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VERIFYING THE IMPLICITIZATION FORMULAE FOR DEGREE n RATIONAL BÉZIER CURVES*1)

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Abstract

This is a continuation of short communication[1]. In [1] a verification of the implicitization equation for degree two rational Bézier curves is presented which does not require the use of resultants. This paper presents these verifications in the general cases, i.e., for degree n rational Bézier curves. Thus some interesting interplay between the structure of the $n \times n$ implicitization matrix and the de Casteljau algorithm is revealed.

Key words: Rational Bézier curve, Implicitization, Resultant, de Casteljau algorithm.

1. Introduction

In order to investigate implicit representations of parametric curves and surfaces, the traditional algebraic geometry theory is always used. Recently some research reports, e.g.[1], show that the rising Blossoming principle[2,3] is more intuitive and efficient than the method of algebraic geometry for the implicitization. This paper is a cotinuation of [1].

Given a degree n plane rational Bézier curve

$$\mathbf{P}(s) = [\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_n] \left[(1-s)^n, \binom{n}{1}(1-s)^{n-1}s, \cdots, s^n\right]^T \quad (0 \le s \le 1), \qquad (1)$$

where $\mathbf{P}_i = (w_i x_i, w_i y_i, w_i)$ are the homogeneous Bézier control points. Denote by

$$\begin{vmatrix} \mathbf{P} \\ \mathbf{P}_i \\ \mathbf{P}_j \end{vmatrix} \equiv w_i w_j \begin{vmatrix} x & y & w \\ x_i & y_i & 1 \\ x_j & y_j & 1 \end{vmatrix}, \qquad (2)$$

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where $\mathbf{P} = (x, y, w)$ is the homogeneous point. Let

$$\overline{L}_{ij} = \binom{n}{i}\binom{n}{j}\begin{vmatrix} \mathbf{P} \\ \mathbf{P}_i \\ \mathbf{P}_j \end{vmatrix}, \tag{3}$$

$$L_{hk} = \sum_{l=\max(0,h+k+1-n)}^{\min(h,k)} \overline{L}_{l,h+k+1-l}, \tag{4}$$

$$F(\mathbf{P}, \mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_n) = F_{n\times n} = \begin{bmatrix} L_{00} & L_{01} & \dots & L_{0,n-1} \\ L_{10} & L_{11} & \dots & L_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ L_{n-1,0} & L_{n-1,1} & \dots & L_{n-1,n-1} \end{bmatrix}. \tag{5}$$

Using polynomial resultants, the implicit equation of (1) can be written[4,5] as [Goldman, Sederberg et al., 1984]

$$f(x, y) = \det\ (F) = 0. \tag{6}$$

In the short communication[1] a proof is presented — which does not rely on the theory of resultants — that $f(x, y) = 0$ for all points on the degree two rational Bézier curve, and hence some insight into the determinantal structure of the implicit equation is given. But in the general cases, i.e., for degree n rational Bézier curves, the proof is not given yet. This paper will present these verifications, and show that the de Casteljau algorithm[2,3] is very useful to investigate the implicitization equations of parametric curves.

2. Verifying the Implicitization Equations

We will prove for all points on the degree n rational Bézier curve,

$$f(x, y) = \det\ (F) = 0.$$

Before the proof we will show the following

Theorem 1. *If subdivide the rational Bézier curve* (1) *at an arbitrary parameter value α, and denote a part of the curve* (1) *that corresponds to* $[0, \alpha]$ *by a new degree n rational Bézier curve*

$$\mathbf{P}^*(s) = [\mathbf{P}_0^*, \mathbf{P}_1^*, \cdots, \mathbf{P}_n^*]\left[(1-s)^n, \binom{n}{1}(1-s)^{n-1}s, \ldots, s^n\right]^T \quad (0 \le s \le 1), \tag{7}$$

then we have

$$\alpha \cdot A(\alpha) \cdot F(\mathbf{P}, \mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n) \cdot A^T(\alpha) = F(\mathbf{P}, \mathbf{P}_0^*, \mathbf{P}_1^*, \cdots, \mathbf{P}_n^*). \tag{8}$$

Where $A(\alpha)$ can be shown to be a lower triangular $n \times n$ matrix, with the elements

$$a_{ij} = \binom{n-j}{i-j}\beta^{i-j}\alpha^{j-1} \quad (i, j = 1, 2, \cdots, n), \tag{9}$$

in which

$$\beta = 1 - \alpha. \tag{10}$$

In order to prove Theorem 1, we need the following two Lemmas.

