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AN INVERSE EIGENVALUE PROBLEM FOR JACOBI MATRICES *1)

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Abstract

Let $T_{1,n}$ be an $n \times n$ unreduced symmetric tridiagonal matrix with eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n.$$

and

$$W_k = \begin{pmatrix} T_{1,k-1} & 0 \\ 0 & T_{k+1,n} \end{pmatrix}$$

is an $(n-1) \times (n-1)$ submatrix by deleting the k^{th} row and k^{th} column, $k = 1, 2, \cdots, n$ from T_n.
Let

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k-1}$$

be the eigenvalues of $T_{1,k-1}$ and

$$\mu_k \leq \mu_{k+1} \leq \cdots \leq \mu_{n-1}$$

be the eigenvalues of $T_{k+1,n}$.
A new inverse eigenvalues problem has put forward as follows: How do we construct an unreduced symmetric tridiagonal matrix $T_{1,n}$, if we only know the spectral data: the eigenvalues of $T_{1,n}$, the eigenvalues of $T_{1,k-1}$ and the eigenvalues of $T_{k+1,n}$?
Namely if we only know the data: $\lambda_1, \lambda_2, \cdots, \lambda_n$, $\mu_1, \mu_2, \cdots, \mu_{k-1}$ and $\mu_k, \mu_{k+1}, \cdots, \mu_{n-1}$ how do we find the matrix $T_{1,n}$? A necessary and sufficient condition and an algorithm of solving such problem, are given in this paper.

Key words: Symmetric tridiagonal matrix, Jacobi matrix, Eigenvalue problem, Inverse eigenvalue problem.

1. Introduction

Let

$$T_n = \begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ 0 & & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

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be an $n \times n$ unreduced symmetric tridiagonal matrix, and denote its submatrix $T_{p,q}, (p < q)$ as follows

$$T_{p,q} = \begin{pmatrix} \alpha_p & \beta_p & & & 0 \\ \beta_p & \alpha_{p+1} & \beta_{p+1} & & \\ & \beta_{p+1} & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{q-1} \\ 0 & & & \beta_{q-1} & \alpha_q \end{pmatrix} \quad p < q$$

We call an unreduced symmetric tridiagonal matrix with $\beta_i > 0$ as a Jacobi matrix.
Consider $T_{1,n}$ and $T_{p,q}$ to be Jacobi matrices. The matrix

$$W_k = \begin{pmatrix} T_{1,k-1} & 0 \\ 0 & T_{k+1,n} \end{pmatrix}$$

is gained by deleting the k^{th} row and the k^{th} column $(k = 1, 2, ..., n)$ from T_n. We put forward an inverse eigenvalue problem to be that: If we don't know the matrix $T_{1,n}$, but we know all eigenvalues of matrix $T_{1,k-1}$, all eigenvalues of matrix $T_{k+1,n}$, and all eigenvalues of matrix $T_{1,n}$, could we construct the matrix $T_{1,n}$. Let $\mu_1, \mu_2, \cdots, \mu_{k-1}, \mu_k, \mu_{k+1}, \cdots, \mu_{n-1}$, and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of matrices $T_{1,k-1}, T_{k+1,n}$ and $T_{1,n}$ respectively.Our problem is that from above 2n-1 data to find other 2n-1 data:

$$\alpha_1, \alpha_2, \cdots, \alpha_n, and\ \beta_1, \beta_2, \cdots, \beta_{n-1}$$

Obviously, when k=1 or k=n this problem has been solved and there are many algorithms to construct $T_{1,n}$ [1],[2],[4],[5],[10] . When we delet the k^{th} row and the k^{th} column from $T_{1,n}$, in cases $k = 2, 3, \cdots, n-1$, it means to delet three numbers α_k, β_{k-1},and β_k ,while in case k=1,or n, it only delets two numbers α_1, β_1 or α_n, β_{n-1}. So there is a difference betweem them. For simplicity, we call the case $k = 2, 3, \cdots, n-1$, above inverse eigenvalue problem as (k) jacobi matrix inverse eigenvalue problem.We also call (1) Jacobi matrix inverse eigenvalue problem, (n) Jacobi matrix inverse eigenvalue problem when $k = 1, k = n$ respectively.More simple we call them as (k)problem, (1)problem and (n)problem, respectively. In sectin 2, some basic theorems such as secular equation, separation theorem are discussed, and the sufficient and necessary condition for (k) problem has an unique solution, when $T_{1,k-1}$ and $T_{k+1,n}$ have no common eigenvalue, are given. In section 3, a discussion of the special case, when $T_{1,k-1}$ and $T_{k+1,n}$ have common eigenvalues, is given. There is a sufficient and necessary condition for (k) problem. The interesting fact is that in this case, if (k) problem has a solution, then there are infinite solutions. In section 4, an algorithm and numerical examples are put forward.

2. The Basic Theorems

Theorem 1. *Let $T_{1,n} = T_n$ be $n \times n$ unreduced symmetric tridiagonal matrix, whose eigenvalues are $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. The matrix*

$$W_k = \begin{pmatrix} T_{1,k-1} & 0 \\ 0 & T_{k+1,n} \end{pmatrix}$$

is gained by deleting the k^{th} row and the k^{th} column from T_n, for $k = 1, 2, \cdots, n$. Let μ_i,$i = 1, 2, \cdots, k-1$ are the eigenvalues of $T_{1,k-1}$ and the corresponding unit eigenvectors are $S_i^{(1)}$, $i = 1, 2 \cdots, k-1$. Let μ_i, $i = k, k+1, \cdots, n-1$ are the eigenvalues of $T_{k+1,n}$ and the corresponding unit eigenvectors are $S_i^{(2)}$,$i = k, k+1, \cdots, n-1$. Denote the $(k-1)^{th}$ component of $S_i^{(1)}$ to be