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THE INVERSE PROBLEM FOR PART SYMMETRIC MATRICES ON A SUBSPACE *1)

Zhen-yun Peng
(*College of Mathematics and Econometrics, Hunan University, Changsha 410082, China*)
(*Department of Mathematics, Loudi Teachers College, Loudi 417000, China*)

Xi-yan Hu Lei Zhang
(*College of Mathematics and Econometrics, Hunan University, Changsha 410082, China*)

Abstract

In this paper, the following two problems are considered:

Problem I. Given $S \in R^{n\times p}, X, B \in R^{n\times m}$, find $A \in SR_{s,n}$ such that $AX = B$, where $SR_{s,n} = \{A \in R^{n\times n} | x^T(A - A^T) = 0$, for all $x \in R(S)\}$.

Problem II. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$ such that $\|\hat{A} - A^*\| = \min_{A\in S_E} \|A - A^*\|$, where S_E is the solution set of Problem I.

The necessary and sufficient conditions for the solvability of and the general form of the solutions of problem I are given. For problem II, the expression for the solution, a numerical algorithm and a numerical example are provided.

Key words: Part symmetric matrix, Inverse problem, Optimal approximation.

1. Introduction

Let $R^{n\times m}, SR^{n\times n}, OR^{n\times n}$ denote the set of real $n \times m$ matrices, real $n \times n$ symmetric matrices and real $n \times n$ orthogonal matrices, respectively. The notation $R(A), N(A), A^+$ and $\|A\|$ stand for the column space, the null space, the Moore-Penrose generalized inverse and the Frobenius norm of a matrix A, respectively. I_k represents the identity matrix of order k. For $A = (a_{ij}) \in R^{n\times m}$ and $B = (b_{ij}) \in R^{n\times m}$, define $A * B = (a_{ij}b_{ij}) \in R^{n\times m}$ as Hardmard product of A and B.

Inverse problem for nonsymmetric matrices and symmetric matrices have studied in [1-5], and a series of perfect results have been obtained. However, inverse problem for matrices between above two kinds of matrices, i.e., inverse problem for part symmetric matrices on a subspace, have not been considered yet. In this paper, we will discuss this problem.

Let $SR_{s,n} = \{A \in R^{n\times n} | x^T(A - A^T) = 0$, for all $x \in R(S)\}$. we considered the following problems:

Problem I. Given $S \in R^{n\times p}, X, B \in R^{n\times m}$, find $A \in SR_{s,n}$ such that $AX = B$.

Problem II. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$ such that

$$\|\hat{A} - A^*\| = \min_{A\in S_E} \|A - A^*\|,$$

where S_E is the solution set of Problem I.

In Section 2, the necessary and sufficient conditions for the solvability of Problem I have been studied, and the general form of S_E has been given. In Section 3, the expression of the solution of Problem II has been provided, and a numerical algorithm and a numerical example are included.

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2. The Solution of Problem I

Let us first introduce some lemmas.

Lemma 1. *Suppose the Singular-Value Decomposition (SVD) of matrix S in Problem I is*

$$S = U_1 \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} V_1^T = U_{11}\Lambda V_{11}^T, \tag{2.1}$$

where $U_1 = (U_{11},\ U_{12}) \in OR^{n\times n}$, $U_{11} \in R^{n\times r}$, $V_1 = (V_{11}, V_{12}) \in OR^{p\times p}$, $V_{11} \in R^{p\times r}$, $\Lambda = diag(\sigma_1, \sigma_2, \ldots, \sigma_r) > 0$, and $r = rank(S)$. Let

$$U_1^T A U_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, A_{11} \in R^{r\times r}. \tag{2.2}$$

Then $A \in SR_{s,n}$ if and only if $A_{11} \in SR^{r\times r}$ and $A_{12} = A_{21}^T \in R^{r\times(n-r)}$.

