such approximants in finding solutions of systems of nonlinear equations is considered, and some numerical examples are given, in which it is shown that the Padé methods are more effective than the Newton methods in some cases.

§1. Introduction

It is well known that univariate Padé approximation(UPA) is a useful tool as rational approximants to a specified power series and has numerous applications in the fields of numerical analysis, theoretical physics and many other subjects(see [1],[2]). The extensions of UPA to the bivariate and multivariate cases were first considered by Chisholm [3] and then followed by Hughes [6],[7], Lutterodt [9],[10], Karlsson and Wallin [8], Cuyt [5] and others. There are numerous possibilities for the extension and generalization by requiring the rational approximants to have certain special properties. Several different definitions for multivariate Padé approximants(MPA) were introduced and much research work was done in the past decade. For a general review on this subject we refer to references [2],[4],[5].

To start with, we introduce some notations. Given a positive integer n, we write $\mathbf{Z}_+^n = \{\alpha : \alpha = [\alpha_1, \cdots, \alpha_n]^T, \alpha_i \in \mathbf{Z}_+, i = 1, \cdots, n\}$, where $\mathbf{Z}_+$ denote the set of all nonnegative integers. The set $\mathbf{Z}_+^n$ is often referred to as the set of multi-indices. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}$, we shall use the standard notations $\mathbf{a} \le \mathbf{b}$ if and only if $a_i \le b_i, i = 1, \cdots, n$, $\mathbf{a} + \mathbf{b} = [a_1 + b_1, \cdots, a_n + b_n]^T$. If $\alpha \in \mathbf{Z}_+^n, \mathbf{x} \in \mathbf{R}^n$, we write $|\alpha| = \sum_{i=1}^n \alpha_i$, $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We also use the notations $0 = [0, \cdots, 0]^T$, $1 = [1, \cdots, 1]^T$ and $\mathbf{e}_i$ = unit vector in the i-th direction $= [0, \cdots, 1, \cdots, 0]^T$.

The general framework of the definition for MPA to a given function $f(\mathbf{x}) = \sum_{\alpha \in \mathbf{Z}_+^n} c_\alpha \mathbf{x}^\alpha$ consists of choosing three multi-index sets N, D and E in $\mathbf{Z}_+^n$ and finding two polynomials $p(\mathbf{x}) = \sum_{\alpha \in N} a_\alpha \mathbf{x}^\alpha$, $q(\mathbf{x}) = \sum_{\alpha \in D} b_\alpha \mathbf{x}^\alpha$, such that $p(\mathbf{x}) - f(\mathbf{x}) q(\mathbf{x}) = \sum_{\alpha \in \mathbf{Z}_+^n \setminus E} e_\alpha \mathbf{x}^\alpha$, $q(0) = 1$.

In the case of UPA, the problem of $(m/l)_f$ Padé approximants is to take $N = \{0, 1, \cdots, m\}$, $D = \{0, 1, \cdots, l\}$ and $E = \{0, 1, \cdots, m + l\}$. However, in the multivariate case, N, D and E can be taken in various manners. Therefore, there may be many different definitions

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for MPA (see [3]–[10]). For the general framework (1.1) of MPA, it is difficult to give simple and easily used conditions under which the existence and uniqueness of MPA are guaranteed. In this paper, we choose N, D and E as follows

$$N = \{\alpha : \alpha \in \mathbf{Z}_+^n,\ |\alpha| \le 1\}, \tag{1.2}$$

$$D = \{\alpha : \alpha \in \mathbf{Z}_+^n,\ |\alpha| \le k,\ \alpha \ne k\mathbf{e}_i,\ i = 1, \cdots, n\}, \tag{1.3}$$

$$E = \{\alpha : \alpha \in \mathbf{Z}_+^n,\ |\alpha| \le k\}, \tag{1.4}$$

where $k \ge 2$. For such N, D and E we study the existence of MPA in §2. The problem of uniqueness is considered in §3. In §4, we investigate the problem of the continuity of Padé approximation operators. In the last section, we consider the problem of application of MPA in solving systems of nonlinear equations.

§2. Existence

For N, D and E defined as in (1.2)–(1.4), let $p(\mathbf{x}) = a_0 + \sum_{|\alpha|=1} a_\alpha \mathbf{x}^\alpha$, $q(\mathbf{x}) = 1 + \sum_{\alpha \in D\setminus\{0\}} b_\alpha \mathbf{x}^\alpha$, Then equation (1.1) is equivalent to the following

$$a_0 = c_0, \quad a_\alpha = c_\alpha + b_\alpha c_0, \quad \alpha \in N \setminus \{0\}, \tag{2.1}$$

$$\sum_{\substack{\alpha+\beta=\gamma \\ \beta \in D\setminus\{0\}}} c_\alpha b_\beta + c_\gamma = 0, \quad \gamma \in E \setminus N. \tag{2.2}$$

Equations (2.1)–(2.2) are linear systems with $|E|$ ($|E|$ stands for the cardinality of E) unknowns and $|E|$ equations. Therefore it is expected to have the unique solution.

Take $\alpha = \mathbf{e}_i$ in (2.1), $\gamma = j\mathbf{e}_i$, $j = 2, \cdots, k$, in (2.2); equations (2.1)–(2.2) have the following equations as their sub–equations

$$a_0 = c_0, \quad a_1^{(i)} = c_1^{(i)} + b_1^{(i)} c_0, \tag{2.3}$$

$$\sum_{s+t=j} c_s^{(i)} b_t^{(i)} + c_j^{(i)} = 0, \quad j = 2, \cdots, k,$$

where $a_1^{(i)} = a_{\mathbf{e}_i}$, $b_t^{(i)} = b_{t\mathbf{e}_i}$, $c_s^{(i)} = c_{s\mathbf{e}_i}$, and $i = 1, \cdots, n$. It is not hard to see that equations (2.3) are the Padé equations for UPA $(1/k-1)_{g_i}$ with $g_i = f(x\mathbf{e}_i)$. Hence problem (1.1) is connected closely with UPA $(1/k-1)_{g_i}$, $i = 1, \cdots, n$. Suppose equations (2.1)–(2.2) are solvable; then (2.3) are solvable also for $i = 1, \cdots, n$. From the theory of the UPA [14], it follows that

$$\operatorname{rank} H_i(1, k-2, k-2) = \operatorname{rank} H_i(2, k-1, k-2), \quad i = 1, \cdots, n, \tag{2.4}$$

where

$$H_i(m, j, k) = \begin{bmatrix} c_m^{(i)} & c_{m-1}^{(i)} & \cdots & c_{m-j}^{(i)} \\ c_{m+1}^{(i)} & c_m^{(i)} & \cdots & c_{m-j+1}^{(i)} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m+k}^{(i)} & c_{m+k-1}^{(i)} & \cdots & c_{m+k-j}^{(i)} \end{bmatrix}, \quad i = 1, \cdots, n.$$

Hence relations (2.4) are necessary conditions for the solvability of (2.1)–(2.2). Now we shall show that if $c_0 \ne 0$, these conditions are also sufficient. In fact, from (2.2) we have

$$b_\gamma = -c_0^{-1} \sum_{\substack{\alpha+\beta=\gamma \\ \alpha\ne 0, \beta \in D}} c_\alpha b_\beta, \quad \gamma \in D \setminus \{0\}. \tag{2.5}$$