Lemma 1. *Let*

$$\phi_1(a,b,c,r) = \sum_{k=0}^{r} \binom{a+1}{k} \binom{b}{r+c-k}, \tag{11}$$

$$\phi_2(a,b,c,r) = \sum_{k=0}^{r} \binom{a-k}{r-k} \binom{b+k}{c+k}. \tag{12}$$

Then

$$\phi_1(a,b,c,r) = \phi_2(a,b,c,r) \quad (0 \le r \le a). \tag{13}$$

Proof. We will perform mathematical induction for a. First, we have

$$\phi_1(0,b,c,r) = \phi_2(0,b,c,r).$$

Now we show that Lemma 1 hold for $a = d+1$ provided it hold when a is replaced by d. In fact, from the postulate of our induction, we have

$$\begin{aligned}\phi_1(d+1,b,c,r) &= \sum_{k=0}^{r} \left[\binom{d+1}{k-1} + \binom{d+1}{k}\right] \binom{b}{r+c-k} \\ &= \sum_{k=0}^{r-1} \binom{d+1}{k} \binom{b+1}{(r-1)+(c+1)-k} + \binom{d+1}{r}\binom{b}{c} \\ &= \sum_{k=0}^{r-1} \binom{d-k}{r-1-k} \binom{b+1+k}{c+1+k} + \binom{d+1}{r}\binom{b}{c} = \phi_2(d+1,b,c,r).\end{aligned}$$

Thus Lemma 1 is proved.

Lemma 2.

$$\phi_1(a,b,0,r) = \phi_2(a,b,0,r) = \binom{a+b+1}{r}. \tag{14}$$

Proof. By the following identical relation [Abramowitz & Stegun'72, P822]:

$$\sum_{k=0}^{r} \binom{p}{k} \binom{q}{r-k} = \binom{p+q}{r}, \tag{15}$$

Lemma 2 is evident.

Proof of Theorem 1. In this paper, we set

$$\binom{n}{p} = 0 \quad (p > n). \tag{16}$$

Thus (4) can be rewritten as

$$L_{hk} = \sum_{l=0}^{\min(h,k)} \overline{L}_{l,h+k+1-l}. \tag{17}$$

Also, we denote a $n \times n$ metrix E by $(e_{i+1,j+1})$, where $e_{i+1,j+1}$ is an element of E that lies in row $i+1$ and column $j+1$, $(i, j = 0, 1, \cdots, n-1)$. Now let

$$F(\mathbf{P}, \mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_n) = (f_{h+1,k+1}), \tag{18}$$

$$F(\mathbf{P}, \mathbf{P}_0^*, \mathbf{P}_1^*, \cdots, \mathbf{P}_n^*) = (f_{i+1,j+1}^*). \tag{19}$$

Without loss of generality, we will prove

$$\alpha(afa^T)_{i+1,j+1} = f_{i+1,j+1}^* \tag{20}$$

for the case of $i \geq j$ only. First, we have

$$(af)_{i+1,k+1} = \sum_{h=0}^{n-1} a_{i+1,h+1} f_{h+1,k+1} = \sum_{h=0}^{i} a_{i+1,h+1} L_{hk}$$
$$= \sum_{h=0}^{i} \binom{n-1-h}{i-h} \beta^{i-h} \alpha^h L_{hk}.$$

let

$$u_{hk} = \binom{n-1-h}{i-h}\binom{n-1-k}{j-k} \alpha^{h+k} \beta^{i+j-h-k} \ (0 \leq h \leq i,\ 0 \leq k \leq j), \tag{21}$$

$$v_{lm} = \alpha^{l+m-1} \beta^{i+j+1-l-m} \begin{vmatrix} \mathbf{P} \\ \mathbf{P}_l \\ \mathbf{P}_m \end{vmatrix}, \tag{22}$$

$$w = \begin{cases} 0 & (i = j) \\ \displaystyle\sum_{h=j+1}^{i} u_{hj} L_{hj} & (i > j), \end{cases} \tag{23}$$

