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ON THE GENERAL INTERPOLATION FORMULAS FOR DISCRETE FUNCTIONAL SPACES (I) *1)

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Abstract

The general interpolation formulas for discrete functional spaces of discrete functions with one index are presented in the note. These are the relationship among the discrete norms in the forms of the summation of powers, the maximum modulo, the Hölder and Lipschitz coefficients for the discrete functions.

§1

The imbedding theorems and the interpolation formulas for the functions of Sobolev's spaces are very important and useful in the linear and nonlinear theory of partial differential equations and systems[1–2]. Similarly the extensions of such interpolation formulas for the spaces of discrete functions play the extremely important part in the study of the finite difference approximations to the problems of linear and nonlinear partial differential equations and systems. In [3–5] some special interpolation formulas for the spaces of discrete functions defined on the finite and infinite segments and domains are established. These interpolation formulas for discrete functional spaces are used in the study of convergence and stability behavior for the finite difference schemes and in the construction of weak, generalized and classical solutions for the various problems of linear and nonlinear partial differential equations and systems.

In the present note, some general interpolation formulas for the discrete functional spaces of discrete functions with one index are presented. These general interpolation formulas give the connected relations among the discrete norms as the types of the summation of powers, the maximum modulo, the Lipschitz and Hölder quotients for different discrete functional spaces.

§2

Let us divide the finite interval $[0,l]$ into the small grid points $\{x_j = jh \mid j = 0,1,\cdots,J\}$, where $Jh = l < \infty, J$ is an integer and h is the stepsize. The discrete

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function $u_h = \{u_j|\ j = 0,1,\cdots,J\}$ is defined on the grid points $\{x_j|\ j = 0,1,\cdots,J\}$. Let us denote $\Delta_+ u_j = u_{j+1} - u_j$ or simply $\Delta u_j = u_{j+1} - u_j$ $(j = 0,1,\cdots,J-1)$ and $\Delta_- u_j = u_j - u_{j-1} (j = 1,2,\cdots,J)$. For the discrete functions $u_h = \{u_j|\ j = 0,1,\cdots,J\}$ and its difference quotients $\delta^k u_h = \left\{\frac{\Delta^k u_j}{h^k}\Big| j = 0,1,\cdots,J-k\right\}$ of order $k = 0,1,\cdots$, the discrete norms can be defined and denoted by the notations as follows: For $1 \le p < \infty$ we have

$$\|\delta^k u_h\|_p = \Big(\sum_{j=0}^{J-k} \Big|\frac{\Delta^k u_j}{h^k}\Big|^p h\Big)^{1/p} \tag{1}$$

and for $p = \infty$, we have

$$\|\delta^k u_h\|_\infty = \max_{j=0,1,\cdots,J-k} \Big|\frac{\Delta^k u_j}{h^k}\Big|, \tag{2}$$

where $k = 0,1,\cdots$. For the norms of discrete functions u_h with negative index $p < 0$, we can define the norms in the following way. Let $s = \left[\frac{1}{|p|}\right]$ and $\lambda = \left\{\frac{1}{|p|}\right\}$ be the integer and decimal parts of the positive value $\frac{1}{|p|}$ respectively, where s is an integer and $0 \le \lambda \le 1$ is a decimal number. For the case $\lambda = 0$, we define the norm of discrete functions with negative index as

$$\|\delta^k u_h\|_p = \max_{j=0,1,\cdots,J-(k+s)} \Big|\frac{\Delta^{k+s} u_j}{h^{k+s}}\Big| = \|\delta^{k+s} u_h\|_\infty, \tag{3}$$

where $k = 0,1,\cdots$ and $s = \left[\frac{1}{|p|}\right] = \frac{1}{|p|}$. For the case $0 < \lambda \le 1$, we define the norm as

$$\|\delta^k u_h\|_p = \max_{\substack{r>m \\ r,m=0,1,\cdots,J-(k+s)}} \frac{\Big|\frac{\Delta^{k+s} u_r}{h^{k+s}} - \frac{\Delta^{k+s} u_m}{h^{k+s}}\Big|}{[(r-m)h]^\lambda}, \tag{4}$$

where $s = \left[\frac{1}{|p|}\right]$, $0 < \lambda = \left\{\frac{1}{|p|}\right\} \le 1$ and $k = 0,1,\cdots$.

For the case of definition (1), the norm of discrete function u_h is in the form of summation of powers. For the case of definitions (2) and (3), these are the discrete norms of maximum modulo type. For the case of definition (4), it is the discrete norm of type of Hölder coefficient. When $\lambda = 1$, it is of the type of Lipschitz coefficient.

The general interpolation formulas for the discrete functional spaces of discrete function with one index for the various types of norms can be stated in the following theorem.

Theorem 1. *For any discrete function $u_k = \{u_j|\ j = 0,1,\cdots,J\}$ defined on the grid points $\{x_j = jh|\ j = 0,1,\cdots,J\}$ of interval $[0,l]$ of finite length $l < \infty$ with positive and negative indices $-(n-k-1/r) \le 1/p \le 1$ and the constants $1 \le q,r \le \infty$ and $0 \le k < n$, there is the relations among the norms as*

$$\|\delta^k u_h\|_p \le C(\|u_h\|_q^{1-\alpha} \|\delta^n u_h\|_r^\alpha + l^{1/p-1/q-k} \|u_h\|_q) \tag{5}$$