Proof. If $A \in SR_{s,n}$, then by $x^T(A - A^T) = 0$, for all $x \in R(S)$, we have

$$S^T(A - A^T) = 0. \tag{2.3}$$

Substitute (2.1) and (2.2) into (2.3), we have $V_1 \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} - A_{11}^T & A_{12} - A_{12}^T \\ A_{21} - A_{21}^T & A_{22} - A_{22}^T \end{pmatrix} U_1^T = 0$, i.e., $\begin{pmatrix} \Lambda(A_{11} - A_{11}^T) & \Lambda(A_{12} - A_{12}^T) \\ 0 & 0 \end{pmatrix} = 0$. Hence $A_{11} \in SR^{r\times r}$ and $A_{12} = A_{21}^T \in R^{r\times(n-r)}$.

Conversely, for all $x \in R(S)$, there exists $y \in R^{p\times 1}$ such that $x = Sy = U_1 \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} V_1^T y$. By $A_{11} = A_{11}^T, A_{12} = A_{21}^T$, we have

$$\begin{aligned} x^T(A - A^T) &= (V_1^T y)^T \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} U_1^T (A - A^T) \\ &= (V_1^T y)^T \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} - A_{11}^T & A_{12} - A_{21}^T \\ A_{21} - A_{12}^T & A_{22} - A_{22}^T \end{pmatrix} U_1^T \\ &= 0. \end{aligned}$$

Hence $A \in SR_{s,n}$.

Lemma 2[2]**.** *Given $Z \in R^{n\times k}, Y \in R^{m\times k}$, and the SVD of Z is*

$$Z = \tilde{U}_1 \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} \tilde{V}_1 = \tilde{U}_{11}\Delta \tilde{V}_{11}^T, \tag{2.4}$$

where $\tilde{U}_1 = (\tilde{U}_{11},\ \tilde{U}_{12}) \in OR^{n\times n}$, $\tilde{U}_{11} \in R^{n\times r_0}$, $\tilde{V}_1 = (\tilde{V}_{11}, \tilde{V}_{12}) \in OR^{k\times k}$, $\tilde{V}_{11} \in R^{k\times r_0}$, $\Delta = diag(\delta_1, \delta_2, \ldots, \delta_{r_0}) > 0, r_0 = rank(Z)$. Then there is a matrix $A \in R^{m\times n}$ such that $AZ = Y$ if and only if $Y\tilde{V}_{12} = 0$. In that case the general solution can be expressed as $A = YZ^+ + \tilde{G}\tilde{U}_{12}^T$, where $\tilde{G} \in R^{m\times(n-r_0)}$ is arbitrary matrix.

Lemma 3[2]**.** *Given $Z, Y \in R^{n\times k}$, and the SVD of Z is of the form (2.4). Then there is a matrix $A \in SR^{n\times n}$ such that $AZ = Y$ if and only if $Z^TY = Y^TZ$ and $Y\tilde{V}_{12} = 0$. In that case the general solution can be expressed as $A = YZ^+ + (YZ^+)^T(I_n - ZZ^+) + \tilde{U}_{12}\tilde{M}\tilde{U}_{12}^T$, where $\tilde{M} \in SR^{(n-r_0)\times(n-r_0)}$ is arbitrary matrix.*

Partition U_1^TX and U_1^TB, where U_1 is the same as (2.1), into the following form

$$U_1^T X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, U_1^T B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, X_1, B_1 \in R^{r\times m}, X_2, B_2 \in R^{(n-r)\times m}. \tag{2.5}$$

Suppose the SVD of X_2 is

$$X_2 = U_2 \begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix} V_2^T = U_{21}\Gamma V_{21}^T \tag{2.6}$$

where $U_2 = (U_{21}, U_{22}) \in OR^{(n-r)\times(n-r)}, U_{21} \in R^{(n-r)\times k_1}, V_2 = (V_{21}, V_{22}) \in OR^{m\times m}, V_{21} \in R^{m\times k_1}, \Gamma = \text{diag}(a_1, a_2, \ldots, a_{k_1}) > 0, k_1 = \text{rank}(X_2)$.

Suppose the SVD of (X_1V_{22}) is

$$X_1 V_{22} = U_3 \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} V_3^T = U_{31}\Omega V_{31}^T \tag{2.7}$$