$$\sigma_{klm} = \binom{n-1-k}{j-k}\binom{n-l-m+k}{i+1-l-m+k}. \tag{24}$$

Then when $i > j$,

$$w = \sum_{l=0}^{j} \sum_{h=j+1}^{i} u_{hj} \overline{L}_{l,h+j+1-l},$$

and hence

$$(afa^T)_{i+1,j+1} = \sum_{k=0}^{n-1} (af)_{i+1,k+1} a_{k+1,j+1}^T = \sum_{k=0}^{j} \sum_{h=0}^{i} u_{hk} L_{hk}$$
$$= \sum_{k=0}^{j} \sum_{h=0}^{k} u_{hk} L_{hk} + \sum_{k=0}^{j-1} \sum_{h=k+1}^{i} u_{hk} L_{hk} + w$$

$$= \sum_{k=0}^{j} \sum_{h=0}^{k} u_{hk} \sum_{l=0}^{h} \overline{L}_{l,h+k+1-l} + \sum_{k=0}^{j-1} \sum_{h=k+1}^{i} u_{hk} \sum_{l=0}^{k} \overline{L}_{l,h+k+1-l} + w$$

$$= \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{h=l}^{k} u_{hk} \overline{L}_{l,h+k+1-l} + \sum_{k=0}^{j-1} \sum_{l=0}^{k} \sum_{h=k+1}^{i} u_{hk} \overline{L}_{l,h+k+1-l} + w$$

$$= \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{h=l}^{i} u_{hk} \overline{L}_{l,h+k+1-l}$$

$$= \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=1+k}^{i+1+k-l} \binom{n}{l} \binom{n}{m} \sigma_{klm} v_{lm} \quad (i, j = 0, 1, \ldots, n-1). \tag{25}$$

On the other hand, we know that

$$f^*_{i+1,j+1} = \sum_{k=0}^{j} \binom{n}{k} \binom{n}{i+j+1-k} \begin{vmatrix} \mathbf{P} \\ \mathbf{P}^*_k \\ \mathbf{P}^*_{i+j+1-k} \end{vmatrix}.$$

By the de Casteljau algorithm,

$$\mathbf{P}^*_k = \sum_{l=0}^{k} \binom{k}{l} \beta^{k-l} \alpha^l \mathbf{P}_l,$$

$$\mathbf{P}^*_{i+j+1-k} = \sum_{m=0}^{i+j+1-k} \binom{i+j+1-k}{m} \beta^{i+j+1-k-m} \alpha^m \mathbf{P}_m.$$

Therefore by applying the identity relation

$$\binom{n}{k} \binom{n}{i+j+1-k} \binom{k}{l} \binom{i+j+1-k}{m}$$
$$= \binom{n}{l} \binom{n}{m} \binom{n-l}{k-l} \binom{n-m}{i+j+1-m-k},$$

we have

$$f^*_{i+1,j+1} = \alpha \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{i+j+1-k} \binom{n}{l} \binom{n}{m} \binom{n-l}{k-l} \binom{n-m}{i+j+1-m-k} v_{lm}$$
$$(i, j = 0, 1, \cdots, n-1). \tag{26}$$

Now let

$$(afa^T)_{i+1,j+1} = \sum_{\substack{0 \le l \le j \\ 1 \le m \le i+j+1 \\ l+m \le i+j+1 \\ l < m}} \binom{n}{l} \binom{n}{m} \xi_{lm} v_{lm}, \tag{27}$$

$$f^*_{i+1,j+1} = \alpha \sum_{\substack{0\le l\le j\\ 1\le m\le i+j+1\\ l+m\le i+j+1\\ l<m}} \binom{n}{l}\binom{n}{m}\eta_{lm}v_{lm}. \tag{28}$$

From (25), it is easy to know that

$$\xi_{lm} = \sum_{k=\max(l,l+m-1-i)}^{\min(j,m-1)} \sigma_{klm} = \begin{cases} \xi^{(1)}_{lm} = \sum_{k=l}^{m-1}\sigma_{klm} & (m<j+1)\\ \xi^{(2)}_{lm} = \sum_{k=l}^{j}\sigma_{klm} & (j+1\le m\le i+1)\\ \xi^{(3)}_{lm} = \sum_{k=l+m-1-i}^{j}\sigma_{klm} & (m>i+1). \end{cases}$$

On the other hand, let

$$\eta_{lm} = \begin{cases} \eta^{(1)}_{lm} & (m<j+1)\\ \eta^{(2)}_{lm} & (j+1\le m\le i+1)\\ \eta^{(3)}_{lm} & (m>i+1). \end{cases}$$

Applying Lemma 1 and 2, we have

$$\begin{aligned} \eta^{(1)}_{lm} &= \sum_{k=l}^{\min(j,i+j+1-m)} \binom{n-l}{k-l}\binom{n-m}{i+j+1-m-k}\\ &\quad - \sum_{k=m}^{j}\binom{n-m}{k-m}\binom{n-l}{i+j+1-l-k}\\ &= \phi_1(n-1-l,n-m,i+1-m,j-l)\\ &\quad - \phi_1(n-1-m,n-l,i+1-l,j-m)\\ &= \phi_2(n-1-l,n-m,i+1-m,j-l)\\ &\quad - \phi_2(n-1-m,n-l,i+1-l,j-m)\\ &= \sum_{k=l}^{j}\sigma_{klm} - \sum_{k=m}^{j}\sigma_{klm} = \xi^{(1)}_{lm},\\ \eta^{(2)}_{lm} &= \sum_{k=l}^{\min(j,i+j+1-m)} \binom{n-l}{k-l}\binom{n-m}{i+j+1-m-k}\\ &= \phi_1(n-1-l,n-m,i+1-m,j-l)\\ &= \phi_2(n-1-l,n-m,i+1-m,j-l) = \xi^{(2)}_{lm},\\ \eta^{(3)}_{lm} &= \sum_{k=l}^{i+j+1-m}\binom{n-l}{k-l}\binom{n-m}{i+j+1-m-k} \end{aligned}$$

$$= \phi_1(n-1-l, n-m, 0, i+j+1-l-m)$$
$$= \phi_2(n+i-l-m, n-1-i, 0, i+j+1-l-m) = \xi_{lm}^{(3)}.$$

Thus

$$\xi_{lm} = \eta_{lm} \quad (\forall l, m). \tag{29}$$

And hence (20) hold, proof of Theorem 1 is completed.

Corollary 1. *For any point on the rational Bézier curve* (1), *we have*

$$f(x, y, w) = \det(F) = 0. \tag{30}$$

Proof. By Theorem 1, we obtain

$$\alpha^n \cdot \{\det(A(\alpha))\}^2 . \det\{F(\mathbf{P}, \mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_n)\} = \det\{F(\mathbf{P}, \mathbf{P}_0^*, \mathbf{P}_1^*, \cdots, \mathbf{P}_n^*)\}$$

or

$$\alpha^{n^2} \cdot \det\{F(\mathbf{P}, \mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_n)\} = \det\{F(\mathbf{P}, \mathbf{P}_0^*, \mathbf{P}_1^*, \cdots, \mathbf{P}_n^*)\} \quad (0 \le \alpha \le 1).$$

Since

$$L_{i,n-1} \equiv L_{n-1,i} \equiv \overline{L}_{i,n} = \binom{n}{i} \begin{vmatrix} \mathbf{P} \\ \mathbf{P}_i \\ \mathbf{P}_n \end{vmatrix} \quad (i = 0, 1, \cdots, n-1),$$

$$\det\{F(\mathbf{P}_n, \mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_n)\} = \begin{vmatrix} L_{00}|_{\mathbf{P}=\mathbf{P}_n} & \cdots & L_{0,n-2}|_{\mathbf{P}=\mathbf{P}_n} & 0 \\ \vdots & & \vdots & \vdots \\ L_{n-2,0}|_{\mathbf{P}=\mathbf{P}_n} & \cdots & L_{n-2,n-2}|_{\mathbf{P}=\mathbf{P}_n} & 0 \\ 0 & \cdots & 0 & 0 \end{vmatrix} = 0.$$

Similarly,

$$\det\{F(\mathbf{P}(\alpha), \mathbf{P}_0^*, \mathbf{P}_1^*, \cdots, \mathbf{P}_n^*)\} \equiv 0 \quad (\forall \alpha).$$

Therefore

$$\det\{F(\mathbf{P}, \mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_n)\} = 0$$

for all s.

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